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NONLOCAL CAUCHY PROBLEM FOR SOBOLEV TYPE FUNCTIONAL INTEGRODIFFERENTIAL EQUATION

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Abstract. In this paper we prove the existence and uniqueness of a mild solution of a functional differential equation of Sobolev type with nonlocal condition using the semigroup theory and the Banach fixed point principle.

1. Introduction

Byszewski [9] studied the problem of existence of solutions of semilinear evolution equations with nonlocal conditions in Banach spaces. Subsequently several authors have investigated the same type of problem to different classes of abstract differential equations in Banach spaces [1, 2, 4, 6, 7, 10, 13, 16, 17]. Brill [8] and Showalter [18] established the existence of solutions of semilinear evolution equations of Sobolev type in Banach spaces. Lightbourne and Rankin [15] discussed the solution of partial functional differential equation of Sobolev type. This type of equations arise in various applications such as in the flow of fluid through fissured rocks, thermodynamics and shear in second order fluids. Recently Byszewski and Acka [11] established the existence and uniqueness and continuous dependence of a mild solution of a semilinear functional differential equation with nonlocal condition of the form

$$u'(t) + Au(t) = f(t, u_t), \qquad t \in [0, a]$$
$$u(s) + [g(u_{t_1}, ..., u_{t_p})](s) = \phi(s), \qquad s \in [-r, 0],$$

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where C_0 semigroup of operators on a general Banach space, f, g and ϕ are given functions and $u_t(s) = u(t+s)$ for $t \in [0,a]$, $s \in [-r,0]$. Balachandran and Park [5] studied the existence and uniqueness of a mild solution for a functional integrodifferential equation with nonlocal conditions

In this paper we shall prove the existence and uniqueness of a mild solution for a functional integrodifferential equation of Sobolev type with nonlocal conditions of the form

$$(1) \qquad (Bu(t))' + Au(t) = f(t, u_t, \int_0^t k(t,\tau,u_\tau)d\tau), \quad t \in [0,a]$$

$$(2) \qquad u(s) + [g(u_{t_1}, .., u_{t_p})](s) = \phi(s), \qquad s \in [-r,0],$$

where B and A are linear operators with domains contained in a Banach space W and ranges contained in a Banach space E and $\phi \in C([-r,0] : E)$ and the nonlinear operators f, k, g are given functions satisfying some assumptions.

2. Preliminaries

In order to prove our main theorem we assume certain conditions on the operators A and B. Let W and E be Banach spaces with norm $|.|$ and $\|.\|$ respectively. The operators $A : D(A) \subset W \to E$ and $B : D(B) \subset W \to E$ satisfy the following hypothesis:

(H_1) A and B are closed linear operators,
(H_2) $D(B) \subset D(A)$ and B is bijective,
(H_3) $B^{-1} : E \to D(B)$ is continuous.

From the above fact and the closed graph theorem imply the boundedness of the linear operator $AB^{-1} : E \to E$. Further $-AB^{-1}$ generates a uniformly continuous semigroup $T(t), t \geq 0$ and so $\max_{t\in I} \|T(t)\|$ is finite. In this sequel the operator norm $\|.\|_{B(E)}$ will be denoted by $\|.\|$. To simplify the notation let us take $I_0 = [-r,0]$, $I = [0,a]$ and $X = C([-r,0] : E)$, $Y = C([-r,a] : E)$, $Z = C([0,a] : E)$. For a continuous function $w : [-r,a] \to E$, we denote w_t a function belonging to X and defined by $w_t = w(t+s)$ for $t \in I$, $s \in I_0$. Let $f : I \times X \times X \to E$, $k : I \times I \times X \to X$ and $\phi \in X$. We denote $M = \max_{t\in I} \|B^{-1}T(t)B\|$, and $R = \|B^{-1}T(t)\|$. We make the following assumptions:

(A_1) For every $u, w \in X$ and $t \in I$, $f(., u_t, w_t) \in Z$.

(A_2) There exists a constant $L > 0$ such that

$$\|f(t, x_t, w_t) - f(t, y_t, u_t)\| \leq L[\|x - y\|_{C([-r,t]:E)} + \|w - u\|_{C([-r,t]:E)}]$$

for $x, y, w, u \in Y, \quad t \in I$.

(A_3) There exists a constant $K > 0$ such that

$$\|k(t, s, x_s) - k(t, s, y_s)\| \leq K\|x - y\|_{C([-r,s]:E)} \text{ for } x, y \in Y, \quad s \in I.$$

(A_4) Let $g : X^p \to X$ and there exists a constant $G > 0$ such that

$$\|[g(w_{t_1}, .., w_{t_p})](s) - [g(u_{t_1}, .., u_{t_p})](s)\| \leq G\|w - u\|_X \text{ for } w, u \in Y, \quad s \in I_0.$$

(A_5) $MG + RLa + RLKa^2 < 1$.

A function $u \in Y$ satisfying the conditions:

$$\text{(i)} \quad u(t) = B^{-1}T(t)B\phi(0) - B^{-1}T(t)B[g(u_{t_1}, .., u_{t_p})](0) + \int_0^t B^{-1}T(t-\tau)f(\tau, u_\tau, \int_0^\tau k(\tau, \theta, u_\theta)d\theta)d\tau, \quad t \in I, \tag{3}$$

$$\text{(ii)} \quad u(s) + [g(u_{t_1}, .., u_{t_p})](s) = \phi(s), \quad s \in I_0$$

is said to be a *mild solution* of the nonlocal Cauchy problem (1)-(2).

3. Existence of a mild solution

THEOREM 3.1. *Assume that the hypotheses $(H_1) - (H_2)$ hold and the functions f and g satisfy conditions $(A_1) - (A_5)$. Then the nonlocal Cauchy problem $(1) - (2)$ has a unique mild solution.*

Proof. Define an operator P on the Banach space Y by the formula

$$(Pu)(t) = \begin{cases} \phi(t) - [g(u_{t_1}, .., u_{t_p})](t), & t \in I_0, \\ B^{-1}T(t)B\phi(0) - B^{-1}T(t)B[g(u_{t_1}, .., u_{t_p})](0) + \\ \int_0^t B^{-1}T(t-\tau)f(\tau, u_\tau, \int_0^\tau k(\tau, \theta, u_\theta)d\theta)d\tau, & t \in I. \end{cases}$$

It is easy to see that P maps Y into itself. Now, we will show that P is a contraction on Y. Consider

$$\begin{aligned} (4) \qquad & (Pw)(t) - (Pu)(t) \\ = \; & [g(w_{t_1}, .., w_{t_p})](t) - [g(u_{t_1}, .., u_{t_p})](t), \text{ for } w, u \in Y, \ t \in [-r, 0) \end{aligned}$$

and

$$\begin{aligned}
& (Pw)(t) - (Pu)(t) \\
(5) \qquad = \; & B^{-1}T(t)B[(g(w_{t_1},..,w_{t_p}))(0) - (g(u_{t_1},..,u_{t_p}))(0)] \\
& + \int_0^t B^{-1}T(t-\tau)[f(\tau, w_\tau, \int_0^\tau k(\tau,\theta,w_\theta)d\theta) \\
& - f(\tau, u_\tau, \int_0^\tau k(\tau,\theta,u_\theta)d\theta)]d\tau, \qquad w,u \in Y, \;\; t \in I.
\end{aligned}$$

From (4) and (A_4),

(6) $\|(Pw)(t) - (Pu)(t)\| \leq G\|w-u\|_Y$, for $w,u \in Y$, $t \in I_0$.

Moreover by (5), (A_2) , (A_3) and (A_4),

$$\begin{aligned}
& \|(Pw)(t) - (Pu)(t)\| \\
\leq \; & \|B^{-1}T(t)B\|\|(g(w_{t_1},..,w_{t_p}))(0) - (g(u_{t_1},..,u_{t_p}))(0)\| \\
& + \int_0^t \|B^{-1}T(t-\tau)\|L[\|w-u\| \\
(7) \qquad & + \int_0^\tau \|k(\tau,\theta,w_\theta) - k(\tau,\theta,u_\theta)\|d\theta]d\tau, \\
\leq \; & MG\|w-u\|_Y + RL\int_0^t [\|w-u\|_{C([-r,s]:E)} \\
& + K\int_0^\tau \|w-u\|_{C([-r,\tau]:E)}d\tau]ds \\
\leq \; & MG\|w-u\|_Y + RLa\|w-u\|_Y + RLKa\int_0^t \|w-u\|_{C([-r,\tau]:E)}ds \\
\leq \; & MG + RLa + RKa^2L\|w-u\|_Y, \quad \text{for } w,u \in Y, \; 0.
\end{aligned}$$

From (6) and (7) we get

(8) $\|Pw - Pu\|_Y \leq q\|w-u\|_Y$, for $w,u \in Y$,

where $q = MG + RLa + RKa^2L$.

Since, $q < 1$ then (8) shows that P is a contraction on Y. Consequently, the operator P satisfies all the assumptions of the Banach contraction mapping theorem. Therefore, in space Y there is a unique fixed point for P and this point is the mild solution of the nonlocal Cauchy problem (1)-(2). □

THEOREM 3.2. *Suppose $(H_1) - (H_3)$ hold and that the functions f and g satisfy assumptions $(A_1) - (A_4)$. Then for each ϕ_1, $\phi_2 \in X$ and* *for the corresponding mild solutions* $u_1,\ u_2$ *of the problems*

$$(9) \qquad (Bu(t))' + Au(t) = f(t, u_t, \int_0^t k(t,\tau,u_\tau)d\tau), \qquad t \in I$$

$$(10) \qquad u(s) + [g(u_{t_1}, .., u_{t_p})](s) = \phi_i(s) \quad s \in I_0,\ (i = 1, 2),$$

the following inequality

$$(11)\ \|u_1 - u_2\|_Y \leq Me^{aRL(1+Ka)}\left[\|\phi_1 - \phi_2\|_X + G\|u_1 - u_2\|_Y\right]$$

is true. Additionally, if $G < \frac{1}{M}e^{-aRL(1+Ka)}$ *then,*

$$(12) \qquad \|u_1 - u_2\|_Y \leq \frac{Me^{aRL(1+Ka)}}{1 - GMe^{aRL(1+Ka)}}\left[\|\phi_1 - \phi_2\|_X\right].$$

Proof. Let $\phi_i\ \ (i = 1, 2)$ be arbitrary functions belonging to X and let $u_i\ (\ i = 1, 2)$ be the mild solutions of problems (9)-(10). Consequently,

$$\begin{aligned}
& u_1(t) - u_2(t) \\
= \ & B^{-1}T(t)B[\phi_1(0) - \phi_2(0)] \\
& - B^{-1}T(t)B[(g((u_1)_{t_1}, .., (u_1)_{t_p}))(0) - (g((u_2)_{t_1}, .., (u_2)_{t_p}))(0)] \\
& + \int_0^t B^{-1}T(t-\tau)[f(\tau, (u_1)_\tau, \int_0^\tau k(\tau,\theta,(u_1)_\theta)d\theta) \\
(13) \qquad & - f(\tau, (u_1)_\tau, \int_0^\tau k(\tau,\theta,(u_2)_\theta)d\theta)]d\tau,\ t \in I
\end{aligned}$$

and

$$\begin{aligned}
& u_1(t) - u_2(t) \\
= \ & [\phi_1(t) - \phi_2(t)] - [(g((u_1)_{t_1}, .., (u_2)_{t_p}))(t) \\
(14) \qquad & - (g((u_2)_{t_1}, .., (u_2)_{t_p}))(t)],\ \text{for } t \in [-r, 0).
\end{aligned}$$

From our assumptions,

$$\begin{aligned}
& \|u_1(\theta) - u_2(\theta)\| \\
\le \; & M\|\phi_1 - \phi_2\|_X + MG\|u_1 - u_2\|_Y \\
+ \; & RL \int_0^\theta [\|u_1 - u_2\|_{C([-r,\theta]:E)} + K \int_0^\tau \|u_1 - u_2\|_{C([-r,\tau]:E)}]ds, \\
\le \; & M\|\phi_1 - \phi_2\|_X + MG\|u_1 - u_2\|_Y \\
+ \; & RL(1 + aK) \int_0^t \|u_1 - u_2\|_{C([-r,s]:E)}ds, \quad \text{for} \quad 0 \le \tau \le \theta \le t \le a.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \sup_{\theta\in[0,t]} \|u_1(\theta) - u_2(\theta)\| \\
\le \; & M\|\phi_1 - \phi_2\|_X + MG\|u_1 - u_2\|_Y \\
+ \; & RL(1 + aK) \int_0^t \|u_1 - u_2\|_{C([-r,s]:E)}ds, \quad \text{for } t \in [0, a]. \qquad (15)
\end{aligned}$$

Simultaneously, by (14) and (A_4),

$$\begin{aligned}
& \|u_1(t) - u_2(t)\| \\
\le \; & M\|\phi_1 - \phi_2\|_X + MG\|u_1 - u_2\|_Y, \quad \text{for} \quad t \in [-r, 0). \qquad (16)
\end{aligned}$$

Since $M \ge 1$, (15) and (16) imply that

$$\begin{aligned}
& \|u_1(t) - u_2(t)\|_{C([-r,t]:E)} \\
\le \; & M\|\phi_1 - \phi_2\|_X + MG\|u_1 - u_2\|_Y \\
+ \; & RL(1 + aK) \int_0^t \|u_1 - u_2\|_{C([-r,s]:E)}ds, \quad \text{for} \quad t \in I. \qquad (17)
\end{aligned}$$

By Gronwall's inequality,

$$\begin{aligned}
& \|u_1(t) - u_2(t)\|_Y \\
\le \; & [M\|\phi_1 - \phi_2\|_X + MG\|u_1 - u_2\|_Y]\, e^{aRL(1+aK)}. \qquad (18)
\end{aligned}$$

and, therefore, (11) holds.

Finally, inequality (12) is a consequence of inequality (11). Hence the proof is complete. □

4. Application

As an application of the Theorem 3.1, we shall consider the system (1) with control parameter

$$(19)\quad (Bu(t))' + Au(t) = Fv(t) + f(t, u_t, \int_0^t k(t,\tau,u_\tau)d\tau), \qquad t \in [0,a]$$

$$(20)\quad u(s) + [g(u_{t_1},..,u_{t_p})](s) = \phi(s), \qquad s \in [-r, 0],$$

where F is a bounded linear operator from V, a Banach space, to E and $v \in L^2(I : V)$. Then the mild solution is given by

$$\begin{aligned} u(t) &= B^{-1}T(t)B\phi(0) - B^{-1}T(t)B[g(u_{t_1},..,u_{t_p})](0) \\ &\quad + \int_0^t B^{-1}T(t-\tau)Fv(\tau)d\tau + \int_0^t B^{-1}T(t-\tau) \\ &\quad \times f(\tau, u_\tau, \int_0^\tau k(\tau,\theta,u_\theta)d\theta)d\tau, \quad t \in I, \\ &\quad u(s) + [g(u_{t_1},..,u_{t_p})](s) = \phi(s), \quad s \in I_0. \end{aligned}$$

We say that the system (19) is *controllable to the origin* if for any given initial function $\phi \in X$ there exists a control $v \in L^2(I : V)$ such that the mild solution $u(t)$ of (19) satisfies $u(a) = 0$.

For the controllability of nonlinear delay systems and Sobolev type systems one can refer the papers [3, 12, 14]. To establish the result we need the following additional hypotheses:

(A_6) The linear operator Q from V into E defined by

$$Qv = \int_0^a B^{-1}T(a-s)Fv(s)ds$$

induces an inverse operator $\tilde{Q}^{-1}$ defined on $L^2(I;V)/kerQ$, such that the operator $F\tilde{Q}^{-1}$ is bounded.

(A_7) $MG + M\|F\tilde{Q}^{-1}\|a[MG + RLa + RLKa^2] + RL + RLKa < 1.$

THEOREM 5.1. *If the hypotheses* $(H_1) - (H_3), (A_1) - (A_4), (A_6)$ *and* (A_7) *are satisfied, then the system* (19) *with* (20) *is controllable.*

Proof. Using the hypothesis (A_6), for an arbitrary function $x(.)$ define the control

$$\begin{aligned} v(t) &= -\tilde{Q}^{-1}[B^{-1}T(a)B\phi(0) - B^{-1}T(a)Bg(u_{t_1},..,u_{t_p})(0)) \\ &+ \int_0^a B^{-1}T(a-s)f(s,u_s,\int_0^s k(s,\tau,u_\tau)d\tau)ds](t). \end{aligned}$$

Now we shall show that, when using this control, the operator defined by

$$(\Phi u)(t) = \begin{cases} B^{-1}T(t)B\phi(0) - B^{-1}T(t)B[g(u_{t_1},..,u_{t_p})](0) \\ +\int_0^t B^{-1}T(t-\tau)Fv(\tau)d\tau \\ +\int_0^t B^{-1}T(t-\tau)f(\tau,u_\tau,\int_0^\tau k(\tau,\theta,u_\theta)d\theta)d\tau, \quad t \in I, \\ \phi(s) - [g(u_{t_1},..,u_{t_p})](s), \quad s \in I_0 \end{cases}$$

has a fixed point. This fixed point is then a solution of equation (19). Substituting $v(t)$ in the above equation we get

$$(\Phi u)(t) = \begin{cases} B^{-1}T(t)B\phi(0) - B^{-1}T(t)B[g(u_{t_1},..,u_{t_p})](0) \\ -\int_0^t B^{-1}T(t-\tau)F\tilde{Q}^{-1} \\ \times [B^{-1}T(a)B\phi(0) - B^{-1}T(a)Bg(u_{t_1},..,u_{t_p})(0)) \\ +\int_0^a B^{-1}T(a-s)f(s,u_s,\int_0^s k(s,\theta,u_\theta)d\theta)ds](\tau)d\tau \\ +\int_0^t B^{-1}T(t-\tau)f(\tau,u_\tau,\int_0^\tau k(\tau,\theta,u_\theta)d\theta)d\tau, \quad t \in I, \\ \phi(s) - [g(u_{t_1},..,u_{t_p})](s), \quad s \in I_0. \end{cases}$$

Clearly, $(\Phi u)(a) = 0$, which means that the control v steers the functional integrodifferential system from the given initial function ϕ to the origin in time a, provided we can obtain a fixed point of the nonlinear operator Φ. The remaining part of the proof is similar to Theorem 3.1 and hence it is omitted. □

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