Journal of the Korea Society of Mathematical Education Series D:
Research in Mathematical Education
Vol. 6, No. 2, September 2002, 159–170

韓國數學教育學會誌 시리즈 D:
<數學教育研究>
제 6 권 제 2 호 2002 년 9 월, 159–170

Conceptualizing the Realistic Mathematics Education Approach in the Teaching and Learning of Ordinary Differential Equations[1]

Kwon, Oh Nam
Department of Mathematics Education, College of Education, Ewha Womans University,
11-1 Daehyun-dong, Seodaemun-ku, Seoul 120-750, Korea;
Email: onkwon@mm.ewha.ac.kr

(Received June 13, 2002)

The undergraduate curriculum in differential equations has undergone important changes in favor of the visual and numerical aspects of the course primarily because of recent technological advances. Yet, research findings that have analyzed students' thinking and understanding in a reformed setting are still lacking. This paper discusses an ongoing developmental research effort to adapt the instructional design perspective of Realistic Mathematics Education (RME) to the teaching and learning of differential equations at Ewha Womans University. The RME theory based on the design heuristic using context problems and modeling was developed for primary school mathematics. However, the analysis of this study indicates that a RME design for a differential equations course can be successfully adapted to the university level.

Keywords: Complement, Distance education, In-class education, Mathematics learning.

INTRODUCTION

During the past decades, there has been a fundamental change in the objectives and nature of mathematics education, as well as a shift in research paradigms. The changes in mathematics education emphasize learning mathematics from realistic situations, students' invention or construction solution procedures, and interaction with other students or the teacher. This shifted perspective has many similarities with the theoretical perspective of Realistic Mathematics Education (RME) developed by Freudenthal (1973; 1991). The RME theory focuses on guided reinvention through mathematizing and takes

[1] Adapted from an invited lecture at the 2nd International Conference on Mathematics Teaching (at undergraduate level), University of Crete, July 2002.

into account students' informal solution strategies and interpretations through experientially real context problems. The heart of this reinvention process involves mathematizing activities in problem situations that are experientially real to students. It is important to note that reinvention is a collective, as well as individual activity, in which whole-class discussions centering on conjecture, explanation, and justification play a crucial role. In the reinvention approach, researchers build upon the work that has been done on symbolizing and modeling in primary-school mathematics (Treffers 1991; Gravemeijer 1994; 1999). Can the framework that was developed for primary school mathematics be adapted to teach differential equations in collegiate mathematics?

For three decades, international comparisons of mathematics achievement have favored primary and secondary students in Korea (Husen 1967; McKnight, Travers, Crosswhite & Swafford 1985a; 1985b; Horvarth 1987; U.S. Department of Education 1997a; 1997b). For instance, Korean eighth grade students ranked second among 41 different nations on the Third International Mathematics and Science Study (TIMSS) (U.S. Department of Education 1996). Superficially, it appears as if Korean students possess advance mathematical knowledge and skills when compare to other students of the same age in different countries. Lew (1999) and Kwon (2002) argued, however, that most Korean students seem quite unable to relate their well-developed manipulative skills to realistic context problems to the real-world situations, as secondary mathematics lessons in Korea put much emphasis on computation and algorithm skills. Korean students, however, are not the only students who have difficulties adapting their mathematical knowledge to real-world situations. Lack of students' understandings of real-world situations and the characteristic of mindless, symbolic manipulation in differential equations has also been noted by a number of mathematicians (e.g., Boyce 1994; Hubbard 1994). The question then becomes how do instructors teach students differential equations in such a meaningful way as to foster students' mathematical growth. RME may give a perspective for conceptualizing this teaching of differential equations since realistic context problems play an essential role from the start and also the point of departure is that context problems can function as anchoring points for the reinvention of mathematics by students themselves (Gravemeijer & Doorman 1999). Such a reinvention process in RME will be paved with realistic context problems that offer students opportunities for progressive mathematizing in differential equations. From the RME perspective, students should learn mathematizing subject matter from realistic situations in differential equations.

The overall purpose of this study is to examine the developmental research efforts to adapt the instructional design perspective of RME to the teaching and learning of differential equations in collegiate mathematics. A differential equations course, highlighting reinvention through progressive mathematization, didactical phenomenology and emergent models design heuristics, was developed. Informed by the instructional design theory of RME and capitalizing on the potential of technology to incorporate qualitative and numerical approaches, this paper offers an approach for conceptualizing the learning and teaching of differential equations that is different from the traditional approach.

THEORETICAL ORIENTATION

Realistic Mathematics Education

RME is rooted in "mathematics as a human activity" and the underlying principles are guided reinvention, didactical phenomenology, and emergent models. These principles are based on Freudenthal's philosophy which emphasizes reinvention through progressive mathematization (Fredenthal 1973; 1991). In RME, context problems are the basis for progressive mathematization, and through mathematizing, the students develop informal context-specific solution strategies from experientially realistic situations (Gravemeijer & Doorman 1999). Thus, it is necessary for the researchers who adapt the instructional design perspective of RME to utilize contextual problems that allow for a wide variety of solution procedures, preferably those which considered together already indicate a possible learning route through a process of progressive mathematization.

Three guiding heuristics for RME instructional design should be considered (Gravemeijer, Cobb, Bowers & Whitenack 2000). The first of these heuristics is re-invention through progressive mathematization. According to the reinvention principle, the students should be given the opportunity to experience a process similar to the process by which the mathematics was invented. The reinvention principle suggests that instructional activities should provide students with experientially realistic situations, and by facilitating informal solution strategies, students should have an opportunity to invent more formal mathematical practices (Freudenthal 1973). Thus, the developer can look at the history of mathematics as a source of inspiration and at informal solution strategies of students who are solving experientially real problems for which they do not know the standard solution procedures yet (Streefland 1991; Gravemeijer 1994) as starting points. Then the developer formulates a tentative learning sequence by a process of progressive mathematization.

The second heuristic is didactical phenomenology. Freudenthal (1973) defines didactical phenomenology as the study of the relation between the phenomena that the mathematical concept represents and the concept itself. In this phenomenology, the focus is on how mathematical interpretations make phenomena accessible for reasoning and calculation. The didactical phenomenology can be viewed as a design heuristic because it suggests ways of identifying possible instructional activities that might support individual activity and whole-class discussions in which the students engage in progressive mathematization (Gravemeijer 1994). Thus the goal of the phenomenological investigation is to create settings in which students can collectively renegotiate increasingly sophisticated solutions to experientially real problems by individual activity and whole-class discussions (Gravemeijer, Cobb, Bowers & Whitenack 2000). RME's third heuristic for instructional design focuses on the role which emergent models play in bridging the gap between informal knowledge and formal mathematics. The term model is understood in a dynamic, holistic sense. As a consequence, the symbolizations that are embedded in the process of modeling and that constitute the model can change over time. Thus, students first develop a model-of a situated activity, and this model later becomes a model-for more sophisticated mathematical reasoning (Gravemeijer & Doorman 1999).

RME's heuristcs of reinvention, didactical phenomenology, and emergent models can serve to guide the development of hypothetical learning trajectories that can be investigated and revised while experimenting in the classroom. A fundamental issue that differentiates RME from an exploratory approach is the manner in which it takes account both of the collective mathematical development of the classroom community and of the mathematical learning of the individual students who participate in it. Thus, RME is aligned with recent theoretical developments in mathematics education that emphasize the socially and culturally situated nature of mathematical activity.

Traditional and Reform-Oriented Approaches in Differential Equations

Traditionally, students who take differential equations in collegiate mathematics are dependent on memorized procedures to solve problems, follow a similar pattern of learning in precalculus mathematics, and follow model procedures given in the textbook or by a teacher. Also, the search for analytic formulas of solution functions in first order differential equations is the typical starting point for developing the concepts and methods of differential equations. This traditional approach emphasizes finding exact solutions to differential equations in closed form, i.e., the dependent variable can be expressed explicitly or implicitly in terms of the independent variable. However, in reality, when modeling a physical or realistic problem with a differential equation, solutions are usually inexpressible in closed form. Therefore, as Hubbard (1994) pointed out, there is a dismaying discrepancy between the view of differential equations as the link between mathematics and science and the standard course on differential equations.

The teaching of differential equations has undergone a vast change over the last ten years because of the tremendous advances in computer technology and the "Reform Calculus" movement. One of the first textbook promoting this reform effort was published by Artigue and Gautheron (1983). More recently, a number of textbooks reflecting on this movement have been written (e.g., Blanchard, Devaney, & Hall 1998; Borelli & Coleman 1998; Kostelich & Armbruster 1997; Hubbard & West 1997). Primary features of these reform-oriented textbooks are content-driven changes made feasible with advances in computer technology. Thus, these textbooks have decreased emphasis on specialized techniques for finding exact solutions to differential equations and have increased the use of computer technology to incorporate graphical and numerical methods for approximating solutions to differential equations (Hubbard, Dcdill, Noonburg & West 1994).

According to Boyce (1995), the primary benefit of incorporating computer technology in differential equations is the visualization of complex relationships that students frequently find too complicated to understand. For example, a typical differential equation, $u'' + 0.2u' + u = \cos\omega t$, $u(0) = 1$, $u'(0) = 0$, can be easily executed with tech-nology, and students can understand the behavior of the system by using technology to draw a three-dimensional plot as a function of both w and t. The main reasons to use computers in a differential equations course are that geometric interpretations of solutions through the use of computer software help students to understand basic concepts such as initial value problems, integral curves, direction fields and flows for dynamical systems (Lu 1995).

In addition, many concepts including phase portrait, stability, stable and unstable manifold, bifurcation and chaos can better be understood by introducing a computer program for teaching and learning. However, the current reform movement in differential equations emphasizes a combination of analytic, graphical, and numerical approaches from the start. Although different from traditional approaches to differential equations, this movement is quite similar to traditional approaches in the way in which conventional graphical and numerical methods are used as the starting point for students' learning, as Rasmussen (1997; 1999) documented. Thus, as is the case with the traditional approach, students typically do not participate in the reinvention or creation of these mathematical ideas associated with graphical and numerical methods, the representation that conventionally accompany these ideas, and the methods themselves. The learning that occurred was characteristic of mindless graphical and numerical manipulation in the reform-oriented approach. In these respects, the learning demonstrates little improvement over traditional approaches where mindless symbolic manipulation was the prevalent mode of operation.

The current curriculum-oriented reform movement in differential equations has some content-based advantages. The approach being developed here seeks to build on and complement these positive aspects by adapting principled perspectives and approaches that have informed the re-thinking of mathematics learning and teaching at the elementary and secondary level to the re-thinking of mathematics learning and teaching of differential equations.

Guided by the RME instructional design theory, students may participate in the reinvention of mathematical idea and methods that comprise a differential equations course. The emphasis on reinvention by no means implies that the instructor is a bystander in the learning process. In fact, the instructor's role might even be more important in this approach than in the traditional dissemination approach to learning. For example, the instructor guides the construction of classroom social and socio-mathematical norms (Yackel, Rasmussen & King 2000) that foster students' reinvention and sophisticated mathematical reasoning in differential equations. Initial work (Trigueros 2000; Yackel et al. 2000; Zandieh & McDonald 1999) suggests that this perspective demonstrates some promise to foster students' mathematics growth in differential equations.

PROJECT CLASSROOM & PRELIMINARY ANALYSIS

A classroom teaching experiment in an introductory course in differential equations was conducted during Fall 2001 at Ewha Womans University with a group of 43 students, most of whom were first-year undergraduate students majoring in mathematics education. Ewha Womans University has over 20,000 students and is one of the most prestigious schools in Korea. Ewha Woman University is also well-known for pre-service teachers education. Over 30% of newly employed in-service secondary mathematics teachers have graduated from Ewha Womans University.

Data based on a methodology for determining the emergence of classroom mathematical practices were collected (Cobb, Stephen, McClain & Gravemeijer 2001). Data from the teaching experiment consisted of videotapes of each class session, including the small group work; field notes made by the observers and the instructor; records of instructional activities and decisions, and copies of students' work such as in-class work, homework assignments, weekly electronic journal entries and reflective portfolios. In addition, experimental curriculum materials as well as programs for the TI-92 calculator were developed. The materials were guided and informed by the RME instructional heuristic and were designed to help students to complete reinvention activities, which occur when students try to devise their own ways of working through a mathematical concept.

In the typical collaborative learning environment of this project, the instructor poses a task, students work in groups of two to four students, and after most groups obtain initial ideas about the task, the class engages in a discussion of students' approaches to the task. Whole-class discussions might continue for 10–15 minutes before another 5–10 minutes segment of small group work took place. This cycle was typically repeated three to four times in a 75-minutes class period. The nature of small group work was not for students to solve a specific problem but to analyze a question and develop reasons to support their thinking. Because of the continuous emphasis on reasoning, whole-class discussions resulted in the emergence of key concepts such as slope fields, phase lines, and bifurcation diagrams.

Research on the design of primary school RME sequences has shown that the concept of emergent models can function as a powerful design heuristic (Gravemeijer 1999). The following example illustrates the RME heuristic that refers to the role models can play in a shift from a model-of a situated activity to a model-for mathematical reasoning in the learning and teaching of differential equations.

Suppose a population of Nomads is modeled by the differential equation $dN/dt = f(N)$.

The graph of *dN/dt* is shown below.

For the following values of the initial population,
What is the long-term value of the population?
Be sure to briefly explain your reasoning.

(1) $N(0)=2$ (2) $N(0)=3$
(3) $N(0)=4$ (4) $N(0)=7$

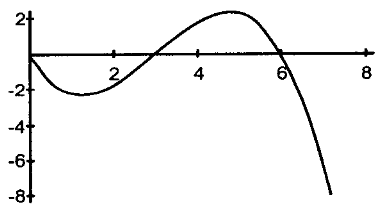

Figure 1. Graph of *dN/dt*

The development from a model-of to a model-for can be illuminated by the four different levels of activity: situational, referential, general, and formal (Gravemeijer & Doorman 1997; Gravemeijer 1997). Each of these four different levels emerged during this teaching experiment.

At the situational level, students' interpretations and solutions depend on understanding how to act in the setting. For example, one participant named Jungsun was trying to figure out how to use the given differential equation to approximate the long-term value of the population for each initial population. This situation means that once she interpreted the differential equation as an experientially realistic context, she understood how to act in the setting. For this level, the TI-92 graphing and symbolic calculator can play an essential role by allowing the slope field to emerge as an initial record of students' reasoning and mathematical activities for their numerical approximations. Then it becomes a tool for fostering students' reasoning about solution functions to differential equations (Figure 2).

At the referential level, models-of is grounded in students' understanding of pragmatic, experientially real settings. Students' activities might be considered referential (that is, referring back to the discrete approximations) when they are initially acting with the slope field as if there is an indication of the differential equation at any conceivable point (Figure 3).

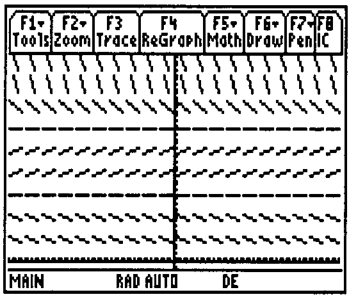

Figure 2. Slope field for *dN/dt*

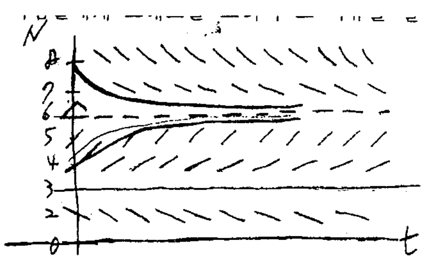

Figure 3. Jungsun's solution graph

At the general level, models-for makes possible a focus on interpretations and solutions independent of situation-specific imagery. Students' interpretations and responses to solution functions are no longer referring back to discrete approximations or specific solutions. Their activities involve holistically interpreting rates of change and solution functions (Figure 4). That is, students' solutions involve simultaneous reasoning about individual solution functions, as well as collections of solution functions.

At the formal level, students' activities are often characterized by the formal use of conventional notation. This fact is a useful and important way to differentiate activity at the general level from activity at the formal level. For example, one student, Miju, uses a dynamic image of the phase line which differentiates activity at the general level from activity at the formal level, thus demonstrating that her reasoning regarding solution functions is at a higher level (Figure 5).

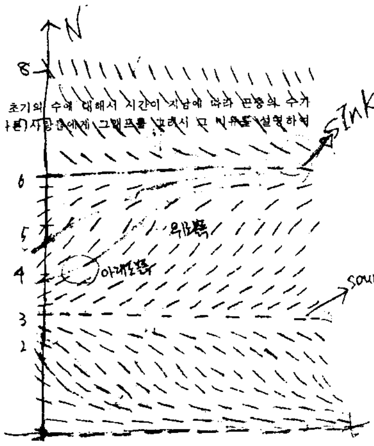

Figure 4. Rami's solution graphs

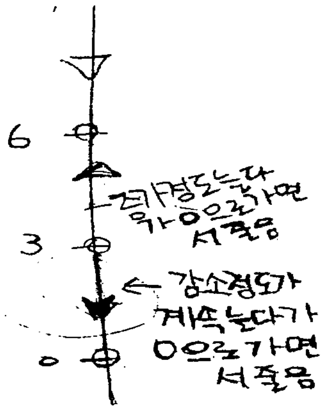

Figure 5. Miju's phase line

Guided and informed by the RME in structional heuristic, students in the differential equations course first act in mathematical situations in progressively more formal ways where the model comes to the fore as a model-of a mathematical context. Then subsequently, the model changes so that it can begin to function as a model-for increasingly sophisticated ways of mathematical reasoning.

CONCLUDING REMARKS

The study of ordinary differential equations is essential for students in many areas of science and technology. Many useful and interesting phenomena in engineering and life sciences that continuously evolve in time can be modeled by ordinary differential equations. Therefore, it is very important for students to have a firm understanding of ordinary differential equations, their solutions, and the different kinds of qualitative behavior the systems of ordinary differential equations can exhibit. Several recent curriculum reform efforts in differential equations are decreasing the traditional emphasis on specialized techniques for finding exact solutions to differential equations and increasing the use of computing technology to incorporate qualitative and numerical methods of analysis. Yet, research findings (e.g., Habre 2000; Rasmussen 1997) on students' thinking and understanding of differential equations are still minimal.

Through conceptualizing RME perspectives to the learning and teaching of differential equations, this research illustrates that when students are engaged in instruction that supports reinventing conventional representations out of mathematizing experiences, slope fields and graphs of solution functions can and do emerge for their mathematical activities. Specifically, students in Korea might more readily adapt their well-developed manipulative skills to experientially real situations with the incorporation of the RME instructional design. Further this research demonstrates how emerging analyses of student thinking and symbol-use can be profitably coordinated to promote students' sophisticated ways of reasoning with mathematical concepts in differential equations. Thus this paper suggests that an RME design for a differential equations course offers an alternative perspective for conceptualizing the learning and teaching of differential equations, even in undergraduate mathematics. This research also implies that researchers should consider, investigate, and adapt principled approaches that have been useful for reform in K-12 mathematics when conceptualizing the reform of undergraduate mathematics.

Research in the teaching and learning of mathematics at the university level is a relatively recent and new phenomenon (Artigue 1999); research in the teaching and learning of differential equations is even newer. The problems in undergraduate mathematics education are not easily solved by just writing or adopting new textbooks. The problems are related to the forms of students' work, the modes of interaction between university teachers and students, and the methods and content which students are assessed. The perspectives reported in this study can complement the growing research base in the teaching and learning of differential equations in both practical and theoretical aspects.

ACKNOWLEDGEMENTS

The author would like to thank Chris Rasmussen for sharing his ideas about structuring and teaching this differential equations course while this research was being conducted.

REFERENCES

Artigue, M. (1999): The teaching and learning of mathematics at the university level: Crucial questions for contemporary research in education. *Notices Amer. Math. Soc.* **46**, 1377–1385.

Artigue, M. & Gautheron, V. (1983): *Systems Differentials, etude graphique*. Paris: CEDIC/F. Nathan.

Blanchard, P.; Devaney, R. & Hall, R. (1998): *Differential equation*. Boston: Brooks/Cole. MATHDI **1998d.**02980

Borrelli, R. & Coleman, C. (1998): *Differential equations: A modeling perspective*. New York: Wiley. MATHDI **1996c.**02196

Boyce, W. E. (1994): New directions in elementary differential equations. *The College Mathematics Journal* **25(5)**, 364–371. MATHDI **1995c.**01961

_____ (1995): Technology in elementary differential equations. In: L. Lum (Ed.), *Proceedings of the Sixth Annual International Conference on Technology in Collegiate Mathematics* (pp. 65–74). Reading, MA: Addison-Wesley Publishing Company.

Cobb, P.; Stephen, M.; McClain, K. & Gravemeijer, K. (2001): Participating in classroom mathematical practices. *Journal of Learning Sciences* **10(1/2)**, 113–163.

Freudenthal, H. (1973): *Mathematics as an educational task*. Dordrecht: Kluwer Academic Publishers.

_____ (1991): *Revisiting mathematics education. China lectures*. Dordrecht: Kluwer Academic Publishers. MATHDI **1991l.**01100

Gravemeijer, K. (1994): Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education* **25(5)**, 443–471. MATHDI

1995c.01615

_____ (1997): Instructional design for reform mathematics education. In: M. Beishuizen, K. P. E. Gravemeijer & E. C. D. M. van Lieshout (Eds.), *The role of contexts and models in the development of mathematical strategies and procedures* (pp. 13–34). Utrecht: Technipress, Culemborg.

_____ (1999): How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning* **1(2)**, 155–177. MATHDI **1999e.**03429

Gravemeijer, K. & Doorman, M. (1999): Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics. An International Journal* **39(1–3)**, 111–129. MATHDI **2000d.**03068

Gravemeijer, K.; Cobb, P.; Bowers, J. & Whitenack, J. (2000): Symbolizing, modeling, and instructional design. In: P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms* (pp. 225–273). Mahwah, NJ: Erlbaum.

Habre, S. (2000): Exploring students' strategies to solve ordinary differential equations in a reformed settings. *Journal of Mathematical Behavior* **18(4)**, 455–472.

Horvath, P. J. (1987): A look at the School International Mathematics Study results in the U.S.A. and Japan. *Mathematics Teacher* **80(5)**, 359–368.

Hubbard, J. H. (1994): What it means to understand a differential equation. *College Mathematics Journal* **25(5)**, 372–384. MATHDI **1995c.**01960

Hubbard, J. H.; McDill, J. M.; Noonburg, A. & West, B. H. (1994): A new look at the airy equation with fences and funnels. *College Mathematics Journal* **25(5)**, 419-431. MATHDI **1995c.**01955 [West, B. (Ed.) (1994): *College Mathematics Journal* **25(5)** published as Special issue on differential equation.]

Hubbard, John & West, Beverly. (1997): Differential equation: A dynamical systems approach. New York: Springer.

Husen, Torsten (Ed.) (1967): *International study of achievement: A comparison of 12 countries.* New York: John Wiley & Sons.

Kostelich, E. & Armbruster, D. (1997): *Introductory differential equations: from linearity to chaos*. Reading, MA: Addison-Wesley. MATHDI **1998c.**02377

Kwon, O. N. (2002): The effects of calculator-based ranger activities on students' graphing ability. *School Science & Mathematics* **102(2)**, 5–15.

Lew, H. C. (1999): New goals and direction for mathematics education in Korea. In: C. Hoyles, C. Morgan & G. Woodhouse (Eds.), *Rethinking the Mathematics Curriculum* (pp. 240–247). London: Falmer Press.

Lu, C. (1995): Teaching ordinary differential equations with a computer program: Phase plane. In: L. Lum (Ed.), *Proceedings of the Sixth Annual International Conference on Technology in Collegiate Mathematics* (pp. 612–617). Reading, MA: Addison-Wesley Publishing Company.

McKnight, C. C.; Travers, K. J.; Crosswhite, F. J. & Swafford, J. O. (1985a): Eighth-grade mathematics in U. S. school: A Report from the Second International Mathematics Study. *Arithmetic Teacher* **32(8)**, 20–26. MATHDI **1985x.**00024

_____; Travers, K. J. & Dossey, J. A. (1985b): Twelfth-grade mathematics in U. S. high schools: A report from the Second International Mathematics Study. *Mathematics Teacher* **78(4)**, 292–300. MATHDI **1985x.**00019

Rasmussen, C. (1997): *Qualitative and numerical methods for analyzing differential equations: A case study of students' understandings and difficulties.* Doctoral dissertation. College Park, MD: University of Maryland.

_____ (1999): *Symbolizing and unitizing in support of students' mathematics growth in differential equations.* Paper presented at the 1999 NCTM Research Precession, San Francisco, CA. Preprint.

Streefland, L. (1991): *Fractions in Realistic Mathematics Education: A Developmental Research.* Dordrecht: Kluwer Academic. MATHDI **1991h.**02018

Treffers, A. (1991): Didactical background of a mathematics program for primary education. In: L. Streefland (Ed.), *Realistic Mathematics Education in Primary School* (pp. 21–57). Utrecht: Cdβ Press.

Trigueros, M. (2000): Students' conceptions of solution curves and equilibrium in systems of differential equations. In: Femandez M. L. (Ed.), *Proceedings of the 22nd annual meeting of the North American chapter of the international group for the psychology of mathematics education* (pp. 93–97). Columbus, OH: ERIC.

U. S. Department of Education, National Center for Education Statistics (1996): *Pursuing Excellence, NCES 97–198*, by Lois Peak. Washington, DC: U. S. Government Printing Office.

U. S. Department of Education, Office of Educational Research and Improvement (1997a): *TIMSS as a Starting Point to Examine Curricula: Guidebook to Examine School Curricula. In Attaining Excellence: A TIMSS Resource Kit.* Washington, DC: U. S. Department of Education, OERI.

U. S. Department of Education, Office of Educational Research and Improvement.(1997b): *TIMSS as a Starting Point to Examine Teaching: Moderator's Guide to Eighth-Grade Mathematics Lessons: United States, Japan, and Germany. In Attaining Excellence: A TIMSS Resource Kit.* Washington, DC: U. S. Department of Education, OERI.

Rasmussen, C. & King, K. (2000): Locating starting points in differential equations: A realistic mathematics education approach. *International Journal of Mathematical Education in Science and Technology* **31(2)**, 161–172.

Zandieh, M. & M. McDonald, M. (1999): Student understanding of equilibrium solution in differential equations. In: F. Hitt & M. Santos (Ed.), *Proceedings of the 21st annual meeting of the North American chapter of the international group for the psychology of mathematics education* (pp. 253–258). Columbus, OH: ERIC. MATHDI **2000d.**02886