

TORSION THEORY, CO-COHEN-MACAULAY AND LOCAL HOMOLOGY

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ABSTRACT. Let A be a commutative ring and M an Artinian A -module. Let σ be a torsion radical functor and (T, F) its corresponding partition of $\text{Spec}(A)$. In [1] the concept of Cohen-Macaulay modules was generalized. In this paper we shall define σ -co-Cohen-Macaulay (abbr. σ -co-CM). Indeed this is one of the aims of this paper, we obtain some satisfactory properties of such modules. Another aim of this paper is to generalize the concept of cograde by using the left derived functor $U_i^{\mathfrak{a}}(-)$ of the \mathfrak{a} -adic completion functor, where \mathfrak{a} is contained in Jacobson radical of A .

1. Introduction, preliminaries and some properties of σ -cograde

Throughout this note A will denote a commutative ring with non-zero identity and σ will be a torsion radical functor over A and $(\mathcal{T}, \mathcal{F})$ will be its torsion theory corresponding σ and (T, F) will be the corresponding partition of $\text{Spec}(A)$ [see 2]. The elements of \mathcal{T} are called torsion modules and the modules in \mathcal{F} are called torsion free modules. We also use T_0 to denote the set of minimal prime elements in T . Let B be another commutative ring and $\phi : A \longrightarrow B$ be a ring homomorphism. We denote the direct image of σ under ϕ by σ_ϕ (See [3, Section 3]). Let \mathfrak{a} be an ideal of A . Let $T = v(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{a} \subseteq \mathfrak{p}\}$ and $(\mathcal{T}_{\mathfrak{a}}, \mathcal{F}_{\mathfrak{a}})$ be the torsion theory corresponding to the partition (T, F) of $\text{Spec}(A)$. We denote the torsion functor corresponding to $(\mathcal{T}_{\mathfrak{a}}, \mathcal{F}_{\mathfrak{a}})$ by $\sigma_{\mathfrak{a}}$. It is easy to see that the partition corresponding to $\sigma_{\mathfrak{a}}$ is T .

We use \mathbb{N} (\mathbb{N}_0) to denote the set of positive (non-negative, respectively) integers. We say that a sequence of elements x_1, x_2, \dots, x_n of A

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is a poor M -cosequence if

$$0 :_M (x_1, x_2, \dots, x_{i-1}) \xrightarrow{x_i} 0 :_M (x_1, x_2, \dots, x_{i-1})$$

is surjective for $i = 1, 2, \dots, n$; it is an M -cosequence if, in addition, $0 :_M (x_1, x_2, \dots, x_n) \neq 0$. In particular, $x \in A$ is called an M -coregular element if $xM = M$. It is easy to see that if x_1, \dots, x_n is an M -cosequence then x_i, x_{i+1}, \dots, x_n is $0 :_M (x_1, \dots, x_{i-1})$ -cosequence for all $i = 1, 2, \dots, n$. Let M be an Artinian A -module. Then $\{x \in A : xM \neq M\} = \bigcup_{\mathfrak{p} \in \text{Att}_A(M)} \mathfrak{p}$, where $\text{Att}_A(M)$ is the set of the attached prime ideal of M (See [5, 2.6]).

PROPOSITION 1.1. (See [11, 1.9 and 1.10]). *Let \mathfrak{a} be an ideal of A and suppose that M be an Artinian A -module. Then*

- (i) $M = \mathfrak{a}M$ if only if \mathfrak{a} contains an M -coregular element.
- (ii) If x_1, x_2, \dots, x_n is a poor M -cosequence in \mathfrak{a} , then $\text{Tor}_n^A(M, \frac{A}{\mathfrak{a}}) \simeq (0 :_M (x_1, \dots, x_n)) \otimes_A \frac{A}{\mathfrak{a}}$ and $\text{Tor}_i^A(M, \frac{A}{\mathfrak{a}}) = 0$ for any $i < n$.
- (iii) If $(0 :_M \mathfrak{a}) \neq 0$, then every maximal M -cosequence in \mathfrak{a} has finite length.

Suppose that M is Artinian and \mathfrak{a} an ideal of A such that $0 :_M \mathfrak{a} \neq 0$. Then we denote by $\text{cograde}_{\mathfrak{a}}(M)$ the length of a maximal M -cosequence in \mathfrak{a} . Note that, in view of 1.1, all such sequences have the same length and $\text{cograde}_{\mathfrak{a}}(M) = \inf\{n \in \mathbb{N}_0 : \text{Tor}_n^A(M, \frac{A}{\mathfrak{a}}) \neq 0\}$ (if $0 :_M \mathfrak{a} = 0$, then $\text{cograde}_{\mathfrak{a}}(M)$ is interpreted as ∞). If $M \neq 0$ is an Artinian A -module and \mathfrak{a} is contained in Jacobson radical of A , then the assumption $0 :_M \mathfrak{a} \neq 0$ is always satisfied (See [4, Corollary page 57]). Furthermore, if (A, \mathfrak{m}) be a quasi local ring and M is a non-zero finitely generated and Artinian A -module. Then $\text{cograde}_{\mathfrak{m}}(M) = 0$.

Let \mathfrak{a} be an ideal of A . Then A -modules are given the \mathfrak{a} -adic topology. The completion of an A -module M is denoted by \hat{M} : thus $\hat{M} = \varprojlim_{n \in \mathbb{N}} \frac{M}{\mathfrak{a}^n M}$.

The left derived functors of the \mathfrak{a} -adic completion functor are denoted by $U_i^{\mathfrak{a}}(-)$ and are called i -th local homology functors. These were studied by Matlis when the ideal \mathfrak{a} is generated by a finite regular sequence and are used in [8] where the ring is Noetherian. Let \mathfrak{a} be ideal of A . In [9] $U_-^{\mathfrak{a}}(M)$ is defined by $\inf\{i \in \mathbb{N}_0 : U_i^{\mathfrak{a}}(M) \neq 0\}$.

PROPOSITION 1.2. (See [9, 2.4]). *Let \mathfrak{a} be an ideal of A . If \mathfrak{a} is contained in Jacobson radical of A or if A is Noetherian, then, for each A -module M*

$$U_-^{\mathfrak{a}}(M) = \inf\{i \in \mathbb{N}_0 : \text{Tor}_i^A(\frac{A}{\mathfrak{a}}, M) \neq 0\}.$$

If M is an Artinian module and \mathfrak{a} is contained in Jacobson radical then $0 :_M \mathfrak{a} \neq 0$. Therefore by Proposition 1.2 we have $U_{-}^{\mathfrak{a}}(M) = \text{cograde}_{\mathfrak{a}}(M)$.

DEFINITION 1.3. Let M be an Artinian A -module. We define σ - $\text{cograde}_A(M) = \inf\{\text{cograde}_{\mathfrak{p}}(M) : \mathfrak{p} \in T\}$.

PROPOSITION 1.4. Let M be an Artinian A -module and \mathfrak{a} an ideal of A . Then

- (i) σ - $\text{cograde}_A(M) = \inf\{\text{cograde}_{\mathfrak{p}}(M) : \mathfrak{p} \in T_0\}$.
- (ii) σ - $\text{cograde}_A(M) = \inf\{\text{cograde}_{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in T\}$.
- (iii) $\sigma_{\mathfrak{a}}\text{-cograde}_A(M) = \text{cograde}_{\mathfrak{a}}(M)$.

Proof. (i) It is evident from the definition that σ - $\text{cograde}_A(M) \leq \inf\{\text{cograde}_{\mathfrak{p}}(M) : \mathfrak{p} \in T_0\}$. We may assume that σ - $\text{cograde}_A(M)$ is finite. Let $\mathfrak{p} \in T$ be a prime ideal of A such that σ - $\text{cograde}_A(M) = \text{cograde}_{\mathfrak{p}}(M)$. There exists $\mathfrak{p}_0 \in T_0$ such that $\mathfrak{p}_0 \subseteq \mathfrak{p}$. It follows that $\inf\{\text{cograde}_{\mathfrak{q}}(M) : \mathfrak{q} \in T_0\} \leq \text{cograde}_{\mathfrak{p}_0}(M) \leq \text{cograde}_{\mathfrak{p}}(M) = \sigma$ - $\text{cograde}_A(M)$.

(ii) Since $\text{cograde}_{\mathfrak{p}}(M) \leq \text{cograde}_{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for all $\mathfrak{p} \in T$, we have σ - $\text{cograde}_A(M) \leq \inf\{\text{cograde}_{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in T\}$. We may assume that σ - $\text{cograde}_A(M)$ is finite. Let $\mathfrak{p} \in T_0$ so that $(0 :_M \mathfrak{p}) \neq 0$ and choose a maximal M -cosequence $\underline{x} = x_1, \dots, x_n$ in \mathfrak{p} . Then $\mathfrak{p} \subseteq \bigcup_{\mathfrak{q} \in \text{Att}_A(0 :_M \underline{x})} \mathfrak{q}$. It follows that there exists $\mathfrak{q} \in \text{Att}_A(0 :_M \underline{x})$ with $\mathfrak{p} \subseteq \mathfrak{q}$ (since $\mathfrak{p} \in T$ and $\mathfrak{p} \subseteq \mathfrak{q}$, it follows that $\mathfrak{q} \in T$ (See [2, 1.3])). Now $\mathfrak{q}A_{\mathfrak{q}} \in \text{Att}(0 :_M \underline{x})_{\mathfrak{q}}$ and $(0 :_M \underline{x})_{\mathfrak{q}} \simeq (0 :_{M_{\mathfrak{q}}} \underline{x})$, the ideal of $\mathfrak{q}A_{\mathfrak{q}}$ consist of non-coregular on $(0 :_{M_{\mathfrak{q}}} \underline{x})$ and \underline{x} is a maximal $M_{\mathfrak{q}}$ -cosequence.

(iii) Similar to the proof of (ii).

If $M_{\mathfrak{p}}$ is finitely generated for some $\mathfrak{p} \in T \cap \text{Supp}_A(M)$ and M be Artinian A -module, then σ - $\text{cograde}_A(M) = 0$. Note that by 1.4(ii) σ - $\text{cograde}_A(M) = \inf\{U_{-}^{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in T\}$. \square

PROPOSITION 1.5. Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of Artinian A -modules. Then one of the following must hold

- (i) σ - $\text{cograde}_A(M'') \geq \sigma$ - $\text{cograde}_A(M') = \sigma$ - $\text{cograde}_A(M)$
- (ii) σ - $\text{cograde}_A(M) \geq \sigma$ - $\text{cograde}_A(M'') = 1 + \sigma$ - $\text{cograde}_A(M')$
- (iii) σ - $\text{cograde}_A(M') \geq \sigma$ - $\text{cograde}_A(M) = \sigma$ - $\text{cograde}_A(M'')$

Proof. Let $n \in \mathbb{N}_0$ and σ - $\text{cograde}_A(M') = n + 1$.

Case 1: $\sigma\text{-cograde}_A(M) = n + 1$. The exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ induces the long exact sequence

(*)

$$\dots \longrightarrow U_i^{\mathfrak{p}A_{\mathfrak{p}}}(M'_p) \longrightarrow U_i^{\mathfrak{p}A_{\mathfrak{p}}}(M_p) \longrightarrow U_i^{\mathfrak{p}A_{\mathfrak{p}}}(M''_p) \longrightarrow U_{i-1}^{\mathfrak{p}A_{\mathfrak{p}}}(M'_p) \longrightarrow \dots$$

for all $\mathfrak{p} \in T$. We have $U_i^{\mathfrak{p}A_{\mathfrak{p}}}(M_p) = U_i^{\mathfrak{p}A_{\mathfrak{p}}}(M'_p) = 0$ for all $\mathfrak{p} \in T$ and $i = 0, 1, \dots, n$. Therefore by (*), $U_i^{\mathfrak{p}A_{\mathfrak{p}}}(M''_p) = 0$ for all $\mathfrak{p} \in T$ and $i = 0, 1, \dots, n$, thus $\sigma\text{-cograde}_A(M'') \geq n + 1$.

Case 2: $\sigma\text{-cograde}_A(M) < n + 1$. By (*), we have $U_i^{\mathfrak{p}A_{\mathfrak{p}}}(M_p) \simeq U_i^{\mathfrak{p}A_{\mathfrak{p}}}(M''_p)$ for all $\mathfrak{p} \in T$ and $i = 0, 1, \dots, n$, thus $\sigma\text{-cograde}_A(M') \geq \sigma\text{-cograde}_A(M'') = \sigma\text{-cograde}_A(M)$.

Case 3: $\sigma\text{-cograde}_A(M) > n + 1$. Hence we have $U_i^{\mathfrak{p}A_{\mathfrak{p}}}(M''_p) \simeq U_{i-1}^{\mathfrak{p}A_{\mathfrak{p}}}(M'_p)$ for all $\mathfrak{p} \in T$ and $i = 1, 2, \dots, n + 1$ and since the sequence $U_{n+2}^{\mathfrak{p}A_{\mathfrak{p}}}(M''_p) \longrightarrow U_{n+1}^{\mathfrak{p}A_{\mathfrak{p}}}(M'_p) \longrightarrow 0$ is exact for all $\mathfrak{p} \in T$, $\sigma\text{-cograde}_A(M'') = n + 2$. The case of $\sigma\text{-cograde}_A(M') = \infty$ is trivial. For the case $\sigma\text{-cograde}_A(M') = 0$ there exists $\mathfrak{p} \in T$ such that $U_0^{\mathfrak{p}A_{\mathfrak{p}}}(M'_p) \neq 0$, if $U_0^{\mathfrak{p}A_{\mathfrak{p}}}(M_p) \neq 0$, then by (*) $\sigma\text{-cograde}_A(M) = 0$ and if $U_0^{\mathfrak{p}A_{\mathfrak{p}}}(M_p) = 0$ then $\sigma\text{-cograde}_A(M'') = 1$. \square

COROLLARY 1.6. *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of Artinian A -modules. Then*

$$\sigma\text{-cograde}_A(M) \geq \min\{\sigma\text{-cograde}_A(M'), \sigma\text{-cograde}_A(M'')\}$$

PROPOSITION 1.7. *Let M be an Artinian A -module. Suppose that $x \in \bigcap_{\mathfrak{p} \in T_0} \mathfrak{p}$ is an M -coregular. Then*

$$\sigma\text{-cograde}_A(0 :_M x) = \sigma\text{-cograde}_A(M) - 1$$

Proof. Since $x \in \mathfrak{p}$ is an M -coregular for all $\mathfrak{p} \in T_0$, by [11, 1.12] $\text{cograde}_{\mathfrak{p}}(0 :_M x) = \text{cograde}_{\mathfrak{p}}(M) - 1$ for all $\mathfrak{p} \in T_0$. Now by Properties of infimum

$$\sigma\text{-cograde}_A(0 :_M x) = \sigma\text{-cograde}_A(M) - 1 \quad \square$$

COROLLARY 1.8. *Let M be an Artinian A -module and $x_1, x_2, \dots, x_n \in \bigcap_{\mathfrak{p} \in T_0} \mathfrak{p}$ is an M -cosequence. Then $\sigma\text{-cograde}(0 :_M (x_1, \dots, x_i)) = \sigma\text{-cograde}_A(M) - i$ for all $i = 1, 2, \dots, n$.*

Proof. We may assume that $\sigma\text{-cograde}_A(M) < \infty$. Since $(0 :_M (x_1, x_2, \dots, x_i)) = (0 :_{(0 :_M (x_1, \dots, x_k))} (x_{k+1}, \dots, x_i))$ for all $1 \leq i \leq n$ and $k \leq i - 1$, the claim can be proved easily using the induction on n . \square

DEFINITION 1.9. (See [3, Definition 1.1]). Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory and M an A -module. We define the $(\mathcal{T}, \mathcal{F})$ -dominant dimension of M , denoted by $(\mathcal{T}, \mathcal{F}) - d_A(M)$, as the least integer n for which the n -th term $E^n(M)$ in a minimal injective resolution for M is not torsion free, if any such integers exist and ∞ otherwise.

In [1] $(\mathcal{T}, \mathcal{F}) - d_A(M)$ was denoted by $\sigma\text{-grade}_A(M)$, where σ is the corresponding torsion functor to the $(\mathcal{T}, \mathcal{F})$. If A be a Noetherian ring and M be finitely generated, then $\sigma\text{-grade}_A(M) = \inf\{\text{grade}_{\mathfrak{p}}(M) : \mathfrak{p} \in T_0\}$ (See [3, 4.3]).

REMARK 1.10. (See [3, p. 75]). Let A be a ring with a torsion theory $(\mathcal{T}, \mathcal{F})$ and $\phi : A \rightarrow B$ a ring homomorphism. Let (T, F) be the partition of $\text{Spec}(A)$ corresponding to $(\mathcal{T}, \mathcal{F})$ and (T^ϕ, F^ϕ) the partition of $\text{Spec}(B)$ corresponding to $(\mathcal{T}^\phi, \mathcal{F}^\phi)$. If $\mathfrak{q} \in \text{Spec}(B)$, then $\mathfrak{q} \in T^\phi$ if and only if $\phi^{-1}(\mathfrak{q}) \in T$. It follows that $\mathfrak{q} \in (T^\phi)_0$ if only if $\phi^{-1}(\mathfrak{q}) \in T_0$.

Let E be the injective hull of the direct sum of all the $\frac{A}{\mathfrak{m}}$ with \mathfrak{m} a maximal ideal of A . The Matlis duality functor is defined by $M^\vee = \text{Hom}_A(M, E)$.

REMARK 1.11. Let M be an Artinian A -module. It follows from [7, 3.2] and the proof of [7, 2.2] that there exists a ring A' with the following properties:

- (i) The ring A' is semi-local commutative Noetherian complete in the topology defined by its Jacobson radical.
- (ii) The module M is, in a natural way, a faithful Artinian module over A' , and moreover a subset of M is an A -module if and only if it is an A' -module.
- (iii) There exists a ring homomorphism $\phi : A \rightarrow A'$ such that the structure of the bi-module M as A -module and A' -module are compatible.

It is easy to deduce from 1.11 (iii) that $(0 :_M I) = (0 :_M I A')$ for every ideal I of A .

PROPOSITION 1.12. (i) Let A be a semi-local Noetherian ring. Suppose that M is finitely generated A -module. Then $\sigma\text{-cograde}_A(M^\vee) = \sigma\text{-grade}_A(M)$.

(ii) Let M be an Artinian A -module. Then $\sigma\text{-cograde}_A(M) = \sigma_\phi\text{-grade}_{A'}(M^*)$ where $\phi : A \rightarrow A'$ is a ring homomorphism in 1.11 and $M^* = \text{Hom}_{A'}(M, E')$ such that E' is the injective hull of the direct sum of all $\frac{A'}{\mathfrak{m}'}$ with \mathfrak{m}' a maximal ideal of A' .

Proof. (i) $M^\vee = \text{Hom}_A(M, E)$ is an Artinian A -module (See [5, 1.6(ii)]). Now, for all $\mathfrak{p} \in T_0$ such that $(0 :_{M^\vee} \mathfrak{p}) \neq 0$, it follows that $\frac{A}{\mathfrak{p}} \otimes M \neq 0$. Now, by [9, 1.5] we have

$$\begin{aligned} \text{cograde}_{\mathfrak{p}}(M^\vee) &= \inf\{i \in \mathbb{N}_0 : \text{Tor}_i^A(\frac{A}{\mathfrak{p}}, M^\vee) \neq 0\} \\ &= \inf\{i \in \mathbb{N}_0 : ((\text{Ext}_A^i(\frac{A}{\mathfrak{p}}, M))^\vee) \neq 0\} \\ &= \inf\{i \in \mathbb{N}_0 : \text{Ext}_A^i(\frac{A}{\mathfrak{p}}, M) \neq 0\} \\ &= \text{grade}_{\mathfrak{p}}(M) \end{aligned}$$

(ii) Let $\mathfrak{p} \in T_0$ such that $(0 :_M \mathfrak{p}) \neq 0$. Then by [12, Remark 3(i)] we have $\text{cograde}_{\mathfrak{p}}(M) = \text{grade}_{\mathfrak{p}A'}(M^*)$. Now, it follows that from 1.10, 1.11 and [5, Theorem 1.6 (iii)]

$$\begin{aligned} \sigma\text{-cograde}_A(M) &= \inf\{\text{cograde}_{\mathfrak{p}}(M) : \mathfrak{p} \in T_0\} \\ &= \inf\{\text{grade}_{\mathfrak{p}A'}(M^*) : \mathfrak{p}A' \in (T^\phi)_0\} \\ &= \sigma_\phi\text{-grade}_{A'}(M^*) \end{aligned}$$

(Note that if $\mathfrak{q} \in (T^\phi)_0$ and $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$, then $\mathfrak{p}A' \in T^\phi$ and $\mathfrak{p}A' \subseteq \mathfrak{q}$. Now by minimality \mathfrak{q} , we have $\mathfrak{q} = \mathfrak{p}A'$). \square

2. σ -Krull dimension

Throughout this section M is an Artinian A -module. Let (A, \mathfrak{m}) be a quasi-local ring. The Krull dimension of M ($K\dim_A(M)$) is defined inductively as follows: when $M = 0$, put $K\dim_A(M) = -1$. Then by induction for any integer $r \geq 0$, put $K\dim_A(M) = r$, if (1) $K\dim_A(M) < r$ is false. (2) for any ascending chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ of submodules of M , there exists an integer n such that $K\dim_A(\frac{M_{i+1}}{M_i}) < r$ for all $i > n$.

Note that if $M \neq 0$, then by [6, Theorem 6], we have

$$K\dim_A(M) = \inf\{i \in \mathbb{N}_0 : \text{there exist } x_1, \dots, x_i \in \mathfrak{m} \text{ such that } (0 :_M (x_1, x_2, \dots, x_i)) \text{ has finite length}\}.$$

In particular, $K\dim_A(M) = 0$ if $M \neq 0$ and it has finite length.

In this section some of the main points concerning Krull dimension of M in [11] are extended.

DEFINITION 2.1. We define $\sigma\text{-}K\dim_A(M) = \sup\{K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in T_0\}$ if this supremum exists, and ∞ otherwise.

Let (A, \mathfrak{m}) be a quasi-local ring. Then

$$\sigma_{\mathfrak{m}}\text{-}K\dim_A(M) = \sup\{K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in T_0 = \{\mathfrak{m}\}\} = K\dim_A(M)$$

COROLLARY 2.2. Let (A, \mathfrak{m}) be a quasi-local ring and $T_0 \subseteq \text{Supp}_A(M)$. Then $\sigma\text{-}K\dim_A(M) \leq K\dim_A(M)$.

Proof. We may assume that $M \neq 0$. Let $x_1, x_2, \dots, x_i \in \mathfrak{m}$ such that $l(0 :_M (x_1, \dots, x_i)) < \infty$. Then $l(0 :_{M_{\mathfrak{p}}} (\frac{x_1}{1}, \frac{x_2}{1}, \dots, \frac{x_i}{1})) < \infty$ for all $\mathfrak{p} \in \text{Spec}(A)$.

Thus, $K\dim_A(M) \geq K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for all $\mathfrak{p} \in T_0$, which completes the proof. \square

PROPOSITION 2.3. Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of Artinian A -modules. Then

- (i) $\sigma\text{-}K\dim_A(M) = \max\{\sigma\text{-}K\dim_A(M'), \sigma\text{-}K\dim_A(M'')\}$. In particular $\sigma\text{-}K\dim_A(\bigoplus_{i=1}^n M) = \sigma\text{-}K\dim_A(M)$ for all $n \in \mathbb{N}$.
- (ii) Let $x \in \bigcap_{\mathfrak{p} \in T_0} \mathfrak{p}$ be a $M_{\mathfrak{p}}$ -coregular element for all $\mathfrak{p} \in T_0$. If $\sigma\text{-}K\dim_A(M)$ is finite, then $\sigma\text{-}K\dim_A(0 :_M x) = \sigma\text{-}K\dim_A(M) - 1$

Proof. (i) For all $\mathfrak{p} \in T_0$, by the exact sequence $0 \longrightarrow M'_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow M''_{\mathfrak{p}} \longrightarrow 0$ and [6, Proposition 1] we have $K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \max\{K\dim_{A_{\mathfrak{p}}}(M'_{\mathfrak{p}}), K\dim_{A_{\mathfrak{p}}}(M''_{\mathfrak{p}})\}$. Now we have

$$\begin{aligned} & \sigma\text{-}K\dim_A(M) \\ &= \sup\{K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in T_0\} \\ &= \sup\{\max\{K\dim_{A_{\mathfrak{p}}}(M'_{\mathfrak{p}}), K\dim_{A_{\mathfrak{p}}}(M''_{\mathfrak{p}})\} : \mathfrak{p} \in T_0\} \\ &= \max\{\sup\{K\dim_{A_{\mathfrak{p}}}(M'_{\mathfrak{p}}) : \mathfrak{p} \in T_0\}, \sup\{K\dim_{A_{\mathfrak{p}}}(M''_{\mathfrak{p}}) : \mathfrak{p} \in T_0\}\} \\ &= \max\{\sigma\text{-}K\dim_A(M'), \sigma\text{-}K\dim_A(M'')\}. \end{aligned}$$

The last assertion is clear.

(ii) For each $\mathfrak{p} \in T_0$, $\frac{x}{1}$ is a $M_{\mathfrak{p}}$ coregular element and $K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ is bounded. Thus by [11, 2.2], we have

$$\begin{aligned} & \sigma_K\dim_A(0 :_M x) \\ &= \sup \left(\left\{ K\dim_{A_{\mathfrak{p}}}(0 :_{M_{\mathfrak{p}}} \frac{x}{1}) : \mathfrak{p} \in T_0, K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) > 0 \right\} \right. \\ & \quad \left. \bigcup \{ K\dim_{A_{\mathfrak{p}}}(0 :_{M_{\mathfrak{p}}} \frac{x}{1}) : \mathfrak{p} \in T_0, K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = -1 \text{ or } 0 \} \right) \\ &= \sup \{ K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) - 1 : \mathfrak{p} \in T_0, K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) > 0 \} \\ &= \sup \{ K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in T_0 \} - 1 \\ &= \sigma_K\dim_A(M) - 1. \end{aligned}$$

Let A be a commutative ring and let I, J be ideals of A . Suppose that M is an Artinian A -module. Then there exists a finitely generated ideal $I_0 \subseteq I$ such that $0 :_M (I \cap J) = 0 :_M (I_0 \cap J)$ and $0 :_M I = 0 :_M I_0$ (See [4, Lemma 3]). \square

PROPOSITION 2.4. *Let I, J be ideals of A . Then*

$$\sigma_K\dim_A(0 :_M (I \cap J)) = \max\{\sigma_K\dim_A(0 :_M I), \sigma_K\dim_A(0 :_M J)\}.$$

Proof. We may assume that I is finitely generated, so suppose $I = (x_1, x_2, \dots, x_n)$. Now, consider the exact sequence

$$0 \longrightarrow 0 :_M I \xrightarrow{\subseteq} 0 :_M (I \cap J) \xrightarrow{\alpha} \bigoplus_{i=1}^n (0 :_M J),$$

where $\alpha(a) = (x_1 a, x_2 a, \dots, x_n a)$ for all $a \in 0 :_M (I \cap J)$, and use 2.3 (i) to establish the result. \square

PROPOSITION 2.5. *Let I be an ideal of A . Then $\sigma_K\dim_A(0 :_M I^n) = \sigma_K\dim_A(0 :_M I)$ for all $n \in \mathbb{N}$.*

Proof. For $n \in \mathbb{N}$, by [11, 2.5] we have

$$\begin{aligned} \sigma_K\dim_A(0 :_M I^n) &= \sup\{K\dim_{A_{\mathfrak{p}}}(0 :_{M_{\mathfrak{p}}} I^n)_{\mathfrak{p}} : \mathfrak{p} \in T_0\} \\ &= \sup\{K\dim_{A_{\mathfrak{p}}}(0 :_{M_{\mathfrak{p}}} (I_{\mathfrak{p}})^n) : \mathfrak{p} \in T_0\} \\ &= \sup\{K\dim(0 :_{M_{\mathfrak{p}}} I_{\mathfrak{p}}) : \mathfrak{p} \in T_0\}. \end{aligned} \quad \square$$

PROPOSITION 2.6. *Let I be an ideal of A . Then*

$$\sigma_K\dim_A(M) = \max\{\sigma_K\dim(0 :_M I), \sigma_K\dim_A(IM)\}.$$

Proof. Since M is an Artinian A -module, by [4, Lemma 3] we have $(0 :_M I) = (0 :_M (x_1, \dots, x_n))$ for some $x_1, x_2, \dots, x_n \in I$. Then consider an exact sequence $0 \longrightarrow 0 :_M I \xrightarrow{\beta} M \xrightarrow{\alpha} \bigoplus_1^n IM$, where β is the inclusion map and $\alpha(a) = (x_1 a, \dots, x_n a)$ for all $a \in M$, and use 2.3(i) to obtain the result. \square

PROPOSITION 2.7. *Let I be an ideal of A and M' be a submodule of M . Then $\sigma_K\dim_A(0 :_{\frac{M}{M'}} I) \leq \sigma_K\dim_A(0 :_M I)$.*

Proof. It follows immediately from the [11, 2.8]. \square

COROLLARY 2.8. *Let I be an ideal of A . Then*

$$\sigma_K\dim_A(0 :_M I) \geq \sigma_K\dim_A\left(\frac{M}{I^n M}\right) \quad \text{for all } n \in \mathbb{N}.$$

Proof. By 2.5 and 2.7, for each $n \in \mathbb{N}$ we have

$$\begin{aligned} \sigma_K\dim_A(0 :_M I) &= \sigma_K\dim(0 :_M I^n) \geq \sigma_K\dim(0 :_{\frac{M}{I^n M}} I^n) \\ &= \sigma_K\dim\left(\frac{M}{I^n M}\right). \end{aligned} \quad \square$$

3. σ -Co-Cohen-Macaulay Modules

Throughout this section M is an Artinian A -module.

PROPOSITION 3.1. *Let M be an A -module and $\text{Supp}(M) \cap T_0 \neq \emptyset$. Then $\sigma_cograde_A(M) \leq \sigma_K\dim_A(M)$.*

Proof. By [11, 2.11] we have

$$\begin{aligned} \sigma_cograde_A(M) &= \inf\{\text{cograde}_{\mathfrak{p}}(M) : \mathfrak{p} \in T_0\} \\ &\leq \inf\{\text{cograde}_{\mathfrak{p}A_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in T_0\} \\ &\leq \sup\{K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in T_0\} = \sigma_K\dim_A(M). \end{aligned} \quad \square$$

DEFINITION 3.2. Let M be a non-zero Artinian A -module. We say that M is a σ -Co-Cohen-Macaulay (σ -C-CM) module if $T_0 \cap \text{Supp}_A(M) = \emptyset$ or $\sigma_cograde_A(M) = \sigma_K\dim_A(M)$.

In particular, M is a Co-Cohen-Macaulay on quasi-local ring (A, \mathfrak{m}) if $\text{cograde}_{\mathfrak{m}}(M) = K\dim_A(M)$.

PROPOSITION 3.3. *Let (A, \mathfrak{m}) be a quasi-local ring. Then M is a Co-CM module if and only if M is $\sigma_{\mathfrak{m}}$ -Co-CM module.*

Proof. We may assume that $T = T_0 = \{\mathfrak{m}\} \subseteq \text{Supp}(M)$. Then, we have

$$\sigma_{\mathfrak{m}}\text{-}K\dim_A(M) = \sup\{K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in T_0\} = K\dim_A(M)$$

and by 1.4 (iii) $\sigma_{\mathfrak{m}}\text{-cograde}_A(M) = \text{cograde}_{\mathfrak{m}}(M)$. \square

PROPOSITION 3.4. *Let M be a non-zero A -module and $M_{\mathfrak{p}}$ is a Noetherian $A_{\mathfrak{p}}$ -module for all $\mathfrak{p} \in T_0 \cap \text{Supp}_A(M)$. Then M is a σ -Co-CM module.*

Proof. We may assume that $T_0 \cap \text{Supp}_A(M) \neq \emptyset$. For all $\mathfrak{p} \in T_0 \cap \text{Supp}_A(M)$, $M_{\mathfrak{p}}$ has a finite length. It follows that $K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$. By 3.1 $\sigma\text{-cograde}_A(M) \leq \sigma\text{-}K\dim_A(M) = 0$, hence M is σ -Co-CM. \square

PROPOSITION 3.5. *Let M be a σ -Co-CM module and $T_0 \cap \text{Supp}_A(M) \neq \emptyset$. Then for all $\mathfrak{p} \in T_0 \cap \text{Supp}_A(M)$, $M_{\mathfrak{p}}$ is a Co-CM $A_{\mathfrak{p}}$ -module.*

Proof.

$$\begin{aligned} \sigma\text{-cograde}_A(M) &\leq \inf\{\text{cograde}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in T_0\} \\ &\leq \sup\{K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) : \mathfrak{p} \in T_0\}. \end{aligned}$$

Since M is a σ -Co-CM, we have $\text{cograde}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = K\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for all $\mathfrak{p} \in T_0 \cap \text{Supp}_A(M)$. \square

PROPOSITION 3.6. *Let x_1, x_2, \dots, x_n be an M -cosequence in $\bigcap_{\mathfrak{p} \in T_0} \mathfrak{p}$, $\sigma\text{-}K\dim_A(M) < \infty$ and M is σ -Co-CM. Then $0 :_M (x_1, \dots, x_n)$ be σ -Co-CM.*

Proof. We may assume that $T_0 \cap \text{Supp}(M) \neq \emptyset$ so that the result is clear by Corollary 1.8 and Proposition 2.3(ii). \square

PROPOSITION 3.7. *Let S be a multiplicatively closed subset of A such that $\mathfrak{p} \cap S = \emptyset$ for all $\mathfrak{p} \in T$ and $T_0 \cap \text{Supp}_A(M) \neq \emptyset$. Then*

- (i) $S^{-1}\sigma\text{-cograde}_{S^{-1}A}(S^{-1}M) = \sigma\text{-cograde}_A(M)$
- (ii) $S^{-1}\sigma\text{-}K\dim_{S^{-1}A}(S^{-1}M) = \sigma\text{-}K\dim_A(M)$

Proof. Let $\mathfrak{p} \in \text{Spec}(A)$ and $\mathfrak{p} \cap S = \emptyset$. Then by 1.10 $\mathfrak{p} \in T$ if and only if $S^{-1}\mathfrak{p} \in T^{\phi}$ in which $\phi : A \rightarrow S^{-1}A$ is the canonical homomorphism. It follows that $\mathfrak{p} \in T_0$ if and only if $S^{-1}\mathfrak{p} \in (T^{\phi})_0$, where $(T^{\phi})_0$ is the set of minimal elements of T^{ϕ} .

- (i) $S^{-1}\sigma\text{-cograde}_{S^{-1}A}(S^{-1}M)$
 $= \inf\{\text{cograde}_{S^{-1}\mathfrak{p}(S^{-1}A)_{S^{-1}\mathfrak{p}}}(S^{-1}M)_{S^{-1}\mathfrak{p}} : S^{-1}\mathfrak{p} \in T^{\phi}\}$
 $= \inf\{(\text{cograde}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})) : \mathfrak{p} \in T\} = \sigma\text{-cograde}_A(M)$

(ii) In a similar way of (i). \square

COROLLARY 3.8. *In the situation of 3.7, M is a σ -Co-CM if and only if $S^{-1}M$ is a $S^{-1}\sigma$ -Co-CM module over $S^{-1}A$.*

Proof. For $\mathfrak{p} \in T$, $(S^{-1}M)_{S^{-1}\mathfrak{p}} = M_{\mathfrak{p}}$ and thus $T_0 \cap \text{Supp}_A(M) \neq \emptyset$ if and only if $(T^{\phi})_0 \cap \text{Supp}_{S^{-1}A}(S^{-1}M) \neq \emptyset$. Now by proposition 3.7 the statement is obvious. \square

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