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Subcategories of Fuzzy Limit Tower Spaces

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ABSTRACT

In this paper, we introduce the notion of fuzzy pseudotopological tower and fuzzy pretopological tower. And we show that the category FPsTR of fuzzy pseudotopological tower spaces and the category FPrTR of fuzzy pretopological tower spaces are bireflective subcategoies of the category FLTR of fuzzy limit tower spaces.

Key Words : fuzzy limit tower, fuzzy pseudotopological tower, fuzzy pretopological tower.

1. Introduction

Probabilistic convergence spaces were first defined in 1989 by Florescu [4], in terms of net convergence. The filter formulation considered here was introduced by G. D. Richardson and D. C. Kent [12]. The fundamental idea is to assign a probability to the convergence of a given filter to a given point. In 1997, P. Brock and D. C. Kent [2], introduce the limit tower whose objects resemble probabilistic convergence spaces.

On the other hand a theory of fuzzy topological space has been developed in various directions. A notion of convergence of prefilters, called fuzzy limitierung, is defined by K. C. Min [10] in 1989 as a generalization of a Q-neighborhood system of a fuzzy point in a fuzzy topological spaces. And a notion of fuzzy limit tower is defined by K. C. Min and me [6] as a fuzzification of limit tower.

In this paper, we introduce the notions of fuzzy pseudotopological tower and fuzzy pretopological tower. And show that the category **FPsTR** of fuzzy pseudotopological tower spaces and the category **FPrTR** of fuzzy pretopological tower spaces are bireflective subcategories of the category **FLTR** of fuzzy limit tower spaces. And we show that the category **FPstop** of fuzzy pseudotopological limit spaces is a bicoreflective subcategory of the category **FPsTR** of fuzzy pseudotpological tower spaces and the category **FPrtop** of fuzzy pseudotopological limit spaces is a bicoreflective subcategory of the category **FPrTR** of fuzzy pretopological tower spaces.

2. Preliminaries

Let X be a set and I be the closed unit interval. A function μ from X into I is called a fuzzy set in X. For any $\lambda \in (0,1)$, a fuzzy set x_λ in X is called a fuzzy point p if $x_\lambda(x)=\lambda$ for $x=x_0$, $x_\lambda(x)=0$ for $x \neq x_0$. We call x_0 the support of p and λ the value of p. We denote a fuzzy point p in X by (x_0, λ) if it is necessary to indicate the support and the value. We say that (x_0, λ) is quasi-coincident with A, denoted by $p=(x_0,\lambda)qA$, if $1-\lambda < \mu_A(x_0)$. For a fuzzy point p in X and a family $\{A_i\}_I$ of fuzzy sets in X, $pq\bigcup_I A_i \Leftrightarrow pqA_i$ for some $i \in I$ and $pqA_1 \cap A_2 \Leftrightarrow pqA_i$ for all $i=1,2$.

For any set X, let $B(X)$ be the collection of all prefilters on X and $H(X)$ be the set of all fuzzy points in X.

Definition 2.1. Let X be a set. A nonempty subset F of I^X is called a prefilter on X if it satisfies the following :

(F1) if $\mu \in F$ and $\mu \leq \nu$, then $\nu \in F$,
(F2) for all $\mu, \nu \in F$, we have $\mu \wedge \nu \in F$,
(F3) $\underline{0} \notin F$.

Definition 2.2. A prefilter U on X is an ultrafilter if there is no other prefilter finer than U.

Definition 2.3. A fuzzy limitierung $\triangle$ is a map from $H(X)$ into $P(B(X))$, the proper set of $B(X)$, subject to the following axioms:
for each fuzzy point $p=x_\lambda$,

(L0) $F \in \triangle(p) \Rightarrow \underline{\alpha} \in F$ for all $\alpha > 1-\lambda$,
(L1) $\langle p \rangle = \{A \in I^X : pqA\} \in \triangle(p)$,
(L2) $F \in \triangle(p)$ and $F \subseteq G \Rightarrow G \in \triangle(p)$,
(L3) $F, G \in \triangle(p) \Rightarrow F \cap G \in \triangle(p)$.

The pair $(X, \triangle)$ is called a fuzzy limit space. If $F \in \triangle(p)$, we say that F converse to p and p is a limit

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of F with respect to $\triangle$.

Definition 2.4. Let X and Y be fuzzy limit spaces. A map $f:X\to Y$ is said to be fuzzy continuous at a fuzzy point p in X if for any prefilter F converges to p, the prefilter $f(F)=\langle\{B\in I^X \mid f(A)\subseteq B \text{ for some } A\in F\}\rangle$ converges to the fuzzy point $f(p)$ in Y.

A map $f:X\to Y$ is said to be fuzzy continuous if it is fuzzy continuous at every fuzzy point p in X.

Let FLim denote the category of all fuzzy limit spaces and all fuzzy continuous maps.

Definition 2.5. A fuzzy limit tower $\overline{\triangle}$ is a family $\{\triangle_\alpha; 0\le\alpha\le\infty\}$ of fuzzy limitierungs such that :

(FT1) $\triangle_\alpha$ is a fuzzy limitierung, $0\le\alpha<\infty$,

(FT2) $\triangle_\infty$ is the indiscrete topology,

(FT3) $\triangle_\alpha=\bigcap_{\beta>\alpha}\triangle_\beta$ for each $0\le\alpha\le\infty$.

The pair $(X,\overline{\triangle})$ is called a fuzzy limit tower space.

Definition 2.6. For any fuzzy limit tower spaces $(X,\overline{\triangle})$ and $(Y,\overline{\triangle'})$, a map $f:(X,\overline{\triangle})\to(Y,\overline{\triangle'})$ is fuzzy continuous if $F\in\triangle_\alpha(x_\lambda)$ then $f(F)\in\triangle'_\alpha(f(x_\lambda))$ for each $0\le\alpha\le\infty$ where $\overline{\triangle}=(\triangle_\alpha)$, $\overline{\triangle'}=(\triangle'_\alpha)$.

Let **FLTR** denote the category of all fuzzy limit tower spaces and all fuzzy continuous maps.

Theorem 2.7. [6] The category **FLTR** is topological category.

Throughout this paper, we refer to [1] for category theory.

3. Fuzzy pseudotopological tower space and Fuzzy pretopological tower space

Definition 3.1. The category **FPsTR** is the full subcategory of the category **FLTR** whose objects are fuzzy pseudotopological tower spaces, that is for fuzzy limit tower $\overline{\triangle}=(\triangle_\alpha)$, for each $\alpha\in[0,\infty]$, $F\in\triangle_\alpha(x_\lambda)$ if and only if every ultrafilter U containing F converges to x_λ, for each $\alpha\in[0,\infty]$.

The category **FPrTR** is the full subcategory of **FLTR** whose objects are the fuzzy pretopological tower spaces, that is for fuzzy limit tower $\overline{\triangle}=(\triangle_\alpha)$ and for all $F\in\triangle_\alpha(x_\lambda)$, $\cap F\in\triangle_\alpha(x_\lambda)$ for any fuzzy points x_λ, for each $\alpha\in[0,\infty]$.

Trivially, **FPrTR** $\subseteq$ **FPsTR** $\subseteq$ **FLTR**.

Theorem 3.2. All the constructs **FLTR**, **FPsTR**, and **FPrTR** are well-fibred topological constructs, and each of the two constructs **FPsTR**, **FPrTR** is initially closed in every preceding construct in the list, hence bireflective in it.

Proof. At first all the construct are easily checked to being well-fibred and by above theorem, the category **FLTR** is topological. Thus we enough to prove that each of two constructs **FPsTR**, **FPrTR** is initially closed in it.

Secondly, we show that **FPsTR** is initially closed in FLTR.

Suppose $\{f_i:(X,\overline{\triangle})\to(X_i,\overline{\triangle_i})\}_{i\in I}$ is initial with $(X_i,\overline{\triangle_i})\in|\mathbf{FPsTR}|$ for each $i\in I$, and

$\overline{\triangle_i}=(\triangle^i_\alpha)$. It suffices to prove that if for all ultrafilter U containing F is $\triangle_\alpha$-converges to x_λ, then we have F is $\triangle_\alpha$-converges to x_λ. Since the source is initial, $f_i(U)\in\triangle^i_\alpha(f_i(x_\lambda))$ and $f_i(F)\subseteq f_i(U)$ for all $i\in I$. But $f_i(U)$ is ultrafilter and $(X_i,\overline{\triangle_i})\in|\mathbf{FPsTR}|$. Thus $f_i(F)\in\triangle^i_\alpha(f_i(x_\lambda))$. Hence $F\in\triangle_\alpha(x_\lambda)$. Lastly, we show that **FPrTR** is initially closed in **FLTR**.

Suppose $\{f_i:(X,\overline{\triangle})\to(X_i,\overline{\triangle_i})\}_{i\in I}$ is initial wit $(X_i,\overline{\triangle_i})\in|\mathrm{FPrTR}|$ for each $i\in I$, and $\overline{\triangle_i}=(\triangle^i_\alpha)$. It is sufficient to prove that for all $F\in\triangle_\alpha(x_\lambda)$, $\cap F\in\triangle_\alpha(x_\lambda)$. Since the source is initial, $f_i(F)\in\triangle^i_\alpha(f_i(x_\lambda))$ for each $i\in I$ and $f_i(\bigcap_{F\in\triangle_\alpha(x_\lambda)}F)=\bigcap_{F\in\triangle_\alpha(x_\lambda)}f_i(F)$.

But $(X,\overline{\triangle_i})\in|\mathrm{FPrTR}|$,

so $\bigcap_{F\in\triangle_\alpha(x_\lambda)}f_i(F)\in\triangle^i_\alpha(f_i(x_\lambda))$.

Thus $\bigcap_{F\in\triangle_\alpha(x_\lambda)}F\in\triangle_\alpha(x_\lambda)$.

Now, we more explicitly present that the bireflection between the category **FPsTR**, and the category **FLTR** and the bireflection between the category **FPrTR** and the category **FLTR**.

Proposition 3.3. The category **FPsTR** is a bireflective subcategory of the category **FLTR**.

Proof. Take any $(X,\overline{\triangle})\in|\mathbf{FLTR}|$ where $\overline{\triangle}=(\triangle_\alpha)$. Define a $\triangle'_\alpha: H(X)\to P(B(X))$ by $F\in\triangle'_\alpha(x_\lambda)$ if and only if for all ultrafilter U containing F, $U\in\triangle_\alpha(x_\lambda)$ or $F=\dot{x}_\lambda$ for each $0\le\alpha<\infty$. And let $\triangle'_\infty=B(X)$. And if $F\in\triangle'_\alpha(x_\lambda)$ and $c>1-\lambda$, then $\underline{c}\in F$. Then $\overline{\triangle'}=(\triangle'_\alpha)$ is a fuzzy pseudotopological tower on X. Since a fuzzy pseudotopological tower is a fuzzy limit tower with pseudotopological property, we enough to show that for all ultrafilter U containing F, $U\in\triangle'_\alpha(x_\lambda)$ then $F\in\triangle_\alpha(x_\lambda)$. But this fact also follows from the definition of $\triangle'_a$. And then to show that $id:(X,\overline{\triangle})\to(X,\overline{\triangle'})$ is a universal map, so let $F\in\triangle_a(x_\lambda)$ and U containing F. Then $U\in\triangle_a(x_\lambda)$ and so $F\in\triangle'_a(x_\lambda)$. And let $(Y,\overline{\triangle^*})\in|\mathbf{FPsTR}|$ and $f:(X,\overline{\triangle})\to(Y,\overline{\triangle^*})$ be a fuzzy continuous map. And for $f:(X,\overline{\triangle'})\to(Y,\overline{\triangle^*})$, let $F\in\triangle'_a(x_\lambda)$ and V be an ultrafilter containing $f(F)$.

Then $f^{-1}(V)$ is an ultrafilter containing F. Hence $f^{-1}(V)\in\triangle_a(x_\lambda)$.

But $f:(X,\overline{\triangle})\to(Y,\overline{\triangle^*})$ is fuzzy continuous and $\overline{\triangle^*}$ is a fuzzy pseudotopological tower.

So $V\in\triangle^*_a(f(x_\lambda))$ and $f(F)\in\triangle^*_a(f(x_\lambda))$. Thus $id:(X,\overline{\triangle})\to(X,\overline{\triangle'})$ is a bireflection.

Proposition 3.4. The category **FPrTR** is a bireflective subcategory of the category **FLTR**.

Proof.

Let $(X,\overline{\triangle})\in|\mathbf{FLTR}|$ where $\overline{\triangle}=(\triangle_a)$.

Define a $\triangle'_a$ by $F\in\triangle'_a(x_\lambda)$ if and only if $\bigcap_{G\in\triangle_a(x_\lambda)}G\subseteq F$ where $a\in[0,\infty)$.

And $\triangle'_\infty=B(X)$.

Then $\overline{\triangle'}$ is a fuzzy pretopological tower. Since a fuzzy pretopological tower is a fuzzy limit tower with pretopological property, we enough to show that for all then $\bigcap_{F\in\triangle'_a(x_\lambda)}F\in\triangle'_a(x_\lambda)$.

But this fact follows from the definition $\triangle'_a$ that is if $F\in\triangle'_a(x_\lambda)$, then $\bigcap_{G\in\triangle_a(x_\lambda)}G\subseteq F$ and $\bigcap_{G\in\triangle_a(x_\lambda)}G\subseteq\bigcap_{F\in\triangle'_a(x_\lambda)}F$ so $\bigcap_{F\in\triangle'_a(x_\lambda)}F\in\triangle'_a(x_\lambda)$.

And to show that $id:(X,\overline{\triangle})\to(X,\overline{\triangle'})$ is a universal map, let $F\in\triangle_a(x_\lambda)$.

Then since $F\supseteq\bigcap_{F\in\triangle_a(x_\lambda)}F$, so $F\in\triangle'_a(x_\lambda)$.

Let $(Y,\overline{\triangle^*})\in|\mathrm{FPrTR}|$, $f:(X,\overline{\triangle})\to(Y,\overline{\triangle^*})$ be a fuzzy continuous map. And for $f:(X,\overline{\triangle'})\to(Y,\overline{\triangle^*})$, let $F\in\triangle'_a(x_\lambda)$.

Then $F\supseteq\bigcap_{G\in\triangle_a(x_\lambda)}G$ and $f(G)\in\triangle^*_a(f(x_\lambda))$.

But $f:(X,\overline{\triangle})\to(Y,\overline{\triangle^*})$ is fuzzy continuous and $\overline{\triangle^*}$ is a fuzzy pretoplogical tower.

So $\bigcap_{G\in\triangle_a(x_\lambda)}f(G)\in\triangle^*_a(f(x_\lambda))$ and $\bigcap_{G\in\triangle_a(x_\lambda)}f(G)\subseteq f(F)$. Thus $f(F)\in\triangle^*_a(f(x_\lambda))$.

Thus $id:(X,\overline{\triangle})\to(X,\overline{\triangle'})$ is a bireflection.

Definition 3.5. The category **FPstop** is the full subcategory of the category **FLim** whose objects are the *fuzzy pseudotopological limit spaces, that is, fuzzy limit spaces satisfying the following axiom:*

<*If every ultrafilter containing prefilter* F converges to x_λ then F converges to x_λ.>

Definition 3.6. The category **FPrtop** is the full subcategory of the category **FLim** whose objects are the fuzzy pretopological limit spaces, that is, fuzzy limit spaces satisfying the following strong axiom:

<For all prefilter F converges to fuzzy point x_λ, $\cap F$ converges to x_λ>.

Then we get the following propositions.

Proposition 3.7. The category **FPstop** is a bicoreflective subcategory of the category **FPsTR**.

Proof. Consider

$$G:FPstop\dashrightarrow FPsTR$$
$$(X,\triangle)\mapsto(X,\overline{\triangle})$$
$$f\mapsto f$$

Define a $\overline{\triangle}=\{\triangle_a\}$, a-th component is $\triangle$, where $0\le a<\infty$. And $\triangle_\infty=B(X)$. Then $\overline{\triangle}$ is a clearly fuzzy pseudotpopological tower on X.

Consider

$$H:FPsTR\dashrightarrow Pstop$$
$$(X,\overline{\triangle})\mapsto(X,\triangle_o)$$
$$f\mapsto f$$

Define $\triangle_o$ be a first component of $(\triangle_a)=\overline{\triangle}$. Then $\triangle_o$ *is a fuzzy pseudotopological limit structure on* X.

And for any fuzzy pseudotopological limit spaces $(X,\triangle)$ and $(Y,\triangle')$, if a map $f:(X,\triangle)\to(Y,\triangle')$ is a fuzzy continuous map in **FPstop** then $f:(X,\overline{\triangle})\to(Y,\overline{\triangle'})$ is fuzzy continuous in **FPsTR**. And for any fuzzy pseudotopological tower spaces $(X,\overline{\triangle_1})$ and $(Y,\overline{\triangle_2})$, if $f:(X,\overline{\triangle})\to(Y,\overline{\triangle'})$ is fuzzy continuous in **FPsTR** then $f:(X,\triangle^1_o)\to(Y,\triangle^2_o)$ is fuzzy continuous in **FPstop** where $\overline{\triangle_1}=(\triangle^1_a)$, $\overline{\triangle_2}=(\triangle^2_a)$, $\triangle^1_o$ and $\triangle^2_o$ are first component of $\triangle^1_a$ and $\triangle^2_a$, respectively. Therefore G and H are functors. And $(\overline{\triangle_o})\subseteq\overline{\triangle}$. Thus $id:(X,(\overline{\triangle_o}))\to(X,\overline{\triangle})$ is a bicoreflection of fuzzy pseudotopological tower spaces $(X,\overline{\triangle})$.

Proposition 3.8. The category **FPrtop** is a bicoreflective subcategory of the category **FPrTR**.

Proof. Consider

$$G:FPrtop\dashrightarrow FPrTR$$
$$(X,\triangle)\mapsto(X,\overline{\triangle})$$
$$f\mapsto f$$

Define a $\overline{\triangle}=\{\triangle_a\}$, a-th component is $\triangle$, where

$0 \leq \alpha < \infty$. And $\triangle_{\infty} = B(X)$.

Then $\overline{\triangle}$ is a clearly fuzzy pretopological tower on X.

Consider

$$H : FPrTR \dashrightarrow \text{FPrtop}$$
$$(X, \overline{\triangle}) \mapsto (X, \triangle_o)$$
$$f \mapsto f$$

Define $\triangle_o$ be a first component of $(\triangle_\alpha) = \overline{\triangle}$.

Then $\triangle_o$ is a fuzzy pretopological limit structure on X. And for any fuzzy pretopological limit spaces $(X, \triangle)$ and $(Y, \triangle')$, if a map $f: (X, \triangle) \to (Y, \triangle')$ is a fuzzy continuous map in **FPrtop** then $f: (X, \overline{\triangle}) \to (Y, \overline{\triangle'})$ is fuzzy continuous in **FPrTR**. And for any fuzzy pretopological tower spaces $(X, \overline{\triangle_1})$ and $(Y, \overline{\triangle_2})$, if $f: (X, \overline{\triangle}) \to (Y, \overline{\triangle'})$ is fuzzy continuous in **FPrTR** then $f: (X, \triangle^1_o) \to (Y, \triangle^2_o)$ is fuzzy continuous in **FPrtop** where $\overline{\triangle_1} = (\triangle^1_\alpha)$, $\overline{\triangle_2} = (\triangle^2_\alpha)$, $\triangle^1_o$ and $\triangle^2_o$ are first component of $\triangle^1_\alpha$ and $\triangle^2_\alpha$, respectively. Therefore G and H are functors. And $(\overline{\triangle_o}) \subseteq \overline{\triangle}$. Thus $id: (X, (\overline{\triangle_o})) \to (X, \overline{\triangle})$ is a bicoreflection of fuzzy pretopological tower spaces $(X, \overline{\triangle})$.

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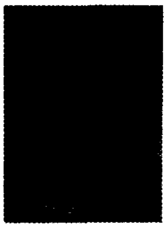
저 자 소 개



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