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NOTE ON THE RESULTS WITH LOWER SEMI-CONTINUITY

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ABSTRACT. In this paper, we introduce the concept of lower semi-continuous from above functions and show that many well-known results, such as Ekland's and Caristi's theorems, remain also true under lower semi-continuous from above functions.

1. Lower semi-continuous from above functions

In what follows, let (X, d) be a metric space. The lower semi-continuo -us condition plays a key role and has been widely used in finding the solution of $\min_{x \in X} f(x)$. See, for example, [1]-[4] and [7]. First, we recall the definition of lower semi-continuity here.

DEFINITION 1.1. A function $f : X \to \overline{R}$ is said to be *lower semi-continuous* at x_0 if, for any sequence (x_n) in X with $x_n \to x_0$,

$$f(x_0) \leq \varliminf_{n \to \infty} f(x_n).$$

Although the lower semi-continuous condition is important, it is not essential for solving some minimization problems. A function which may not be necessarily lower semi-continuous can still obtain its infimum.

The purpose of this paper is to give a generalization of lower semi-continuous functions and to show that many well-known results, such as Ekland's and Caristi's theorems ([5], [6]) are also true under the condition of the lower semi-continuity from above. Let us introduce the following definition:

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DEFINITION 1.2. A function $f : X \to \overline{R}$ is said to be *lower semi-continuous from above* at x_0 if $x_n \to x_0$ and $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq \cdots$ imply that

$$f(x_0) \leq \lim_{n\to\infty} f(x_n).$$

It is obvious that the lower semi-continuity implies the lower semi-continuity from above. The following example shows that the reverse is not true. Thus the lower semi-continuity from above is weaker than the lower semi-continuity.

EXAMPLE 1.3. Let $f : R \to R$ be defined as follows:

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < 0, \\ x^2 + 1 & \text{if } x \geq 0. \end{cases}$$

It is easy to check that the function f is lower semi-continuous from above at 0, but not lower semi-continuous at 0.

Example 1.3 also shows that the epi-graph of f (shortly, *epi*(f)) is not closed. For definition of epi-graph, see [7]. It is well-known that the lower semi-continuity of a function is equivalent to the closedness of its epi-graph ([7]).

PROPOSITION 1.4. *Let D be a compact subset of X and a function $f : D \to \overline{R}$ be lower semi-continuous from above and bounded from below. Then there exists $x_0 \in D$ such that $f(x_0) = \inf_{x\in D} f(x)$.*

Proof. Since D is compact and f is bounded from below, there exists a sequence (x_n) in D such that $x_n \to x_0 \in D$, $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq \cdots$ and $f(x_n) \to \inf_{x\in D} f(x)$. By the lower semi-continuity from above, we have

$$f(x_0) \leq \lim_{n\to\infty} f(x_n) = \inf_{x\in D} f(x).$$

Hence $f(x_0) = \inf_{x\in D} f(x)$. This completes the proof. $\square$

In normed linear spaces, we can introduce the concept of the weak lower semi-continuity from above.

Definition 1.5. Let X be a normed linear space. A function $f : X \to \overline{R}$ is said to be *weak lower semi-continuous from above* at x_0 if $x_n \rightharpoonup x_0$ and $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq \cdots$ imply that

$$f(x_0) \leq \lim_{n\to\infty} f(x_n),$$

where $\rightharpoonup$ represents the weak convergence in X.

It is well-known that, for convex functions, the lower semi-continuity is equivalent to the weak lower semi-continuity (see [7]), but we can not prove that the lower semi-continuity from above is also equivalent to the weak lower semi-continuity from above. We conjecture that this might be true.

The following results can be viewed as generalizations of the corresponding results for lower semi-continuous convex functions:

Theorem 1.6. *Let X be a real reflexive Banach space and $f : D(f) \to \overline{R}$ be a proper lower semi-continuous from above and convex function. Suppose that $\lim_{\|x\|\to\infty} f(x) = +\infty$. Then there exists $x_0 \in D(f)$ such that*

$$f(x_0) = \inf_{x\in D(f)} f(x).$$

Proof. Take $x_n \in D(f)$ for $n = 1, 2, \cdots$ such that

$$\begin{aligned}
&f(x_1) \leq \inf_{x\in D(f)} f(x) + \frac{1}{2},\\
&f(x_2) \leq \min\Big\{f(x_1), \inf_{x\in D(f)} f(x) + \frac{1}{2^2}\Big\},\\
&f(x_3) \leq \min\Big\{\min_{x\in co\{x_1,x_2\}} f(x), \inf_{x\in D(f)} f(x) + \frac{1}{2^3}\Big\},\\
&\cdots,\\
&f(x_{n+1}) \leq \min\Big\{\min_{x\in co\{x_1,x_2,\cdots,x_n\}} f(x), \inf_{x\in D(f)} f(x) + \frac{1}{2^{n+1}}\Big\},\\
&\cdots.
\end{aligned}$$

By assumption, since $\lim_{\|x\|\to\infty} f(x) = +\infty$, the sequence (x_n) is bound -ed. Since X is reflexive, without loss of generality, we may assume $x_n \rightharpoonup y_0$ (otherwise, taking subsequence).

In view of $y_0 \in \overline{co\{x_k, k \geq n\}}$ for $n = 1, 2, \cdots$, there exist a sequence (n_k) of positive integers with $n_1 < n_2 < \cdots$ and $y_{n_k} \in co\{x_{n_{i_k}}, \cdots, x_{n_k}\}$, $n_{k-1} < n_{i_k} \leq n_k$, $k \geq 2$, such that $y_{n_k} \to y_0$. By construction of the sequence (x_n), we know that $(f(y_{n_k}))$ is decreasing and so it follows from the lower semicontinuity from above of f that

$$f(y_0) \leq \lim_{k \to \infty} f(y_{n_k}) = \inf_{x \in D(f)} f(x).$$

This completes the proof. □

THEOREM 1.7. *Let X be a real normed linear space and $f : X \to \overline{R}$ be a lower semi-continuous from above and convex function. Suppose that there exist $x_0 \in D(f)$ and $r_0 > 0$ such that*

$$\inf_{x \in B(x_0, r_0)} f(x) > -\infty.$$

Then there exist $g \in X^*$ and $b \in R$ such that $f(x) > g(x) + b$ for all $x \in X$.

Proof. Since $\inf_{x \in B(x_0, r_0)} f(x) > -\infty$, there exists $a \in R$ such that $f(x) > a + 1$ for all $x \in B(x_0, r_0)$. It is easy to see that $(x_0, a) \notin \overline{epi(f)}$. Since $\overline{epi(f)}$ is closed convex, there exist $g \in X^*$ and $l \in R$ such that

$$g(x_0) + la < g(x) + lf(x)$$

for all $x \in X$. It is obvious that $l > 0$ and so

$$f(x) > -\frac{1}{l} g(x) + \frac{1}{l}(g(x_0) + la)$$

for all $x \in X$. This completes the proof. □

2. Ekland's and Caristi's theorems

In this section, we show that the well-known results of Ekland and Caristi are also true under the condition of the lower semi-continuity from above.

THEOREM 2.1 (Ekland's Variational Principle). *Let (X,d) be a complete metric space, and let $f : X \to \overline{R}$ be lower semi-continuous from above and bounded from below. Then, for each $\epsilon > 0$, $\lambda > 0$ and $f(u_0) \le \inf_{x\in X} f(x) + \epsilon$, there exists $u_1 \in X$ such that*

(1) $f(u_1) \le f(u_0)$,

(2) $f(u) > f(u_1) - \frac{\epsilon}{\lambda} d(u, u_1)$ *for all* $u \neq u_1$.

Proof. Put $x_0 = u_0$. We construct a sequence (x_n) in X inductively as follows: Assume that we have $x_n \in X$ satisfying (1). If $f(u) > f(x_n) - \frac{\epsilon}{\lambda} d(u, x_n)$ for all $u \neq x_n$, then we put $x_{n+1} = x_n$. Otherwise, we set

$$S_n = \{x : f(x) \le f(x_n) - \frac{\epsilon}{\lambda} d(x, x_n)\}.$$

Take $x_{n+1} \in S$ such that

$$f(x_{n+1}) - \inf_{x\in S_n} f(x) \le \frac{1}{2}[f(x_n) - \inf_{x\in S_{n-1}} f(x)].$$

It is easy to see that $(f(x_n))$ is decreasing, and we have

$$\frac{\epsilon}{\lambda} d(x_n, x_{n+1}) \le f(x_n) - f(x_{n+1}).$$

Therefore, it follows that (x_n) is a Cauchy sequence and so let $u_1 = \lim_{n\to\infty} x_n$.

Next, we show that u_1 satisfies our conclusions (1) and (2). In fact, (1) is obvious. Now we prove (2). Since $(f(x_n))$ is decreasing, by the lower semi-continuity from above of f, we have

$$f(u_1) \le \lim_{n\to\infty} f(x_n).$$

If (2) is not true, then there exists $x \in X$ such that

$$f(x) \le f(u_1) - \frac{\epsilon}{\lambda} d(u_1, x).$$

By construction of the sequence (x_n), we have $f(u_1) \le f(x_n) - \frac{\epsilon}{\lambda} d(u_1, x_n)$. Therefore, it follows that

$$f(x) \le f(x_n) - \frac{\epsilon}{\lambda} d(u_1, x_n) - \frac{\epsilon}{\lambda} d(u_1, x) \le f(x_n) - \frac{\epsilon}{\lambda} d(x_n, x).$$

Thus we have $x \in S_n$ for $n = 1, 2, \cdots$ and hence $f(x) \ge \inf_{y\in S_n} f(y)$, which contradicts

$$f(x) < f(u_1) \le \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \inf_{x\in S_n} f(x).$$

Therefore, the conclusion (2) is true. This completes the proof. □

THEOREM 2.2 (Caristi's Fixed Point Theorem). *Let (X, d) be a complete metric space and a function $\phi : X \to R^+$ be lower semi-continuous from above. Suppose that a mapping $T : X \to X$ satisfies the following:*

$$d(x, Tx) \leq \phi(x) - \phi(Tx)$$

for all $x \in X$. Then there exists $x_0 \in X$ such that $Tx_0 = x_0$.

Proof. Take $\epsilon < 1$ and $\lambda = 1$. By Theorem 2.1, there exists $x_0 \in X$ such that $\phi(x_0) \leq \inf_{x \in X} \phi(x) + \epsilon$ and $\phi(x) > \phi(x_0) - \epsilon d(x, x_0)$ for all $x \neq x_0$.

Now, we prove $Tx_0 = x_0$. If x_0 is not a fixed point of T, then we have

(1) $d(x_0, Tx_0) \leq \phi(x_0) - \phi(Tx_0)$,
(2) $\phi(Tx_0) > \phi(x_0) - \epsilon d(Tx_0, x_0)$.

Therefore, we have

$$d(x_0, Tx_0) < \epsilon d(x_0, Tx_0),$$

which is a contradiction. This completes the proof. □

The proof of Caristi's fixed point theorem actually shows the existence of infinite fixed points of the mapping T if we know that ϕ does not obtain its infimum on X, which is called *Caristi's infinite fixed points theorem.*

Now we state its precise form.

THEOREM 2.3 (Caristi's Infinite Fixed Points Theorem). *Let (X, d) be a complete metric space and a function $\phi : X \to R^+$ be lower semi-continuous from above. Suppose that ϕ does not obtain its infimum on X and a mapping $T : X \to X$ satisfies the following:*

$$d(x, Tx) \leq \phi(x) - \phi(Tx)$$

for all $x \in X$. Then T has infinite fixed points in X.

Proof. Suppose that T only has finite fixed points. Let $\mathrm{Fix}(T)$ denote the set of all fixed points of the mapping T. By Theorem 2.2, $\mathrm{Fix}(T)$ is non-empty. Since ϕ does not obtain its infimum on X, we have

$$\inf_{x \in X} \phi(x) < \min_{x \in \mathrm{Fix}(T)} \phi(x).$$

Taking
$$\epsilon < \min\{1, \min_{x\in \mathrm{Fix}(T)} \phi(x) - \inf_{x\in X} \phi(x)\}$$
and $\lambda = 1$, then, by Theorem 2.1, there exists $x_0 \in X$ such that
$$\phi(x_0) \le \inf_{x\in X} \phi(x) + \epsilon$$
and
$$\phi(x) > \phi(x_0) - \epsilon d(x, x_0)$$
for all $x \in X$ with $x \neq x_0$. It is obvious that $x_0 \notin \mathrm{Fix}(T)$.

On the other hand, by the same argument as in Theorem 2.2, we know that $Tx_0 = x_0$, which is a contradiction. Therefore, the mapping T has infinite fixed points in X. This completes the proof. □

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