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CONSTRUCTION OF UNBOUNDED DIRICHLET FORMS ON STANDARD FORMS OF VON NEUMANN ALGEBRAS

CHANGSOO BAHN* AND CHUL KI KO**

ABSTRACT. We extend the construction of Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebra given in [13] to the case of unbounded operators affiliated with the von Neumann algebra. We then apply our result to give Dirichlet forms associated to the momentum and position operators on quantum mechanical systems.

1. Introduction

Recently Cipriani developed the theory of noncommutative Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras [6]. Employing this theory, Park made a general construction method of Dirichlet forms for any bounded analytic elements [13]. The purpose of this work is to extend the construction method to unbounded elements.

In order to describe the content of this paper, let us recall the construction method of Park [13]. For a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ with a cyclic and separating vector ξ_0 for $\mathcal{M}$, let $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ be the natural standard form and σ_t be the modular automorphism associated with the pair $(\mathcal{M}, \xi_0)$ [4]. For any admissible function f (Definition 3.3) and any bounded self-adjoint analytic element x in $\mathcal{M}$, the author constructed a (bounded) Dirichlet form

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$\mathcal{E}(\cdot,\cdot):\mathcal{H}\times\mathcal{H}\to\mathbb{C}$ of the following type:

$$(1.1)\qquad \mathcal{E}(\xi,\eta) = \int\langle(\sigma_{t-i/4}(x)-j(\sigma_{t-i/4}(x)))\xi, (\sigma_{t-i/4}(x)-j(\sigma_{t-i/4}(x)))\eta\rangle f(t)\,dt.$$

In this paper we extend this method to unbounded operator x affiliated to $\mathcal{M}$ satisfying Assumption 3.1. The problem is how to give meaning the term $\sigma_{t-i/4}(x)$ for the unbounded operator x. To overcome this problem, we approximate x using the spectral decomposition of $|x|$ (see (3.6)). Assumption 3.1 means that in case x is symmetric, the cyclic and separating vector ξ_0 belongs to the domain of x^2, i.e., $\xi_0\in D(x^2)$ (Remark 3.2). This assumption enables us to give the meaning of $\sigma_{t-i/4}(x)$ (see (3.14)) and establish the form $\mathcal{E}$ given by (3.3) on suitable dense domain. Our main result (Theorem 3.4) is that this extended form is a (unbounded) Dirichlet form which generates a Markovian semigroup on $\mathcal{H}$.

As an application of this extended method, we consider the von Neumann algebra $\mathcal{L}(\mathfrak{h})$ of all bounded operators on $\mathfrak{h}=L^2(\mathbf{R},dx)$ and the faithful normal state ω on $\mathcal{L}(\mathfrak{h})$ given by $\omega(A)=Tr(\rho A)$ with density operator $\rho=\frac{e^{-\beta H}}{Tr(e^{-\beta H})}$, $\beta>0$ where the Hamiltonian operator $H=-\frac{1}{2}\triangle+V$. When the potential V is a real valued bounded below polynomial satisfying (4.1), we construct a Dirichlet form associated with the momentum operator P or the position operator Q (Theorem 4.5).

Many papers have been devoted to study of quantum Markov semigroups and noncommutative Dirichlet forms [6, 7, 8, 9, 10]. Although we have quite well developed theory on an abstract level, the progress in concrete application (especially on the von Neumann algebras with non-tracial states) is slow. We would like to mention a few recent works in this direction. Majewski and Zegarlinski used the generalized conditional expectation to construct generators of spin-flip type dynamics for quantum spin systems [11, 12]. As mentioned before, Park gave a general construction method of Dirichlet forms on standard forms of von Neumann algebras and applied the method to construct translation invariant Markovian semigroups for quantum spin systems [13]. Quantum-Ornstein-Uhlenbek semigroups were constructed by means of noncommutative Dirichlet forms in [5]. It should be mentioned that the form defined in [5] is a special case we have made (see Remark 4.6). In [2, 3], the authors constructed Dirichlet forms and associated Markovian semigroups on CCR and CAR algebras with respect to quasi-free states.

We organize the paper as follows. In section 2, we briefly review the theory of noncommutative Dirichlet forms and Markovian semigroups in the sense of Cipriani [6]. In section 3, for closed unbounded operators satisfying Assumption 3.1, we construct the noncommutative Dirichlet forms using the limiting processes. In section 4 we apply the result in Section 3 to construct Dirichlet forms on quantum mechanical systems (Theorem 4.5).

2. Review on noncommutative Dirichlet forms

In this section we briefly review on the theory of Dirichlet forms and Markovian semigroups on standard forms of von Neumann algebras. For details, we refer the reader to [6].

Let $\mathcal{M}$ be a σ-finite von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$. That is equivalent to the existence of a faithful normal state on $\mathcal{M}$ or to the existence of a cyclic and separating vector in some faithful representation of $\mathcal{M}$. A *self-dual cone* $\mathcal{P}$ in $\mathcal{H}$ is a subset satisfying the property

$$\{\xi \in \mathcal{H} : \langle \xi, \eta \rangle \geq 0,\ \forall \eta \in \mathcal{P}\} = \mathcal{P}.$$

Then $\mathcal{P}$ is a closed convex cone and $\mathcal{H}$ is the complexification of the real subspace $\mathcal{H}^J = \{\xi \in \mathcal{H} : \langle \xi, \eta \rangle \in \mathbb{R},\ \forall \eta \in \mathcal{P}\}$, whose elements are called *J-real*: $\mathcal{H} = \mathcal{H}^J \oplus i\mathcal{H}^J$. Such a $\mathcal{P}$ gives a rise to a structure of ordered Hilbert space $\mathcal{H}^J$ (denoted by $\leq$) and to an anti-unitary involution J on $\mathcal{H}$, which preserves $\mathcal{P}$ and $\mathcal{H}^J$: $J(\xi + i\eta) := \xi - i\eta, \quad \forall \xi, \eta \in \mathcal{H}^J$. Any J-real element $\xi \in \mathcal{H}^J$ can be decomposed uniquely as a difference $\xi = \xi_+ - \xi_-, \quad \xi_+, \xi_- \in \mathcal{P}$ and $\langle \xi_+, \xi_- \rangle = 0$.

A *standard form* $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ of the von Neumann algebra $\mathcal{M}$ acting faithfully on the Hilbert space $\mathcal{H}$ consists of a self-dual, closed, convex cone $\mathcal{P}$ in $\mathcal{H}$ and the anti-unitary involution J satisfying the following properties:

(a) $J\mathcal{M}J = \mathcal{M}'$;
(b) $JAJ = A^*, \quad \forall A \in \mathcal{M} \cap \mathcal{M}'$;
(c) $J\xi = \xi, \quad \forall \xi \in \mathcal{P}$;
(d) $AJAJ(\mathcal{P}) \subset \mathcal{P}, \quad \forall A \in \mathcal{M}$,

where $\mathcal{M}'$ is the commutant of $\mathcal{M}$, i.e., the set of all bounded operators on $\mathcal{H}$ commute with each $A \in \mathcal{M}$.

The semigroup $\{T_t\}_{t\geq 0}$ on $\mathcal{H}$ is said to be *J-real* if $T_tJ = JT_t$ for any $t \geq 0$ and it is called *positive preserving* if $T_t\mathcal{P} \subset \mathcal{P}$ for any $t \geq 0$. Let us fix a cyclic and separating vector ξ_0 in $\mathcal{P}$. The semigroup $\{T_t\}_{t\geq 0}$ is said to be *sub-Markovian* (w.r.t. ξ_0) if $0 \le \xi \le \xi_0 \Rightarrow 0 \le T_t\xi \le \xi_0$ for any $t \ge 0$. The semigroup $\{T_t\}_{t\ge 0}$ is called *Markovian* if T_t is sub-Markovian and $T_t\xi_0 = \xi_0$ for any $t \ge 0$.

Next we consider a complex valued sesquilinear form $\mathcal{E}$ defined on a dense domain $D(\mathcal{E})$ in $\mathcal{H}$: $\mathcal{E}(\cdot,\cdot) : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{C}$ and also the associated quadratic form $\mathcal{E}[\cdot] : \mathcal{E}[\xi] = \mathcal{E}(\xi,\xi)$, $\forall \xi \in D(\mathcal{E})$. If $\mathcal{E}[\xi] \ge 0$ for all $\xi \in D(\mathcal{E})$, $\mathcal{E}$ is called *positive*, and if $\mathcal{E}[\xi] \ge -b\|\xi\|^2$ for some b, we say that $\mathcal{E}$ is *semi-bounded*. For a given semi-bounded quadratic form $\mathcal{E}$, one considers the inner product given by $\langle \xi,\eta\rangle_\lambda = \mathcal{E}(\xi,\eta) + \lambda\langle \xi,\eta\rangle$ for $\lambda > b$. The form $\mathcal{E}$ is *closed* if $D(\mathcal{E})$ is a Hilbert space for some of the above norms. The form $\mathcal{E}$ is called *closable* if it admits a closed extension.

Associated to a semi-bounded closed form $\mathcal{E}$, there is a unique self-adjoint operator K such that $D(K) \subset D(\mathcal{E})$ and for $\eta, \xi \in D(K)$

$$\mathcal{E}(\eta,\xi) = \langle \eta, K\xi\rangle, \tag{2.1}$$

and a strongly continuous semigroup

$$T_t = e^{-tK} \tag{2.2}$$

on $\mathcal{H}$.

The quadratic form $(\mathcal{E}[\cdot], D(\mathcal{E}))$ is said to be *J-real* if

$$JD(\mathcal{E}) \subset D(\mathcal{E}) \quad \text{and} \quad \mathcal{E}[J\xi] = \overline{\mathcal{E}[\xi]} \tag{2.3}$$

for all $\xi \in D(\mathcal{E})$.

Let $Proj(\xi, Q)$ be the projection of $\xi \in \mathcal{H}^J$ onto the closed convex set $Q \subset \mathcal{H}^J$ and for $\xi, \eta \in \mathcal{H}^J$ define

$$\begin{aligned} \xi \vee \eta &:= Proj(\xi, \eta + \mathcal{P}), \\ \xi \wedge \eta &:= Proj(\xi, \eta - \mathcal{P}). \end{aligned} \tag{2.4}$$

DEFINITION 2.1. A J-real, positive, densely defined quadratic form $(\mathcal{E}, D(\mathcal{E}))$ is called *Markovian* w.r.t. $\xi_0 \in \mathcal{P}$ if

$$\xi \in D(\mathcal{E}) \cap \mathcal{H}^J \quad \text{implies} \quad \xi \wedge \xi_0 \in D(\mathcal{E}) \quad \text{and} \quad \mathcal{E}[\xi \wedge \xi_0] \le \mathcal{E}[\xi]$$

equivalently if $\xi_0 \in D(\mathcal{E})$ and $\mathcal{E}(\xi_0,\xi) \ge 0$ for all $\xi \in D(\mathcal{E}) \cap \mathcal{P}$ and

$$\xi \in D(\mathcal{E}) \cap \mathcal{H}^J \quad \text{implies} \quad \xi_\pm \in D(\mathcal{E}) \quad \text{and} \quad \mathcal{E}(\xi_+,\xi_-) \le 0.$$

A closed Markovian form is called a *Dirichlet form.*

The following is one of the main results in [6], which is the characterization of sub-Markovian semigroups associated to Dirichlet forms with respect to $\xi_0 \in \mathcal{P}$.

THEOREM 2.2. (Theorem 4.11 of [6]) *Let* $(\mathcal{E}, D(\mathcal{E}))$ *be a J-real, positive, densely defined closed form and let* $\{T_t\}_{t\geq 0}$ *be the associated J-real strongly continuous, symmetric semigroup on* $\mathcal{H}$. *Then the following conditions are equivalent:*

(a) $\mathcal{E}$ *is a Dirichlet form;*
(b) $\{T_t\}_{t\geq 0}$ *is sub-Markovian.*

REMARK 2.3. In particular if $\{T_t\}_{t\geq 0}$ is sub-Markovian and $K\xi_0 = 0$ then it is Markovian.

3. Construction of Dirichlet forms: main results

In this section we construct Dirichlet forms on the natural standard forms of von Neumann algebras with any unbounded operators satisfying Assumption 3.1. The forms generate Markovian semigroups.

Let $\mathcal{M}$ denote a σ-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with the inner product $\langle \cdot, \cdot \rangle$, anti-linear and linear in first and second variable respectively. Let ξ_0 be a cyclic and separating vector for $\mathcal{M}$. We use Δ and J to denote respectively, the modular operator and the modular conjugation associated with the pair $(\mathcal{M}, \xi_0)$. The associated modular automorphism group is denoted by $\sigma_t : \sigma_t(A) = \Delta^{it} A \Delta^{-it}, A \in \mathcal{M}$. Also $j : \mathcal{M} \to \mathcal{M}'$ is the antilinear *-isomorphism defined by $j(A) = JAJ$, $A \in \mathcal{M}$, where $\mathcal{M}'$ is the commutant of $\mathcal{M}$. The natural positive cone $\mathcal{P}$ associated with the pair $(\mathcal{M}, \xi_0)$ is defined as the closure of the set

$$\{Aj(A)\xi_0 : A \in \mathcal{M}\}.$$

Then the form $(\mathcal{M}, \mathcal{H}, \mathcal{P}, J)$ is the standard form associated with the pair $(\mathcal{M}, \xi_0)$. Denote by ω a vector state on $\mathcal{M}$ associated with ξ_0 : $\omega(A) = \langle \xi_0, A\xi_0 \rangle, A \in \mathcal{M}$. Clearly ω satisfies the KMS condition [4]. For details we refer to Section 2.5 of [4].

Throughout this section, we assume that x satisfies the following properties.

ASSUMPTION 3.1. (a) x *is a* (unbounded) *densely defined closed operator affiliated to* $\mathcal{M}$;
(b) ξ_0 *belongs to the domains of* $|x|^2$ *and* $|x^*|^2$, *i.e.,* $\xi_0 \in D(|x|^2) \cap D(|x^*|^2)$, *where* $\xi_0 \in D(A^2)$ *means* $\xi_0 \in D(A)$ *and* $A\xi_0 \in D(A)$.

REMARK 3.2. If x is a symmetric operator, then the condition (b) of Assumption 3.1 is equal to $\xi_0 \in D(x^2)$. Let us mention that we will consider a (unbounded) closed operator x which is not necessary symmetric (see Remark 4.6).

In order to express Dirichlet forms, let us introduce the notion of an admissible function[13].

DEFINITION 3.3. [13] An analytic function $f : D \to \mathbb{C}$ on a domain D containing the strip Im $z \in [-1/4, 1/4]$ is said to be admissible if the following properties hold:

(a) $f(t) \geq 0$ for $\forall t \in \mathbb{R}$;
(b) $f(t + i/4) + f(t - i/4) \geq 0$ for $\forall t \in \mathbb{R}$;
(c) there exist $M > 0$ and $p > 1$ such that the bound

$$|f(t + is)| \leq M(1 + |t|)^{-p}$$

holds uniformly in $s \in [-1/4, 1/4]$.

Let us mention that the function $f(t)$ defined by

$$f(t) = \frac{2}{\sqrt{2\pi}} \int (e^{k/4} + e^{-k/4})^{-1} e^{-\frac{1}{2}k^2} e^{-ikt}\, dk$$

is an admissible function [13].

Next, we introduce a dense subset of $\mathcal{H}$. For any $B \in \mathcal{M}$ and $m \in \mathbb{N}$, define B_m by

$$B_m = \sqrt{\frac{m}{\pi}} \int \sigma_t(B) e^{-mt^2}\, dt. \tag{3.1}$$

Then B_m is an entire analytic element for σ_t, $\|B_m\| \leq \|B\|$ $\forall m \in \mathbb{N}$ and $B_m \to B$ strongly as $m \to \infty$. In fact

$$\sigma_z(B_m) = \sqrt{\frac{m}{\pi}} \int \sigma_t(B) e^{-m(t-z)^2}\, dt \tag{3.2}$$

is well defined for all $z \in \mathbb{C}$ and for each $m \in \mathbb{N}$, $z \mapsto \sigma_z(B_m)$ is strongly analytic. See the proof of Proposition 2.5.22 in [4]. Denote

$$\mathcal{M}_0 := \text{ the *-algebra generated by } B_m,\ \forall B \in \mathcal{M},\ \forall m \in \mathbb{N}.$$

Since ξ_0 is a cyclic vector for $\mathcal{M}$ and $B_m \to B$ strongly, $\mathcal{M}_0\xi_0$ is a dense subset in $\mathcal{H}$.

For given admissible function and any σ_t-analytic (bounded) self adjoint operator, Park constructed a noncommutative Dirichlet form [13]. Extending the method, we would like to construct noncommutative Dirichlet forms with any unbounded operator x satisfying Assumption

3.1. We denote by $x^{\#}$ either x or x^*. Notice that x is an unbounded closed operator affiliated to $\mathcal{M}$ and $\xi_0 \in D(x) \cap D(x^*)$. We are able to choose $x_n \in \mathcal{M}$, $\forall n \in \mathbb{N}$ such that $x_n\xi_0 \to x\xi_0$ and again σ_t-analytic elements $x_{nm} \in \mathcal{M}_0$, $\forall n, m \in \mathbb{N}$ (see (3.7) and (3.9)). By the limiting processes we define $\sigma_{t-i/4}(x^{\#})$, $\forall t \in \mathbb{R}$ on a suitable domain containing the dense set $\mathcal{M}_0\xi_0$(see Corollary 3.8 and (3.14)).

We are ready to describe the Dirichlet forms. For given admissible function f and a closed operator x satisfying Assumption 3.1, define a sesquilinear form $(\mathcal{E}, \mathcal{M}_0\xi_0)$ by

$$\begin{aligned} D(\mathcal{E}) &= \mathcal{M}_0\xi_0, \\ \mathcal{E}(\xi,\eta) &= \int \langle (\sigma_{t-i/4}(x) - j(\sigma_{t-i/4}(x^*)))\xi, \\ &\qquad (\sigma_{t-i/4}(x) - j(\sigma_{t-i/4}(x^*)))\eta\rangle f(t)dt \\ &\quad + \int \langle (\sigma_{t-i/4}(x^*) - j(\sigma_{t-i/4}(x)))\xi, \\ &\qquad (\sigma_{t-i/4}(x^*) - j(\sigma_{t-i/4}(x)))\eta\rangle f(t)dt \end{aligned} \tag{3.3}$$

for any $\xi, \eta \in \mathcal{M}_0\xi_0$. Also we define the associated quadratic form $\mathcal{E}[\cdot]$ by

$$\begin{aligned} \mathcal{E}[\xi] &= \int \|(\sigma_{t-i/4}(x) - j(\sigma_{t-i/4}(x^*)))\xi\|^2 f(t)dt \\ &\quad + \int \|(\sigma_{t-i/4}(x^*) - j(\sigma_{t-i/4}(x)))\xi\|^2 f(t)dt \end{aligned} \tag{3.4}$$

for any $\xi \in \mathcal{M}_0\xi_0$. For each $n, m \in \mathbb{N}$, let $(\mathcal{E}_{nm}, \mathcal{H})$ be the form given by

$$\begin{aligned} \mathcal{E}_{nm}(\xi,\eta) &= \int \langle (\sigma_{t-i/4}(x_{nm}) - j(\sigma_{t-i/4}(x^*_{nm})))\xi, \\ &\qquad (\sigma_{t-i/4}(x_{nm}) - j(\sigma_{t-i/4}(x^*_{nm})))\eta\rangle f(t)dt \\ &\quad + \int \langle (\sigma_{t-i/4}(x^*_{nm}) - j(\sigma_{t-i/4}(x_{nm})))\xi, \\ &\qquad (\sigma_{t-i/4}(x^*_{nm}) - j(\sigma_{t-i/4}(x_{nm})))\eta\rangle f(t)dt, \end{aligned} \tag{3.5}$$

where $x^{\#}_{nm}$ has been defined in (3.9) and (3.7). Since $x^{\#}_{nm} \in \mathcal{M}_0$, it follows from Theorem 3.3 of [13] and Theorem 2.1 of [2] that $(\mathcal{E}_{nm}, \mathcal{H})$ is a Dirichlet form for each $n, m \in \mathbb{N}$. We will show that $\mathcal{E}(\xi, \eta) = \lim_{n\to\infty} \lim_{m\to\infty} \mathcal{E}_{nm}(\xi, \eta)$ on the domain $\mathcal{M}_0\xi_0$ (Lemma 3.9).

Now we state main result. The form $(\mathcal{E}, \mathcal{M}_0\xi_0)$ given by (3.3) is well defined (Lemma 3.9) and closable (Proposition 3.12).

THEOREM 3.4. *For given admissible function f and a closed operator x satisfying Assumption 3.1, let $(\mathcal{E}, \mathcal{M}_0\xi_0)$ be defined as in* (3.3). *Let $\tilde{K}$ be the self adjoint operator associated with the closure $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ of $(\mathcal{E}, \mathcal{M}_0\xi_0)$. Then the following properties hold*:

(a) $\xi_0 \in D(\tilde{K})$ *and* $\tilde{K}\xi_0 = 0$;

(b) $\overline{\mathcal{E}}$ *is J-real, i.e.,* $\overline{\mathcal{E}}[J\xi] = \overline{\mathcal{E}}[\xi], \forall \xi \in D(\overline{\mathcal{E}})$;

(c) $\overline{\mathcal{E}}(\xi_+, \xi_-) \leq 0 \quad$ *for* $\quad \forall \xi \in \mathcal{H}^J \cap D(\overline{\mathcal{E}})$.

Furthermore the form $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ is a Dirichlet form.

By Theorem 2.2 and Theorem 3.4 the semigroup $\{T_t\}_{t\geq 0}$, $T_t = e^{-t\tilde{K}}$ associated to the Dirichlet form $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ is Markovian.

Let us briefly describe the basic idea of the proof of Theorem 3.4. We divide it into three steps. As mentioned before, we first give the meaning of $\sigma_{t-i/4}(x^\#)$ on a suitable domain (Corollary 3.8 and (3.14)), and show that the form $(\mathcal{E}, \mathcal{M}_0\xi_0)$ defined as in (3.3) is well defined (Lemma 3.9). Secondly by obtaining a positive symmetric operator K on $\mathcal{M}_0\xi_0$ such that $\mathcal{E}(\xi, \eta) = \langle \xi, K\eta \rangle$ ξ, $\eta \in \mathcal{M}_0\xi_0$, we prove that the form is closable. Finally we prove that the closure of $(\mathcal{E}, \mathcal{M}_0\xi_0)$ is a Dirichlet form. It should be mentioned that in the final step we used the argument similar to that of [2].

Let us establish the first step. The closed operator x has the polar decomposition $x = U|x|$ and the spectral decomposition of $|x|$

$$|x| = \int_0^\infty \lambda dP_\lambda. \tag{3.6}$$

For each $n \in \mathbb{N}$, define the operators y_n and x_n on $\mathcal{H}$

$$\begin{aligned} y_n &= \int_0^n \lambda dP_\lambda, \\ x_n &= U y_n. \end{aligned} \tag{3.7}$$

Since x is affiliated to $\mathcal{M}$, for all $n \in \mathbb{N}$ the operators U, y_n and x_n belong to $\mathcal{M}$.

Let us establish technical results which will be used in the sequel. In the following, $A^\#$ means either A or A^*.

LEMMA 3.5. *The following properties hold*:

(a) $\xi_0 \in D(x) \cap D(x^*)$;

(b) $\lim_{n\to\infty} x_n^\# \xi_0 = x^\# \xi_0$;

(c) $\lim_{n\to\infty} (x_n^\#)^* x_n^\# \xi_0 = |x^\#|^2 \xi_0$;

(d) $\lim_{n,m\to\infty} (x_n^\# - x_m^\#)^* (x_n^\# - x_m^\#) \xi_0 = 0$.

Proof. The lemma follows from Assumption 3.1 and the spectral theorem. □

LEMMA 3.6. *The inclusion* $\mathcal{M}_0\xi_0 \subset D(x) \cap D(x^*)$ *holds, and*

$$\lim_{n\to\infty} x_n^{\#}\xi = x^{\#}\xi, \quad \xi \in \mathcal{M}_0\xi_0.$$

Proof. Let $\xi = A\xi_0$, $A \in \mathcal{M}_0$. Notice that $\xi = j(\sigma_{-i/2}(A^*))\xi_0$ and

$$\begin{aligned} x_n\xi &= x_n j(\sigma_{-i/2}(A^*))\xi_0 \\ &= j(\sigma_{-i/2}(A^*))x_n\xi_0. \end{aligned} \tag{3.8}$$

It follows from (3.8) and Lemma 3.5(b) that

$$\begin{aligned} \lim_{n\to\infty} x_n\xi &= j(\sigma_{-i/2}(A^*))x\xi_0 \\ &= xj(\sigma_{-i/2}(A^*))\xi_0 \\ &= x\xi. \end{aligned}$$

Here we have used the fact that x is affiliated to $\mathcal{M}$. Also the method used above gives the result for x^*. □

Recall the definitions of x_n, y_n, $\forall n \in \mathbb{N}$ in (3.7) and $A^{\#}$ is either A or A^*. Define the operators $x_{nm}^{\#}$ by

$$x_{nm}^{\#} = \sqrt{\frac{m}{\pi}} \int \sigma_t(x_n^{\#})e^{-mt^2}dt, \quad n, m = 1, 2, \cdots. \tag{3.9}$$

It is easy to check that [4]

$$x_{nm}^{\#} \in \mathcal{M}_0, \quad n, m = 1, 2, \cdots \tag{3.10}$$

and (Lemma 3.6)

$$\lim_{n\to\infty}\lim_{m\to\infty} x_{nm}^{\#}\xi_0 = x^{\#}\xi_0. \tag{3.11}$$

To give the meaning of $\sigma_{t-i/4}(x^{\#})$, we need next results.

LEMMA 3.7. (a) *Let* $C = \max\{\|x\xi_0\|, \|x^*\xi_0\|\}$. *The inequality*

$$\|\sigma_{t-i/4}(x_{nm}^{\#})\xi_0\| \le C \tag{3.12}$$

holds uniformly in $t \in \mathbb{R}$ *and* $n, m \in \mathbb{N}$.

(b) *For any* $t \in \mathbb{R}$, $\lim_{n\to\infty}\lim_{m\to\infty} \sigma_{t-i/4}(x_{nm}^{\#})\xi_0$ *exists.*

Proof. (a) Notice that $\|x^{\#}_{nm}\xi_0\| \leq \|x^{\#}\xi_0\| \leq C, \quad n, m = 1, 2, \cdots$ and $\Delta\xi_0 = \xi_0$. We obtain that

$$\begin{aligned}\|\sigma_{t-i/4}(x^{\#}_{nm})\xi_0\|^2 &= \langle x^{\#}_{nm}\xi_0, \Delta^{1/2}x^{\#}_{nm}\xi_0\rangle \\ &= \langle x^{\#}_{nm}\xi_0, J(x^{\#}_{nm})^*\xi_0\rangle \\ &\leq \|x^{\#}_{nm}\xi_0\|\|(x^{\#}_{nm})^*\xi_0\| \\ &\leq C^2. \end{aligned} \tag{3.13}$$

(b) Using (3.13), we obtain that for $n, m, l, k \in \mathbb{N}$

$$\begin{aligned}&\|\sigma_{t-i/4}(x^{\#}_{nm})\xi_0 - \sigma_{t-i/4}(x^{\#}_{lk})\xi_0\|^2 \\ &\leq \|(x^{\#}_{nm} - x^{\#}_{lk})\xi_0\|\|(x^{\#}_{nm} - x^{\#}_{lk})^*\xi_0\|.\end{aligned}$$

The part (b) of lemma follows from the above result and (3.11). □

COROLLARY 3.8. (a) *For any $\xi \in \mathcal{M}_0\xi_0$, there exists a constant C_ξ independent of $t \in \mathbb{R}$ and $n, m \in \mathbb{N}$ such that*

$$\|\sigma_{t-i/4}(x^{\#}_{nm})\xi\| \leq C_\xi.$$

(b) *For any $\xi = A\xi_0, A \in \mathcal{M}_0$ and $t \in \mathbb{R}$, $\lim_{n\to\infty}\lim_{m\to\infty} \sigma_{t-i/4}(x^{\#}_{nm})\xi$ exists and*

$$\lim_{n\to\infty}\lim_{m\to\infty} \sigma_{t-i/4}(x^{\#}_{nm})\xi = j(\sigma_{-i/2}(A^*)) \lim_{n\to\infty}\lim_{m\to\infty} \sigma_{t-i/4}(x^{\#}_{nm})\xi_0.$$

Proof. It follows directly from the Lemma 3.7 and the relation (3.8). □

Now we define the operators $\sigma_{t-i/4}(x^{\#})$ for each $t \in \mathbb{R}$ by

$$D(\sigma_{t-i/4}(x^{\#})) = \{\xi \in \mathcal{M}\xi_0 : \lim_{n\to\infty}\lim_{m\to\infty} \sigma_{t-i/4}(x^{\#}_{nm})\xi \quad \text{exists}\}, \tag{3.14}$$

$$\sigma_{t-i/4}(x^{\#})\xi = \lim_{n\to\infty}\lim_{m\to\infty} \sigma_{t-i/4}(x^{\#}_{nm})\xi, \ \xi \in D(\sigma_{t-i/4}(x^{\#})).$$

By Corollary 3.8 (b), $\mathcal{M}_0\xi_0 \subset D(\sigma_{t-i/4}(x^{\#}))$ and so $\sigma_{t-i/4}(x^{\#})$ is densely defined. Also we have that

$$j(\sigma_{t-i/4}(x^{\#}))\xi = \lim_{n\to\infty}\lim_{m\to\infty} j(\sigma_{t-i/4}(x^{\#}_{nm}))\xi, \ \xi \in D(\sigma_{t-i/4}(x^{\#})).$$

LEMMA 3.9. *For any $\xi, \eta \in \mathcal{M}_0\xi_0$,*

$$\mathcal{E}(\xi, \eta) = \lim_{n\to\infty}\lim_{m\to\infty} \mathcal{E}_{nm}(\xi, \eta). \tag{3.15}$$

Proof. The lemma follows from Corollary 3.8, (3.14) and the dominated convergence theorem. □

The following lemma will be used in the proof of Proposition 3.11.

LEMMA 3.10. *For any $A \in \mathcal{M}_0$, the limit*

$$\lim_{n\to\infty}\lim_{m\to\infty}(x^{\#}_{nm})^* A\sigma_{-i/2}(x^{\#}_{nm})\xi_0$$

exists.

Proof. Notice that $x^{\#}_{nm} \in \mathcal{M}_0$, $\forall n, m \in \mathbb{N}$ and for each $n \in \mathbb{N}$

$$\lim_{m\to\infty} x^{\#}_{nm}(x^{\#}_{nm})^*\xi_0 = x^{\#}_n(x^{\#}_n)^*\xi_0.$$

We write that for $A \in \mathcal{M}_0$ and $n, m, l, k \in \mathbb{N}$

$$\begin{aligned}(3.16)\quad &\|x^*_{nm}A\sigma_{-i/2}(x_{nm})\xi_0 - x^*_{lk}A\sigma_{-i/2}(x_{lk})\xi_0\|\\ &\le \|(x_{nm}-x_{lk})^*A\sigma_{-i/2}(x_{nm})\xi_0\| + \|x^*_{lk}A\sigma_{-i/2}(x_{nm}-x_{lk})\xi_0\|\\ &\equiv \Lambda_1+\Lambda_2.\end{aligned}$$

Applying the KMS condition to Λ_1 and using Schwarz inequality, we obtain that

$$\begin{aligned}\Lambda_1^2 &= \omega(\sigma_{i/2}(x^*_{nm})A^*(x_{nm}-x_{lk})(x_{nm}-x_{lk})^*A\sigma_{-i/2}(x_{nm}))\\ &= \omega(A^*(x_{nm}-x_{lk})(x_{nm}-x_{lk})^*A\sigma_{-i/2}(x_{nm}x^*_{nm}))\\ &\le \omega(A^*((x_{nm}-x_{lk})(x_{nm}-x_{lk})^*)^2A)^{1/2}\\ &\qquad\cdot\omega(\sigma_{i/2}(x_{nm}x^*_{nm})A^*A\sigma_{-i/2}(x_{nm}x^*_{nm}))^{1/2}\\ &\le \|(x_{nm}-x_{lk})(x_{nm}-x_{lk})^*A\xi_0\|\|A\|\|\sigma_{-i/2}(x_{nm}x^*_{nm})\xi_0\|\\ (3.17)\quad &\le \|A\|\|j(\sigma_{-i/2}(A^*))\|\|(x_{nm}-x_{lk})(x_{nm}-x_{lk})^*\xi_0\|\|x_{nm}x^*_{nm}\xi_0\|.\end{aligned}$$

Here we have used $A\xi_0 = j(\sigma_{-i/2}(A^*))\xi_0$ and $j(\sigma_{-i/2}(A^*)) \in \mathcal{M}'$. If m and k tend to infinity, the right hand in (3.17) converges to

$$(3.18)\qquad \|A\|\|j(\sigma_{-i/2}(A^*))\|\|(x_n-x_l)(x_n-x_l)^*\xi_0\|\|x_nx^*_n\xi_0\|.$$

It follows from Lemma 3.5 (c), (d) and (3.18) that

$$\lim_{n,\,l\to\infty}\ \lim_{m,\,k\to\infty}\Lambda_1 = 0.$$

By the similar calculation we have

$$\lim_{n,\,l\to\infty}\ \lim_{m,\,k\to\infty}\Lambda_2 = 0.$$

Also the method used above gives the result for x^*_{nm}. □

In order to show that the form $(\mathcal{E}, \mathcal{M}_0\xi_0)$ is closable, we introduce a densely defined positive operator K such that $\mathcal{E}(\xi,\eta) = \langle\xi, K\eta\rangle$, $\xi,\eta \in \mathcal{M}_0\xi_0$.

PROPOSITION 3.11. *The operator K on $\mathcal{M}_0\xi_0$ defined by*

$$K\xi = \lim_{n\to\infty}\lim_{m\to\infty} K_{nm}\xi, \quad \xi \in \mathcal{M}_0\xi_0$$

is densely defined, symmetric and positive, where

$$K_{nm} = K_{nm}^{(1)} + K_{nm}^{(2)}, \quad n, m = 1, 2, \cdots, \tag{3.19}$$

$$\begin{aligned} K_{nm}^{(1)} &= \int \sigma_{t+i/4}(x_{nm}^*)\sigma_{t-i/4}(x_{nm})f(t)dt \\ &+ \int j((\sigma_{t-i/4}(x_{nm}^*))^*)j((\sigma_{t+i/4}(x_{nm}))^*)f(t)dt \\ &- \int \sigma_{t+i/4}(x_{nm}^*)j((\sigma_{t+i/4}(x_{nm}))^*)f(t)dt \\ &- \int j((\sigma_{t-i/4}(x_{nm}^*))^*)\sigma_{t-i/4}(x_{nm})f(t)dt \end{aligned}$$

and $K_{nm}^{(2)}$ defined replacing x_{nm} by x_{nm}^* in $K_{nm}^{(1)}$.

Proof. Notice that for any $\xi \in \mathcal{M}_0\xi_0$, $n, m \in \mathbb{N}$ and $z \in \mathbb{C}$

$$\begin{aligned} z &\mapsto \sigma_z(x_{nm})\xi, \\ z &\mapsto j((\sigma_z(x_{nm}))^*)\xi \end{aligned} \tag{3.20}$$

are strongly analytic. Using Corollary 3.8 and Cauchy integral theorem, we obtain that for fixed $n, m \in \mathbb{N}$, $\xi = A\xi_0 = j(\sigma_{-i/2}(A^*))\xi_0$, $A \in \mathcal{M}_0, \eta \in \mathcal{H}$

$$\begin{aligned} \langle \eta, K_{nm}^{(1)}\xi\rangle &= \int \langle \eta, j(\sigma_{-i/2}(A^*))\sigma_t(x_{nm}^*)\sigma_{t-i/2}(x_{nm})\xi_0\rangle f(t-i/4)dt \\ &+ \int \langle \eta, AJ\sigma_t(x_{nm})\sigma_{t-i/2}(x_{nm}^*)\xi_0\rangle f(t+i/4)dt \\ &- \int \langle \eta, \sigma_t(x_{nm}^*)A\sigma_{t-i/2}(x_{nm})\xi_0\rangle f(t-i/4)dt \\ &- \int \langle \eta, \sigma_t(x_{nm})A\sigma_{t-i/2}(x_{nm}^*)\xi_0\rangle f(t+i/4)dt. \end{aligned} \tag{3.21}$$

Comparing (3.19) with (3.21), we get that for any $\xi = A\xi_0$, $A \in \mathcal{M}_0$

$$
\begin{aligned}
K^{(1)}_{nm}\xi &= \int j(\sigma_{-i/2}(A^*))\sigma_t(x^*_{nm})\sigma_{t-i/2}(x_{nm})\xi_0\, f(t-i/4)dt \\
&\quad + \int AJ\sigma_t(x_{nm})\sigma_{t-i/2}(x^*_{nm})\xi_0\, f(t+i/4)dt \\
&\quad - \int \sigma_t(x^*_{nm})A\sigma_{t-i/2}(x_{nm})\xi_0\, f(t-i/4)dt \qquad (3.22) \\
&\quad - \int \sigma_t(x_{nm})A\sigma_{t-i/2}(x^*_{nm})\xi_0\, f(t+i/4)dt.
\end{aligned}
$$

It follows from Lemma 3.10 and the definition of admissible function that there exist

$$\lim_{n\to\infty}\lim_{m\to\infty} K^{(1)}_{nm}\xi, \quad \xi \in \mathcal{M}_0\xi_0$$

and also

$$\lim_{n\to\infty}\lim_{m\to\infty} K^{(2)}_{nm}\xi, \quad \xi \in \mathcal{M}_0\xi_0.$$

Since $\mathcal{M}_0\xi_0$ is dense, K is densely defined. Clearly we have $\langle K_{nm}\xi, \eta\rangle = \langle \xi, K_{nm}\eta\rangle$ for any ξ, $\eta \in \mathcal{M}_0\xi_0$, which implies that $\langle K\xi, \eta\rangle = \langle \xi, K\eta\rangle$ for any ξ, $\eta \in \mathcal{M}_0\xi_0$. Also it follows from (3.3) and (3.19) that

$$\langle \xi, K\eta\rangle = \mathcal{E}(\xi, \eta) \quad \text{for any } \ \xi, \eta \in \mathcal{M}_0\xi_0.$$

This implies that K is a positive operator on $\mathcal{M}_0\xi_0$. Therefore K is a symmetric and positive operator on $\mathcal{M}_0\xi_0$. □

PROPOSITION 3.12. *The form* $(\mathcal{E}, \mathcal{M}_0\xi_0)$ *defined as in* (3.3) *is closable.*

Proof. By Proposition 3.11, K is a positive, symmetric operator on $\mathcal{M}_0\xi_0$ and $\mathcal{E}(\xi, \eta) = \langle \xi, K\eta\rangle$ for ξ, $\eta \in \mathcal{M}_0\xi_0$. Thus $(\mathcal{E}, \mathcal{M}_0\xi_0)$ is closable by Theorem X.23 of [16]. □

Denote by $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ the closure form of $(\mathcal{E}, \mathcal{M}_0\xi_0)$. In order to show that the form is a Dirichlet form, we need the next lemma.

LEMMA 3.13. *For any* $\xi \in \mathcal{M}_0\xi_0 \cap \mathcal{H}^J$, ξ_+, $\xi_- \in D(\overline{\mathcal{E}})$ *and*

$$\overline{\mathcal{E}}[\xi_\pm] = \lim_{n\to\infty}\lim_{m\to\infty} \mathcal{E}_{nm}[\xi_\pm].$$

Proof. Let $\xi = A\xi_0$, $A \in \mathcal{M}_0$. Let s_+ and s_- be the projections onto $\mathcal{M}'\xi_+$ and $\mathcal{M}'\xi_-$ respectively, where s_+, $s_- \in \mathcal{M}$. See Theorem 4.7 of [1]. We write that

$$\xi_{m,\pm} = (s_\pm A)_m\xi_0,$$

$$(s_\pm A)_m = \sqrt{\frac{m}{\pi}}\int \sigma_t(s_\pm A)e^{-mt^2}dt.$$

Notice that $\|(s_\pm A)_m\| \leq \|A\|, m \in \mathbb{N}$ and that for any $\eta \in \mathcal{H}$, $(s_\pm A)_m\eta \to s_\pm A\eta$ as $m \to \infty$. See also the proof of Proposition 2.18 in [4]. Since $\Delta^{it}\mathcal{P} \subset \mathcal{P}$ for all $t \in \mathbb{R}$, we get that for each $m \in \mathbb{N}$

$$\xi_{m,\pm} = (s_\pm A)_m\xi_0 = j((s_\pm A)_m)\xi_0, \tag{3.23}$$

which implies $\xi_{m,\pm} \in \mathcal{M}_0\xi_0 \subset D(\mathcal{E})$ and

$$\sigma_{t-i/4}(x^\#)\xi_{m,\pm} = j((s_\pm A)_m)\sigma_{t-i/4}(x^\#)\xi_0. \tag{3.24}$$

Notice that $\xi_{m,\pm} \to \xi_\pm$ as $m \to \infty$ and the form $\mathcal{E}$ on $\mathcal{M}_0\xi_0$ is closable. By (3.24), the dominated convergence theorem and (3.3), we obtain that $\mathcal{E}[\xi_{m,\pm} - \xi_{m',\pm}] \to 0$ as $m,\ m' \to \infty$, which implies $\xi_+,\ \xi_- \in D(\overline{\mathcal{E}})$.

Next we will prove that $\overline{\mathcal{E}}_{nm}[\xi_\pm]$ converges to $\overline{\mathcal{E}}[\xi_\pm]$ as $m, n \to \infty$. Define $\delta(\sigma_{t-i/4}(x^\#))$ on $D(\sigma_{t-i/4}(x)) \cap D(\sigma_{t-i/4}(x^*))$ by

$$\delta(\sigma_{t-i/4}(x^\#)) := \sigma_{t-i/4}(x^\#) - j(\sigma_{t-i/4}(x^\#)).$$

By the definition of $\delta(\sigma_{t-i/4}(x^\#))$ and $\xi_\pm = s_\pm A\xi_0 = j(s_\pm A)\xi_0$, we have the facts that $\xi_\pm \in D(\sigma_{t-i/4}(x)) \cap D(\sigma_{t-i/4}(x^*))$ and $\delta(\sigma_{t-i/4}(x^\#))\xi_\pm = j(s_\pm A)\sigma_{t-i/4}(x^\#)\xi_0 - (s_\pm A)j(\sigma_{t-i/4}((x^\#)^*))\xi_0$.

Using Corollary 3.8 and the above relations we conclude that there exists a constant $C_1 > 0$ such that the bound

$$\|\delta(\sigma_{t-i/4}(x^\#_{nm}))\xi_\pm\| + \|\delta(\sigma_{t-i/4}(x^\#))\xi_\pm\| \leq C_1 \tag{3.25}$$

holds uniformly in $n, m \in \mathbb{N}$ and $t \in \mathbb{R}$. It follows from (3.25) and Corollary 3.8 that

$$\begin{aligned} & |\mathcal{E}_{nm}[\xi_\pm] - \overline{\mathcal{E}}[\xi_\pm]| \\ \leq\ & 2C_1 \int \|\delta(\sigma_{t-i/4}(x_{nm}))\xi_\pm - \delta(\sigma_{t-i/4}(x))\xi_\pm\| f(t)dt \\ \leq\ & C_2 \int \|(\sigma_{t-i/4}(x_{nm}) - \sigma_{t-i/4}(x))\xi_0\| f(t)dt \\ \to\ & 0 \quad \text{as } m, n \to \infty. \end{aligned}$$

Here we have used Lemma 3.7 and the dominated convergence theorem. Thus the proof is completed. □

Proof of Theorem 3.4 (a) Notice that for each $n, m \in \mathbb{N}$,

$$j(\sigma_{t-i/4}((x^*_{nm})^\#)))\xi_0 = \sigma_{t-i/4}(x^\#_{nm})\xi_0,$$

which implies $K_{nm}\xi_0 = 0$(see (3.19)). The part (a) follows from the above facts and Proposition 3.11.

(b) Since $\mathcal{M}_0\xi_0$ is a form core, it is enough to consider $\xi \in \mathcal{M}_0\xi_0$. We have known that $JA\xi_0 = \sigma_{-i/2}(A^*)\xi_0 \in \mathcal{M}_0\xi_0$, $A \in \mathcal{M}_0$. So $\xi \in \mathcal{M}_0\xi_0$ implies $J\xi \in \mathcal{M}_0\xi_0$. A direct calculation shows that for any $\xi \in \mathcal{M}_0\xi_0$,

$$\big(\sigma_{t-i/4}(x_{nm}^{\#}) - j(\sigma_{t-i/4}((x_{nm}^{\#})^*))\big)J\xi$$
$$= J(j(\sigma_{t-i/4}(x_{nm}^{\#})) - \sigma_{t-i/4}((x_{nm}^{\#})^*))\xi.$$

Using the above relations and (3.3), we get that for any $\xi \in \mathcal{M}_0\xi_0$

$$\overline{\mathcal{E}}[J\xi] = \overline{\mathcal{E}}[\xi].$$

(c) Let $\xi \in \mathcal{H}^J \cap D(\overline{\mathcal{E}})$ be given. Choose a sequence $\{\xi_k\} \subset \mathcal{M}_0\xi_0 \cap \mathcal{H}^J$ such that $\xi_k \to \xi$ in $\mathcal{H}$ and $\mathcal{E}[\xi_k - \xi_l] \to 0$ as $k,\ l \to \infty$. Since $\|\xi_{k,\pm} - \xi_\pm\| \le \|\xi_k - \xi\|$ and $|\xi| = \xi_+ + \xi_-$ (Proposition 1.2 in [7]), we have that $|\xi_k| \to |\xi|$ as $k \to \infty$. Furthermore by Lemma 3.13

$$\overline{\mathcal{E}}[|\xi_k|] = \lim_{n\to\infty}\lim_{m\to\infty}\mathcal{E}_{nm}[|\xi_k|]$$

for all $k \in \mathbb{N}$. Notice that $\overline{\mathcal{E}}(\xi_+, \xi_-) \le 0$ is equivalent to $\overline{\mathcal{E}}[|\xi|] \le \overline{\mathcal{E}}[\xi]$. Since x_{nm} is a σ_t-analytic element, $\mathcal{E}_{nm}$ satisfies for all $n, m, k \in \mathbb{N}$, $\mathcal{E}_{nm}[|\xi_k|] \le \mathcal{E}_{nm}[\xi_k]$ by Theorem 3.3 of [13]. By the lower semi-continuity of $\overline{\mathcal{E}}$, we get that

$$\begin{aligned}
\overline{\mathcal{E}}[|\xi|] &\le \liminf_{k\to\infty}\overline{\mathcal{E}}[|\xi_k|] \\
&= \liminf_{k\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\mathcal{E}_{nm}[|\xi_k|] \\
&\le \liminf_{k\to\infty}\lim_{n\to\infty}\lim_{m\to\infty}\mathcal{E}_{nm}[\xi_k] \\
&= \lim_{k\to\infty}\mathcal{E}[\xi_k] \\
&= \overline{\mathcal{E}}[\xi].
\end{aligned}$$

Thus $|\xi| \in D(\overline{\mathcal{E}})$ and $\overline{\mathcal{E}}[|\xi|] \le \overline{\mathcal{E}}[\xi]$. This completes the proof of the property (c).

Since $\overline{\mathcal{E}}[\cdot] \ge 0$ and $\overline{\mathcal{E}}(\xi, \xi_0) = 0$, $\forall \xi \in D(\overline{\mathcal{E}})$, by Proposition 4.5 (b) and Proposition 4.10(ii) of [6], the properties (b) and (c) imply that $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ is a Dirichlet form. □

4. Applications

In this section, we denote by $\mathfrak{h}$ the separable Hilbert space $L^2(\mathbb{R}, dx)$, i.e., $\mathfrak{h} = L^2(\mathbb{R}, dx)$, where dx is the Lebesgue measure on $\mathbb{R}$. Let $H = -\frac{1}{2}\triangle + V$ as an operator on $\mathfrak{h}$, where $\triangle = \frac{d^2}{dx^2}$ is the Laplacian for the variable $x \in \mathbb{R}$ and V is a real valued polynomial satisfying the condition:

$$\text{(4.1)} \quad \text{there exist } k > 0,\ c \in \mathbb{R} \text{ such that } V(x) \geq kx^2 + c,\ \forall x \in \mathbb{R}.$$

Then H is a semi-bounded self adjoint operator with the core $C_0^{co}(\mathbb{R})$, the space of the continuous functions vanishing at infinity with compact support (Theorem X.38 in [16]), and for any $\beta > 0$, $e^{-\beta H}$ is an invertible, trace class operator on $\mathfrak{h}$ (Theorem 9.2 in [17]).

Now the von Neumann algebra of all bounded linear operators on $\mathfrak{h}$ is denoted by $\mathcal{L}(\mathfrak{h})$ and the faithful normal state ω on $\mathcal{L}(\mathfrak{h})$ is given by

$$\text{(4.2)} \qquad \begin{aligned} \omega(A) &= Tr(\rho A), \quad A \in \mathcal{L}(\mathfrak{h}), \\ \rho &= \frac{e^{-\beta H}}{Tr(e^{-\beta H})}. \end{aligned}$$

The modular group is given by $\sigma_t(A) = \rho^{it} A \rho^{-it},\ A \in \mathcal{L}(\mathfrak{h})$. The state ω satisfies σ_t-KMS condition [4].

With the aid of the faithful normal state ω, the space $\mathcal{L}(\mathfrak{h})$ may be converted into a pre-Hilbert space by the inner product

$$\langle A, B\rangle = \omega(A^* B) = Tr((A\rho^{1/2})^*(B\rho^{1/2})),\ A, B \in \mathcal{L}(\mathfrak{h}).$$

The completion of this space is defined as the representation space $\mathcal{H}_\omega$. Next let us consider the definition of the representative $\pi_\omega(A)$ of $A \in \mathcal{L}(\mathfrak{h})$. We specify their action on the dense subspace of $\mathcal{H}_\omega$ by

$$\pi_\omega(A)B = AB,\ A, B \in \mathcal{L}(\mathfrak{h}).$$

Clearly $\pi_\omega(A)$ is a linear operator on $\mathcal{L}(\mathfrak{h})$ and $\|\pi_\omega(A)B\|_{\mathcal{H}_\omega} \leq \|A\|\|B\|_{\mathcal{H}_\omega}$. Hence $\pi_\omega(A)$ has a bounded closure, which we also denote by $\pi_\omega(A)$. π_ω is a homomorphism on $\mathcal{L}(\mathfrak{h})$, i.e., $\pi_\omega(AB) = \pi_\omega(A)\pi_\omega(B),\ A, B \in \mathcal{L}(\mathfrak{h})$. Denote by $\xi_0 = \mathbf{1}$ and $\mathcal{M} = \pi_\omega(\mathcal{L}(\mathfrak{h}))''$, and this gives the correct identification of ω:

$$\langle \xi_0, \pi_\omega(A)\xi_0\rangle = \omega(A),\ A \in \mathcal{L}(\mathfrak{h}).$$

$(\mathcal{H}_\omega, \pi_\omega, \xi_0)$ is the cyclic representation of $(\mathcal{L}(\mathfrak{h}), \omega)$[4]. From now on we suppress ω and π_ω from notations. Thus $\mathcal{H} := \mathcal{H}_\omega$, $A := \pi_\omega(A)$ and $\sigma_t := \pi_\omega(\sigma_t)$. ξ_0 is a cyclic and separating vector for the von Neumann algebra $\mathcal{M}$.

We next introduce unbounded operators on $\mathcal{H}$. Let X be a closed and densely defined operator on $\mathfrak{h}$ with domain $D(X)$. Its adjoint operator X^* is then also closed and densely defined. For each $A \in \mathcal{L}(\mathfrak{h})$, XA is closed, but not necessarily densely defined. If XA is bounded then

$XA\xi_0$ belongs to $\mathcal{H}$. We define an (unbounded) operator $\pi(X)$ on $\mathcal{H}$ as follows:

$$D_0(\pi(X)) = \{A\xi_0 \in \mathcal{H} : XA \text{ is bounded on } \mathfrak{h},\ A \in \mathcal{L}(\mathfrak{h})\}; \tag{4.3}$$
$$\pi(X)A\xi_0 = XA\xi_0, \quad A\xi_0 \in D_0(\pi(X)).$$

The following lemma gives the meaning of $\pi(X)$ as a densely defined operator on $\mathcal{H}$.(See also Lemma 2.1 of [5]).

LEMMA 4.1. *Let X be a closed and densely defined operator on $\mathfrak{h}$. $\pi(X)$ defined as in (4.3) is closable and densely defined. Denote also by $\pi(X)$ the closure with domain $D(\pi(X))$. The closure $\pi(X)$ is a closed and densely defined operator on $\mathcal{H}$ affiliated to $\mathcal{M}$, and $\pi(X^*) \subset \pi(X)^*$. Moreover if $X\rho^{1/4}$ is a bounded operator on $\mathfrak{h}$ then $\xi_0 \in D(\pi(X))$.*

Proof. Since $D(X)$ is dense in $\mathfrak{h}$, we can choose a Hilbert basis $\phi = \{\phi_n\}$ for $\mathfrak{h}$ contained in $D(X)$. The linear space $C_{00}([\phi])$ generated by the operators $\langle\phi_n, \cdot\rangle\phi_m$, $n, m \in \mathbb{N}$ is a weak* dense subspace of $\mathcal{L}(\mathfrak{h})$ whose norm closure is the Banach space of compact operators on $\mathfrak{h}$. This implies that $C_{00}([\phi])\xi_0$ is dense in $\mathcal{H}$. Clearly $X\langle\phi_n, \cdot\rangle\phi_m$ is an everywhere defined bounded operator on $\mathfrak{h}$, which implies $C_{00}([\phi])\xi_0 \subset D_0(\pi(X))$. Hence $\pi(X)$ is densely defined on $\mathcal{H}$.

If $A\xi_0 \in D_0(\pi(X^*))$, $B\xi_0 \in D_0(\pi(X))$ then X^*A, XB and A^*XB are bounded operators on $\mathfrak{h}$. But $(X^*A)^*B$ extends the everywhere defined operator A^*XB, so the two operators must coincide, and we have $\langle A\xi_0, \pi(X)B\xi_0\rangle = Tr((A\rho^{1/2})^*(XB\rho^{1/2})) = Tr((X^*A\rho^{1/2})^*(B\rho^{1/2})) = \langle\pi(X^*)A\xi_0, B\xi_0\rangle$. It follows from the density of $\mathcal{L}(\mathfrak{h})$ in $\mathcal{H}$ and the definition of $\pi(X)$ that $\pi(X)^* \supset \pi(X^*)$. Therefore $\pi(X)^*$ is densely defined, so $\pi(X)$ is closable.

The affiliation properties easily follow from the fact that the commutant algebra $\mathcal{M}'$ is $\{\pi'(B) \in \mathcal{L}(\mathcal{H}) : \pi'(B)A\xi_0 = AB\xi_0,\ A, B \in \mathcal{L}(\mathfrak{h})\}$.

To show the last statement, we consider the polar decomposition $X = U|X|$ and the spectral decomposition of $|X|$

$$|X| = \int_0^\infty \lambda dP_\lambda.$$

For each $n \in \mathbb{N}$, define the operators e_n on $\mathfrak{h}$

$$e_n = \int_{-n}^{n} dP_\lambda$$

and $X_n = Xe_n$. Clearly $e_n\xi_0$ belongs to $D_0(\pi(X))$ and $e_n\xi_0 \to \xi_0$ in $\mathcal{H}$. Note that $\rho^{1/4}$ is a trace class operator and $X\rho^{1/4}$ is bounded. Applying the fact

$$(X_n - X_m)\rho^{1/2} = ((X_n - X_m)\rho^{1/4})\rho^{1/4}, \tag{4.4}$$

we have that $Tr(((X_n - X_m)\rho^{1/2})^*((X_n - X_m)\rho^{1/2}))$ converges to 0 as $n, m \to \infty$. Thus $\pi(X)e_n\xi_0$ converges in $\mathcal{H}$. Since $\pi(X)$ is closed $\xi_0 \in D(\pi(X))$. □

Let P and Q be the self adjoint operators on $\mathfrak{h}$ given by

$$(Pf)(x) = -i\frac{d}{dx}f(x), \quad (Qf)(x) = xf(x) \tag{4.5}$$

with common core $C_0^\infty(\mathbb{R})$. Then by Lemma 4.1, $\pi(P)$ and $\pi(Q)$ are densely defined, unbounded closed symmetric operators on $\mathcal{H}$ affiliated to $\mathcal{M}$.

In order to show the Proposition 4.3, we will use the lemma listed below. The following lemma is well known to experts. (See the example of p.270 in [16].)

LEMMA 4.2. *There exists a positive constant α such that the following inequality holds:*

$$(-\triangle)^2 + V^2 \le (-\triangle + V + \alpha\mathbf{1})^2 \tag{4.6}$$

as quadratic form on $C_0^\infty(\mathbb{R})$. Moreover $-\triangle(-\triangle+V+\alpha\mathbf{1})^{-1}$ and $x^2(-\triangle+V+\alpha\mathbf{1})^{-1}$ are bounded on $\mathfrak{h}$, where x^2 stands for a multiplication operator on $\mathfrak{h}$.

Proof. Since V is a real valued bounded below polynomial we can choose a constant $\alpha > 1$ such that

$$V(x) \ge -\alpha/4, \; V(x) + \alpha/4 \ge V''(x)(= \frac{d^2}{dx^2}V(x)), \; \forall x \in \mathbb{R}. \tag{4.7}$$

Let $V_1 = V + \alpha/2$. Then $V_1(x) > 0$ and $V_1^2(x) \ge V^2(x)$ for all $x \in \mathbb{R}$. By a simple calculation, one gets that

$$\begin{aligned}(-\triangle + V_1)^2 &= (-\triangle)^2 + V_1^2 + 2(i\frac{d}{dx})V_1(i\frac{d}{dx}) - V_1'' \\ &\ge (-\triangle)^2 + V_1^2 - V_1'',\end{aligned} \tag{4.8}$$

where V_1 and V_1'' stand for multiplication operators on $\mathfrak{h}$. Notice that $-\triangle + V + \alpha\mathbf{1}$ has an inverse. It follows from (4.7) and (4.8) that

$$\begin{aligned}(-\triangle + V + \alpha\mathbf{1})^2 &= (-\triangle + V_1)^2 + \alpha(-\triangle) + \alpha(V_1 + \frac{\alpha}{4}\mathbf{1}) \\ &\ge (-\triangle)^2 + V_1^2 - V_1'' + \alpha(V_1 + \frac{\alpha}{4}\mathbf{1}) \\ &\ge (-\triangle)^2 + V^2.\end{aligned}$$

The above fact and (4.1) imply that $-\triangle(-\triangle+V+\alpha\mathbf{1})^{-1}$ and $x^2(-\triangle+V+\alpha\mathbf{1})^{-1}$ are bounded on $\mathfrak{h}$. □

Since the potential V is a polynomial satisfying (4.1), for each $\beta > 0$ and $n \in \mathbb{N}$, the kernel of $e^{-\beta H}$ is n-times differentiable and decays exponentially at infinity [15]. So for $f \in C_0^\infty(\mathbb{R})$, $x^n e^{-\frac{\beta}{2}H} f$ and $(i\frac{d}{dx})^n e^{-\frac{\beta}{2}H} f$ are continuous functions on $\mathbb{R}$. In fact they are everywhere defined on $\mathfrak{h}$.

PROPOSITION 4.3. *Let $\beta > 0$ and $n = 1, 2$. The operators $x^n e^{-\beta H}$, $(-i\frac{d}{dx})^n e^{-\beta H}$ are bounded and also Hilbert Schumidt class. Here x^n stands for a multiplication operator on $\mathfrak{h}$.*

Proof. The boundedness of operators directly follows from Lemma 4.2. The method used in (4.4) gives that the operators are Hilbert Schumidt class. □

Actually Proposition 4.3 holds for arbitrary $n \in \mathbb{N}$. One can prove it modifying the path integral method used in the proof of Proposition 4.1 in [14] or Proposition 2.6 in [15]. But we need only $n = 1, 2$. The following proposition gives us the operators on $\mathcal{H}$ satisfying Assumption 3.1.

PROPOSITION 4.4. *ξ_0 belongs to the domains of $|\pi(P)|^2$, $|\pi(P)^*|^2$, $|\pi(Q)|^2$ and $|\pi(Q)^*|^2$.*

Proof. By Remark 3.2 and the symmetricity of $\pi(P)$, ξ_0, $\pi(P)\xi_0 \in D(\pi(P))$ is equivalent to $\xi_0 \in D(|\pi(P)|^2) \cap D(|\pi(P)^*|^2)$. It follows from Proposition 4.3 that for each $n = 1, 2$, $Tr((P^n\rho^{1/2})^*(P^n\rho^{1/2}))$ is finite. Hence by (4.4) and the closability of $\pi(P)$, ξ_0, $\pi(P)\xi_0 \in D(\pi(P))$. Also the method used above gives $\xi_0 \in |\pi(Q)|^2 = |\pi(Q)^*|^2$. □

We have showed that the unbounded operators $\pi(P)$ and $\pi(Q)$ satisfy Assumption 3.1 from Lemma 4.1 and Proposition 4.4. Using the method developed in Section 3 we construct the Dirichlet forms and Markovian Semigroups on the standard form of $\mathcal{M}$.

THEOREM 4.5. *For given admissible function f and a closed operator $\pi(P)$ or $\pi(Q)$ given as in (4.3) and (4.5), let $(\mathcal{E}, \mathcal{M}_0\xi_0)$ be defined by changing x into $\pi(P)$ or $\pi(Q)$ in (3.3). Let K be the self adjoint operator associated with $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$. Then the following properties hold:*

(a) *$\xi_0 \in D(K)$ and $K\xi_0 = 0$;*
(b) *$\overline{\mathcal{E}}$ is J-real, i.e., $\overline{\mathcal{E}}[J\xi] = \overline{\mathcal{E}}[\xi]$;*
(c) *$\overline{\mathcal{E}}(\xi_+, \xi_-) \le 0$ for $\forall \xi \in \mathcal{H}^J \cap D(\overline{\mathcal{E}})$.*

Furthermore the form $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ is a Dirichlet form.

REMARK 4.6. Let us discuss the Harmonic oscillator. We consider the potential $V(x) = \frac{1}{2}x^2 - \frac{1}{2}$. Let a and a^* be the annihilation and creation operators with common core, the Schwarz space $\mathcal{S}(\mathbf{R})$, i.e.,

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}(x + \frac{d}{dx}) = \frac{1}{\sqrt{2}}(Q + iP), \\ a^* &= \frac{1}{\sqrt{2}}(x - \frac{d}{dx}) = \frac{1}{\sqrt{2}}(Q - iP) \end{aligned}$$

and also the number operator is defined by N, i.e., $N = a^*a = aa^* - \mathbf{1}$. In fact the Hamiltonian $H = -\frac{1}{2}\Delta + V(x)$ equals to N and thus the density operator ρ is $e^{-\beta N}$, $\beta > 0$. Let $\phi_0 = \pi^{-1/4}e^{-x^2/2}$ and $\phi_n = (n!)^{-1/2}(a^*)^n\phi_0$. Then $\{\phi_n\}_{n=0}^{\infty}$ are just the Hermite functions which form an orthonormal basis for $L^2(\mathbb{R})$ and hold the properties:

$$\begin{aligned} &a^*\phi_n = \sqrt{n+1}\,\phi_{n+1} \quad n = 0, 1, 2, ..., \\ &a\phi_n = \sqrt{n}\,\phi_{n-1} \quad n = 1, 2, ..., \\ &a\phi_0 = 0. \end{aligned}$$

Also the operators a, a^* and N have the relations on $\mathcal{S}(\mathbf{R})$:

$$a\rho = e^{-\beta}\rho a, \quad a^*\rho = e^{\beta}\rho a^*. \tag{4.9}$$

By Lemma 4.1, $Tr(Ne^{-\beta N}) < \infty$ implies that $\pi(a)$ and $\pi(a^*)$ are closed and densely defined operators, and also ξ_0 belongs to $D(|\pi(a)|^2) \cap D(|\pi(a^*)|^2)$. Let us change $\pi(P)$ or $\pi(Q)$ into $\pi(a)$ and a normalized admissible function f,i.e., $\int_{\mathbb{R}} f(t)dt = 1$ in Theorem 4.5. We obtain the Dirichlet form $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ independent of admissible function f; for any $\xi \in D(\overline{\mathcal{E}})$

$$\overline{\mathcal{E}}[\xi] = \|(\mu\pi(a) - \lambda j(\pi(a^*)))\xi\|^2 + \|(\mu\pi(a^*) - \lambda j(\pi(a)))\xi\|^2, \tag{4.10}$$

where $\mu = \lambda^{-1} = e^{\beta/4}$. This Dirichlet form is the form in Proposition 4.1 of [5].

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Changsoo Bahn
Institute of Natural Science
Yonsei University
Seoul 120-749, Korea
E-mail: bahn@yonsei.ac.kr

Chul Ki Ko
Department of Mathematics
Seoul National University
Seoul 151-747, Korea
E-mail: kochulki@hotmail.com