J. Korean Math. Soc. **39** (2002), No. 6, pp. 881–898

A GENERALIZATION OF HOMOLOGICAL ALGEBRA

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Abstract. Our aim in this paper is to introduce a generalization of some notions in homological algebra. We define the concepts of chain U-complex, U-homology, chain (U, U')-map, chain (U, U')-homotopy and $\mathcal{U}$-functor. We also obtain some interesting results. We use these results to find a generalization of Lambek Lemma, Snake Lemma, Connecting Homomorphism and Exact Triangle.

1. Introduction

Defining the kernel of a hypergroup homomorphism as the reciprocal image of the intersection of all ultra-closed subhypergroups of its codomain, Freni and Sureau in [3] introduced the notion of exact sequence of hypergroups (note that, in general, a hypergroup does not have a zero element). If some natural conditions (which are always valid for groups) are satisfied then the existence of a Ker-Coker sequence in a category of hypergroups is established. They used these results to find a homology in a supercategory of the category of groups.

Suppose that we have the following exact sequence of R-modules and R-homomorphisms.

$$\cdots \longrightarrow C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \longrightarrow \cdots .$$

Then $\mathrm{Im}(\partial_{i+1}) = \mathrm{Ker}(\partial_i)$ or $\mathrm{Im}(\partial_{i+1}) = \partial_i^{-1}(\{0\})$. It is a natural question to ask what does happen if we substitute a submodule U_{i-1} of C_{i-1} instead of the trivial submodule $\{0\}$ in the above definition. In [2], Davvaz and Parnian introduced the concept of U-exact sequences, which is a modification of the standard notion of exact sequences and answered the above question. The authors then generalized some results from the standard case to the modified case. In [1], Anvariyeh

Received January 6, 2002. Revised June 14, 2002.
2000 Mathematics Subject Classification: 16E05, 16E30.
Key words and phrases: chain U-complex, chain (U, U')-homotopy, U-functor, U-homology.

and Davvaz continued working on this topic and focusing on application of U-exactness, and studied U-split sequences. They established several relationships between U-split and projective modules. Note that the notion of exact sequences is a fundamental concept, and it has been widely used in many areas such as Ring and Module Theory, Group Theory, Homological Theory, Algebraic Topology, and Complex Theory.

Our aim in this paper is to introduce a generalization of some notions in homological algebra. We define the concepts of chain U-complex, U-homology, chain (U, U')-map, chain (U, U')-homotopy and $\mathcal{U}$-functor. If one were to choose several results to call Fundamental Lemmas of Homological Algebra, then he would include the Lambek Lemma, Snake Lemma, Connecting Homomorphism and Exact Triangle on his list. We use the above concepts to find a generalization of these important Lemmas and Theorems. We assume as known the basic notions of homological algebra. For concepts related to decision procedures the reader may consult the books [4], [6].

2. Chain U-complexes and chain (U, U')-homotopy

In this section first we introduce the notions of chain U-complex and p-th U-homology generalizing the notions of chain complex and p-th homology, and then we establish chain (U, U')-maps among various chain U-complexes.

DEFINITION 2.1. We are given two family $\{C_p\}, \{U_p\}, p \in Z$, of R-modules, where every C_p contains U_p, and a family of R-module homomorphisms $\{\partial_p : C_p \longrightarrow C_{p-1}\}$. The chain $\{C_p, U_p, \partial_p\}$ is called a *chain U-complex* if the following conditions hold:

i) $\partial_p \partial_{p+1}(C_{p+1}) \subseteq U_{p-1}$,
ii) $\mathrm{Im}\partial_p \supseteq U_{p-1}$.

We put $C = \{C_p\}$, $\partial = \{\partial_p\}$ and show a chain U-complex as follows:

$$(C, U, \partial) : \cdots \longrightarrow C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \longrightarrow \cdots .$$

It will be clear to see that every chain complex is a chain 0-Complex, where 0 is a sequence of zero submodules. Also every chain $\{C_p, U_p, \partial_p\}$ with property $\partial_p \partial_{p+1}(C_{p+1}) = U_{p-1}$ is a chain U-complex. If (C, U, ∂) be a chain U-complex, then $\mathrm{Im}\partial_{p+1} \subseteq \partial^{-1}(U_{p-1})$. In fact, this condition is equivalent to the first condition of Definition 2.1.

We shall now introduce the p-th U-homology module of C. Assume that $Z_p(C,U,\partial) = \partial_p^{-1}(U_{p-1})$ and $B_p(C,U,\partial) = \mathrm{Im}\partial_{p+1}$. Hence we can associate with C the module

$$H_p(C,U,\partial) = \frac{Z_p(C,U,\partial)}{B_p(C,U,\partial)}, \quad p \in Z.$$

Then $H_p(C,U,\partial)$ is called the *p-th U-homology module* of C.

DEFINITION 2.2. Let (C,U,∂) be a chain U-complex and (C',U',∂') a chain U'-complex. The sequence $F = \{F_p : C_p \longrightarrow C'_p\}$ is called a *chain* (U,U')*-map* if the following diagram is commutative. In other words, $F_p(U_p) \subseteq U'_p$ and $F_{p-1}\partial_p = \partial'_p F_p$:

$$\begin{array}{lccccccc}
(C,U,\partial) & \cdots \longrightarrow & C_{p+1} & \xrightarrow{\partial_{p+1}} & C_p & \xrightarrow{\partial_p} & C_{p-1} & \longrightarrow \cdots \\
 & & \downarrow F_{p+1} & & \downarrow F_p & & \downarrow F_{p-1} & \\
(C',U',\partial') & \cdots \longrightarrow & C'_{p+1} & \xrightarrow{\partial'_{p+1}} & C'_p & \xrightarrow{\partial'_p} & C'_{p-1} & \longrightarrow \cdots .
\end{array}$$

PROPOSITION 2.3. *Let (C,U,∂) be a chain U-complex such that $\partial_p\partial_{p+1}(C_{p+1}) = U_{p-1}$ and (C',U',∂') a chain U'-complex. If $F = \{F_p\}$ is a chain map, then it is also a chain (U,U')-map.*

Proof. We show that $F_{p-1}(U_{p-1}) \subseteq U'_{p-1}$. Assume that $u \in U_{p-1}$. Then $x = F_{p-1}(u) \in F_{p-1}(U_{p-1})$. Since $u \in U_{p-1}$ there exists $c \in C_{p+1}$ such that $u = \partial_p\partial_{p+1}(c)$. Therefore

$$\begin{aligned}
x &= F_{p-1}(u) = F_{p-1}(\partial_p\partial_{p+1}(c)) = (F_{p-1}\partial_p)(\partial_{p+1}(c)) \\
&= (\partial'_p F_p)(\partial_{p+1}(c)) = \partial'_p(F_p\partial_{p+1}(c)) = \partial'_p(\partial'_{p+1}F_{p+1}(c)) \\
&= \partial'_p\partial'_{p+1}(F_{p+1}(c)).
\end{aligned}$$

Hence $x \in U'_{p-1}$ and the proof is completed. □

LEMMA 2.4. *Z_p and B_p are invariant under a chain (U,U')-map $\{F_p\}$, i.e.,*

i) $F_p(Z_p(C,U,\partial)) \subseteq Z_p(C',U',\partial')$,
ii) $F_p(B_p(C,U,\partial)) \subseteq B_p(C',U',\partial')$.

Proof. i) Suppose that $x \in Z_p(C,U,\partial)$ then $\partial_p(x) \in U_{p-1}$, and so $F_{p-1}(\partial_p(x)) \in U'_{p-1}$, which implies that $\partial'_p F_p(x) \in U'_{p-1}$. Hence $F_p(x) \in \partial_p'^{-1}(U'_{p-1}) = Z_p(C',U',\partial')$.

ii) If $x \in B_p(C,U,\partial)$ then there exists $y \in C_{p+1}$ such that $x = \partial_{p+1}(y)$. So $F_p(x) = F_p(\partial_{p+1}(y)) = (F_p\partial_{p+1})(y) = (\partial'_{p+1}F_{p+1})(y) = \partial'_{p+1}(F_{p+1}(y)) \in \mathrm{Im}\partial'_{p+1}$. Hence $F_p(x) \in B_p(C',U',\partial')$. □

THEOREM 2.5. *Let (C, U, ∂) be a chain U-complex and (C', U', ∂') a chain U'-complex. If $F = \{F_p\}$ is a chain (U, U')-map then it induces R-module homomorphism $H(F) = \{H_p(F)\} = \{F_p^*\}$ as follows:*

$$\begin{array}{rcl} F_p^* : \ H_p(C, U, \partial) & \longrightarrow & H_p(C', U', \partial') \\ x + B_p(C, U, \partial) & \longrightarrow & F_p(x) + B_p(C', U', \partial'). \end{array}$$

Proof. We show that F_p^* is well-defined. Suppose that $x + B_p(C, U, \partial) = y + B_p(C, U, \partial)$. Then $x - y \in B_p(C, U, \partial)$, and so $F_p(x - y) \in B_p(C', U', \partial')$, which implies $F_p(x) - F_p(y) \in B_p(C', U', \partial')$. Therefore $F_p(x) + B_p(C', U', \partial') = F_p(y) + B_p(C', U', \partial')$ and hence $F_p^*(x) = F_p^*(y)$. The fact that F_p^* is a homomorphism is just a reformulation of the defini -tion. □

Let $G : (C', U', \partial') \longrightarrow (C'', U'', \partial'')$ be a chain (U', U'')-map. Then one obtains $(GF)_p^* = G_p^* F_p^*$, also we have $I^* = I$, where I is the identity map.

Let (C, U, ∂) be a chain U-complex and (C', U', ∂') a chain U'-complex. A chain (U, U')-map $F = \{F_p\}$ is called an isomorphism if F_p is an R-module isomorphism and $F^{-1} = \{F_p^{-1}\}$ is a chain (U', U)-map. If there exists an isomorphism of (C, U, ∂) onto (C', U', ∂'), we say that (C, U, ∂) is isomorphic to (C', U', ∂'). It is easy to see that isomorphism relation of chain U-complexes is an equivalence relation.

PROPOSITION 2.6. *If two chain U-complex and chain U'-complex are isomorphic then $U_p \simeq U'_p$ for all p.*

Proof. The proof is straightforward and omitted. □

DEFINITION 2.7. Let (C, U, ∂) be a chain U-complex and (C', U', ∂') a chain U'-complex and $F, G : C \longrightarrow C'$ two chain (U, U')-maps. Then F and G are chain (U, U')-homotopic, denoted by $F \simeq G$, if there is a sequence $D = \{D_p\}$, where $D_p : C_p \longrightarrow C'_{p+1}$ is an R-module homomorphism, such that for all $p \in Z$,

i) $\partial'_{p+1} D_p + D_{p-1} \partial_p = F_p - G_p$,

ii) $D_p(U_p) \subseteq U'_{p+1}$.

The sequence $D = \{D_p\}$ is called a *chain (U, U')-homotopy.*

LEMMA 2.8. *The (U, U')-homotopy relation "$\simeq$" is an equivalence relation.*

Proof. Plainly " $\simeq$" is reflexive and symmetric. To check transitivity, let $F \simeq G$ and $G \simeq H$. Then

$$\partial'_{p+1}D_p + D_{p-1}\partial_p = F_p - G_p, \quad \partial'_{p+1}D'_p + D'_{p-1}\partial_p = G_p - H_p$$

and $D_p(U_p) \subseteq U'_{p+1}$, $D'_p(U_p) \subseteq U'_{p+1}$. An easy calculation shows the sequence $D'' = \{D''_p\}$, where $D''_p = D_p + D'_p$ is a chain (U, U')-homotopy and F, H are chain (U, U')-homotopic. □

LEMMA 2.9. *Let (C, U, ∂), (C', U', ∂') and (C'', U'', ∂'') be chain U-complex, U'-complex and U''-complex, respectively. If $F \simeq G : C \longrightarrow C'$ and $F' \simeq G' : C' \longrightarrow C''$, then*

$$F'F \simeq G'G : C \longrightarrow C''.$$

Proof. Suppose that $F_p - G_p = \partial'_{p+1}D_p + D_{p-1}\partial_p$, $F'_p - G'_p = \partial''_{p+1}D'_p + D'_{p-1}\partial'_p$ and $D_p(U_p) \subseteq U'_{p+1}$, $D'_p(U'_p) \subseteq U''_{p+1}$. We have

$$\begin{aligned} & F'_pF_p - G'_pG_p \\ = \; & F'_p(F_p - G_p) + (F'_p - G'_p)G_p \\ = \; & F'_p(\partial'_{p+1}D_p + D_{p-1}\partial_p) + (\partial''_{p+1}D'_p + D'_{p-1}\partial'_p)G_p \\ = \; & F'_p\partial'_{p+1}D_p + F'_pD_{p-1}\partial_p + \partial''_{p+1}D'_pG_p + D'_{p-1}\partial'_pG_p \\ = \; & \partial''_{p+1}F'_{p+1}D_p + F'_pD_{p-1}\partial_p + \partial''_{p+1}D'_pG_p + D'_{p-1}G_{p-1}\partial_p \\ = \; & \partial''_{p+1}(F'_{p+1}D_p + D'_pG_p) + (F'_pD_{p-1} + D'_{p-1}G_{p-1})\partial_p. \end{aligned}$$

Now, we put $D''_p = F'_{p+1}D_p + D'_pG_p$. Hence the first condition of Definition 2.7 is satisfied. For the second condition we must prove that $D''_p(U_p) \subseteq U''_{p+1}$. Since G is a chain (U, U')-map, we have $G(U_p) \subseteq U'_p$ and so $D'_pG(U_p) \subseteq D'_p(U'_p) \subseteq U''_{p+1}$. On the other hand, since F' is a chain (U', U'')-map, we get $F'_{p+1}D_p(U_p) \subseteq F'_{p+1}(U'_{p+1}) \subseteq U''_{p+1}$. Therefore $D''_p(U_p) \subseteq U''_{p+1}$, and the proof is completed. □

The essential fact about chain (U, U')-homotopies is given in the following.

THEOREM 2.10. *If the two chain (U, U')-maps $F, G : C \longrightarrow C'$ are (U, U')-homotopic, then $F^*_p = G^*_p(H_p(F) = H_p(G))$.*

Proof. Suppose that $F_p - G_p = \partial'_{p+1}D_p + D_{p-1}\partial_p$ and $D_p(U_p) \subseteq U'_{p+1}$. Let $x \in Z_p(C, U, \partial)$. Then $F_p(x) - G_p(x) = \partial'_{p+1}D_p(x) + D_{p-1}\partial_p(x)$. Since $\partial_p(x) \in U_{p-1}$, we have $D_{p-1}(\partial_p(x)) \in U'_p$. Using Definition 2.7 we obtain $D_{p-1}(\partial_p(x)) \in U'_p \subseteq \mathrm{Im}\partial'_{p+1} = B_p(C', U', \partial')$. Also we have

$\partial'_{p+1}D_p(x) \in B_p(C',U',\partial')$. Therefore $F_p(x) - G_p(x) \in B_p(C',U',\partial')$, which implies

$$F_p(x) + B_p(C',U',\partial') = G_p(x) + B_p(C',U',\partial').$$

Hence $F_p^* = G_p^*$. □

DEFINITION 2.11. A chain (U,U')-map $F : (C,U,\partial) \longrightarrow (C',U',\partial')$ is called a *chain* (U,U')*-equivalence* if there exists a chain (U,U')-map $G : (C',U',\partial') \longrightarrow (C,U,\partial)$ such that $FG \simeq I_C$ and $GF \simeq I_{C'}$. Two chain U-complex and U'-complex are called *chain* (U,U')*-equivalent* if there exists a chain (U,U')-equivalence between them.

COROLLARY 2.12. *If chain U-complex (C,U,∂) and chain U'-complex (C',U',∂') are chain (U,U')-equivalent, then for all p, $H_p(C,U,\partial) = H_p(C',U',\partial')$.*

Proof. Suppose F is a chain (U,U')-equivalence between (C,U,∂) and (C',U',∂'). Then there exists a chain (U',U)-map $G : (C',U',\partial') \longrightarrow (C,U,\partial)$ such that $FG = I_C$ and $GF = I_{C'}$. By Theorem 2.10 we have $(FG)_p^* = I_c^*$ and $(GF)_p^* = I_{c'}^*$. On the other hand, we know $(FG)_p^* = F_p^*G_p^*$ and $(GF)_p^* = G_p^*F_p^*$. Therefore F_p^* is an isomorphism. Hence $H_p(C,U,\partial) = H_p(C',U',\partial')$. □

3. U-exact sequences

In this section we will use the redefined definition of a U-exact sequence as in [2], and will mainly give generalizations of Lambek Lemma [5] and Snake Lemma (see [4], [6]).

DEFINITION 3.1. A sequence of R-modules and R-module homomorphisms

$$\cdots \longrightarrow C_{p+2} \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \longrightarrow \cdots$$

is said to be U_p*-exact* (where U_p is a submodule of C_p) at C_{p+1} if

$$\mathrm{Im}\partial_{p+2} = \partial_{p+1}^{-1}(U_p).$$

Let $U = \{\cdots, U_{p+1}, U_p, \cdots\}$. A U-exact sequence is a sequence U_p-exact at each of its modules. Similar to the chain U-complex, we denote every U-exact sequence by (C,U,∂). Every U-exact sequence satisfies the condition $\partial_p\partial_{p+1}(C_{p+1}) \subseteq U_{p-1}$. Therefore every U-exact sequence such that $U_{p-1} \subseteq \mathrm{Im}\partial_p$ is a chain U-complex. A chain U-complex (C,U,∂) is a U-exact sequence if and only if for all p, $H_p(C,U,\partial) = 0$.

DEFINITION 3.2. The U-exact sequence (C, U, ∂) is said to be *isomorphic* to the U'-exact sequence (C', U', ∂') if there exists a chain (U, U')-map $F = \{F_p\}$ such that every F_p is an R-module isomorphism.

PROPOSITION 3.3. *If two U-exact and U'-exact sequences are isomorphic then $U \simeq U'$.*

Proof. The proof is similar to the proof of Proposition 10 in [1]. □

Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be $\{0\}$-exact at A, U-exact at B and $\{0\}$-exact at C. Then to simplify, we say the sequence is short U-exact. Some properties of short U-exact sequences are given in [1].

PROPOSITION 3.4. *Let (C, U, ∂) be a chain of R-modules and R-module homomorphisms, and (C', U', ∂') be a U'-exact sequence. If in sequence $F = \{F_p\}$, every F_p is an R-module isomorphism such that $F(U) = U'$ and the following diagram is commutative, then (C, U, ∂) is a U-exact sequence.*

$$\begin{array}{llccccccl} (C,U,\partial): & \cdots \longrightarrow & C_{p+1} & \xrightarrow{\partial_{p+1}} & C_p & \xrightarrow{\partial_p} & C_{p-1} & \longrightarrow \cdots \\ & & \downarrow F_{p+1} & & \downarrow F_p & & \downarrow F_{p-1} & \\ (C',U',\partial'): & \cdots \longrightarrow & C'_{p+1} & \xrightarrow{\partial'_{p+1}} & C'_p & \xrightarrow{\partial'_p} & C'_{p-1} & \longrightarrow \cdots . \end{array}$$

Proof. We show that $\mathrm{Im}\partial_{p+1} = \partial_p^{-1}(U_{p-1})$. Assume that $x \in \mathrm{Im}\partial_{p+1}$. Then there exists $c \in C_{p+1}$ such that $x = \partial_{p+1}(c)$. We have $F_p\partial_{p+1}(c) = \partial'_{p+1}F_{p+1}(c)$ and so $\partial'_pF_p\partial_{p+1}(c) = \partial'_p\partial'_{p+1}F_{p+1}(c) \in U'_{p-1}$, which implies that $F_{p-1}(\partial_p\partial_{p+1}(c)) \in U'_{p-1}$. Since $F_{p-1}(U_{p-1}) = U'_{p-1}$ and F_{p-1} is an isomorphism we get $\partial_p\partial_{p+1}(c) \in U_{p-1}$ or $\partial_p(x) \in U_{p-1}$. Hence $\mathrm{Im}\partial_{p+1} \subseteq \partial_p^{-1}(U_{p-1})$.

Conversely, suppose that $x \in \partial_p^{-1}(U_{p-1})$. Then $\partial_p(x) \in U_{p-1}$. We have $F_{p-1}\partial_p(x) \in U'_{p-1}$ and so $\partial'_pF_p(x) \in U'_{p-1}$. Also we have $F_p(x) \in \partial_p'^{-1}(U'_{p-1}) = \mathrm{Im}\partial'_{p+1}$. Therefore there exists $y \in C'_{p+1}$ such that $F_p(x) = \partial'_{p+1}(y)$. Since F_{p+1} is onto, there exists $z \in C_{p+1}$ such that $y = F_{p+1}(z)$. Therefore $F_p(x) = \partial'_{p+1}(F_{p+1}(z))$ and so $F_p(x) = F_p\partial_{p+1}(z) = F_p(\partial_{p+1}(z))$. Since F_p is one to one, we get $x = \partial_{p+1}(z)$. Hence $x \in \mathrm{Im}\partial_{p+1}$ and the proof is completed. □

From Propositions 2.6 and 3.4 we easily deduce the following corollary.

Corollary 3.5. *Let (C, U, ∂) be a chain U-complex and (C', U', ∂') be a U'-exact sequence. If (C, U, ∂) and (C', U', ∂') are isomorphic, then (C, U, ∂) is a U-exact sequence.*

Let $A \xrightarrow{f} B \xrightarrow{g} C$ be U-exact at B. Then for simplify, we say the sequence is U-exact. Now we shall give a generalization of Lambek Lemma (see [3]).

Lemma 3.6 (A Generalization of Lambek Lemma). *Let*

$$\begin{array}{ccccc} A' & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & A'' \\ \downarrow \psi & & \downarrow \varphi & & \downarrow \theta \\ B' & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & B'' \end{array}$$

be a commutative diagram such that the first row is U'-exact and the second row is U-exact. If V and W be submodules of B and B', respectively, such that $\mathrm{Im}\psi \supseteq W$ and $\beta_1(W) \supseteq V$. Then φ induces an isomorphism

$$\Phi : \frac{(\theta\alpha_2)^{-1}(U)}{\alpha_2^{-1}(U') + \varphi^{-1}(V)} \xrightarrow{\sim} \frac{\mathrm{Im}\varphi \cap \mathrm{Im}\beta_1}{\mathrm{Im}\varphi\alpha_1}.$$

Proof. We show that φ induces a homomorphism of this kind. Let $x \in (\theta\alpha_2)^{-1}(U)$; plainly $\varphi(x) \in \mathrm{Im}\varphi$. Since $\beta_2\varphi(x) = \theta\alpha_2(x) \in U$ and $\beta_2^{-1}(U) = \mathrm{Im}\beta_1$, we get $\varphi(x) \in \mathrm{Im}\beta_1$. Now, we define $\Phi(x + \alpha_2^{-1}(U') + \varphi^{-1}(V)) = \varphi(x) + \mathrm{Im}\varphi\alpha_1$. First we show that Φ is well-defined. Assume that $x + \alpha_2^{-1}(U') + \varphi^{-1}(V) = y + \alpha_2^{-1}(U') + \varphi^{-1}(V)$. Then $x - y \in \alpha_2^{-1}(U') + \varphi^{-1}(V)$ and so there exist $a \in \alpha_2^{-1}(U')$, $b \in \varphi^{-1}(V)$ with $x-y = a+b$, hence $\varphi(x)-\varphi(y) = \varphi(a)+\varphi(b)$. Since $a \in \alpha_2^{-1}(U')$ we have $a \in \mathrm{Im}\alpha_1$, thus $\varphi(a) \in \mathrm{Im}\varphi\alpha_1$. Since $\varphi(b) \in V$, we get $\varphi(b) \in \beta_1(W)$. Thus there exists $c \in W$ such that $\varphi(b) = \beta_1(c)$. Also there exists $d \in A'$ such that $\psi(d) = c$. Thus $\varphi(b) = \beta_1(c) = \beta_1(\psi(d)) = \varphi\alpha_1(d)$, which implies $\varphi(b) \in \mathrm{Im}\varphi\alpha_1$, hence $\varphi(x) - \varphi(y) = \varphi(a) + \varphi(b) \in \mathrm{Im}\varphi\alpha_1$. Therefore Φ is well-defined. Clearly Φ is a homomorphism. To show it is epimorphic, let $y \in \mathrm{Im}\varphi \cap \mathrm{Im}\beta_1$. There exists $x \in A$ with $\varphi(x) = y$, so $\theta\alpha_2(x) = \beta_2\varphi(x) = \beta_2(y)$. Also we have $y \in \beta_2^{-1}(U)$ or $\beta_2(y) \in U$. Therefore we obtain $\theta\alpha_2(x) \in U$ or $x \in (\theta\alpha_2)^{-1}(U)$.

Finally we show that Φ is monomorphism. Assume that $x+\alpha_2^{-1}(U')+\varphi^{-1}(V) \in \mathrm{Ker}\Phi$. Then $\Phi(x + \alpha_2^{-1}(U') + \varphi^{-1}(V)) = \mathrm{Im}\varphi\alpha_1$ or $\varphi(x) + \mathrm{Im}\varphi\alpha_1 = \mathrm{Im}\varphi\alpha_1$, which implies $\varphi(x) \in \mathrm{Im}\varphi\alpha_1$. There exists $z \in A'$ such that $\varphi(x) = \varphi\alpha_1(z)$ and so $x - \alpha_1(z) \in \mathrm{Ker}\varphi$, thus there exists $t \in \mathrm{Ker}\varphi$

with $x = t + \alpha_1(z)$. We have $\alpha_1(z) \in \mathrm{Im}\alpha_1 = \alpha_2^{-1}(U')$ and $t \in \varphi^{-1}(V)$. Therefore $x \in \alpha_2^{-1}(U') + \varphi^{-1}(V)$. □

COROLLARY 3.7. *Let*

$$\begin{array}{ccccc} A' & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & A'' \\ \downarrow \psi & & \downarrow \varphi & & \downarrow \theta \\ B' & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & B'' \end{array}$$

be a commutative diagram such that the first row is U'-exact and the second row is U-exact. Then φ induces an isomorphism $\Phi : \frac{(\theta\alpha_2)^{-1}(U)}{\alpha_2^{-1}(U')+\varphi^{-1}(0)} \xrightarrow{\sim} \frac{\mathrm{Im}\varphi\cap\mathrm{Im}\beta_1}{\mathrm{Im}\varphi\alpha_1}$.

LEMMA 3.8. *Suppose we have the following commutative diagram of R-module homomorphisms in which the first row be U-exact at A_2, V-exact at A_3 and $\{0\}$-exact at A_4, the second row be $\{0\}$-exact at B_2, U'-exact at B_3 and V'-exact at B_4 :*

$$\begin{array}{ccccccccc} A_5 & \xrightarrow{f_5} & A_4 & \xrightarrow{f_4} & A_3 & \xrightarrow{f_3} & A_2 & \xrightarrow{f_2} & A_1 \\ \downarrow \alpha_5 & & \downarrow \alpha_4 & & \downarrow \alpha_3 & & \downarrow \alpha_2 & & \downarrow \alpha_1 \\ B_5 & \xrightarrow{g_5} & B_4 & \xrightarrow{g_4} & B_3 & \xrightarrow{g_3} & B_2 & \xrightarrow{g_2} & B_1. \end{array}$$

i) *If α_2, α_4 are monic and α_5 is epic then α_3 is monic.*
ii) *If α_2, α_4 are epic and α_1 is monic then α_3 is epic.*
iii) *If every vertical arrow but α_3 is an isomorphism then α_3 is also isomorphism.*

Proof. The proof is straightforward and omitted. □

LEMMA 3.9 (A Generalization of Snake Lemma). *Let*

$$\begin{array}{ccccccc} & & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \end{array}$$

be a commutative diagram such that

1) *The first row is U'-exact and the second row is U-exact,*
2) $W \subseteq \mathrm{Im}\alpha$,
3) $U \subseteq \mathrm{Im}\gamma$,
4) $g(V) \subseteq U$, $f(W) = V$,
5) $U' \subseteq \gamma^{-1}(U)$.

Then there is a "connecting homomorphism" $\omega : \frac{\gamma^{-1}(U)}{U'} \longrightarrow \mathrm{Coker}\alpha$ *such that the following sequence is exact:*

$$\alpha^{-1}(W) \xrightarrow{f'_*} \beta^{-1}(V) \xrightarrow{g'_*} \frac{\gamma^{-1}(U)}{U'} \xrightarrow{\omega} \mathrm{Coker}\alpha \xrightarrow{f_*} \mathrm{Coker}\beta \xrightarrow{g_*} \mathrm{Coker}\gamma.$$

Proof. It is very easy to see $g'(\beta^{-1}(V)) \subseteq \gamma^{-1}(U)$ and $f'(\alpha^{-1}(W)) \subseteq \beta^{-1}(V)$. Therefore we have the following sequence:

$$\alpha^{-1}(W) \overset{f'|_{\alpha^{-1}(W)}}{\longrightarrow} \beta^{-1}(V) \overset{g'|_{\beta^{-1}(V)}}{\longrightarrow} \gamma^{-1}(U).$$

Suppose π is the canonical homomorphism $\pi : \gamma^{-1}(U) \longrightarrow \frac{\gamma^{-1}(U)}{U'}$. We denote $f'_* = f'|_{\alpha^{-1}(W)}$ and $g'_* = \pi g'|_{\beta^{-1}(V)}$. Then

$$(I) \qquad \alpha^{-1}(W) \xrightarrow{f'_*} \beta^{-1}(V) \xrightarrow{g'_*} \frac{\gamma^{-1}(U)}{U'}.$$

On the other hand, f and g induce the following homomorphism: $f^* : \frac{A}{\mathrm{Im}\alpha} \longrightarrow \frac{B}{\mathrm{Im}\beta}$, $g^* : \frac{B}{\mathrm{Im}\beta} \longrightarrow \frac{C}{\mathrm{Im}\gamma}$ by $a + \mathrm{Im}\alpha \longrightarrow f(a) + \mathrm{Im}\beta$ and $b + \mathrm{Im}\beta \longrightarrow g(b) + \mathrm{Im}\gamma$. Hence we have the following sequence:

$$(II) \qquad \mathrm{Coker}\alpha \xrightarrow{f^*} \mathrm{Coker}\beta \xrightarrow{g^*} \mathrm{Coker}\gamma.$$

Now, we show that there exists a homomorphism $\omega : \frac{\gamma^{-1}(U)}{U'} \longrightarrow \mathrm{Coker}\alpha$ "connecting" the sequence (I) and (II). In fact, ω is defined as follows.

Assume that $z + U' \in \frac{\gamma^{-1}(U)}{U'}$. Choose $b' \in B'$ with $g'(b') = z$. Since $g\beta(b') = \gamma g'(b') = \gamma(z) \in U$, we get $\beta(b') \in g^{-1}(U)$ and so $\beta(b') \in \mathrm{Im}f$. Since f is one to one, $f : A \longrightarrow \mathrm{Im}f$ is bijective. Therefore there exists a unique element $a \in A$ such that $\beta(b') = f(a)$, which implies $a = f^{-1}\beta(b')$. Define $\omega(z + U') = a + \mathrm{Im}\alpha$. We show that ω is well-defined, that is, $\omega(z + U')$ is independent of the choice of $b' \in B'$. Indeed, let $b'' \in B'$ with $g'(b'') = z$. Then there exists $a' \in A$ such that $\beta(b'') = f(a')$. We obtain $g'(b' - b'') = 0$, and so $b' - b'' \in \mathrm{Ker}g' \subseteq g'^{-1}(U') = \mathrm{Im}f'$. Hence there exists $\bar{a} \in A'$ with $b' - b'' = f'(\bar{a})$. Since $\beta f'(\bar{a}) = f\alpha(\bar{a})$, $\beta(b' - b'') = f\alpha(\bar{a})$, which implies $f(a - a') = f\alpha(\bar{a})$, so $a - a' = \alpha(\bar{a}) \in \mathrm{Im}\alpha$. Therefore $a + \mathrm{Im}\alpha = a' + \mathrm{Im}\alpha$. Clearly ω is a homomorphism. Since the proof of the exactness is rather long, it will be convenient to divide into several steps.

Step 1. $\mathrm{Im}f'_* = \mathrm{Ker}g'_*$.

Suppose $b' \in \mathrm{Ker}g'_*$. Then $g'(b') + U' = U'$, and so $b' \in g'^{-1}(U')$. Hence there exists $x \in A'$ such that $b' = f'(x)$. Now, it is enough to show that $x \in \alpha^{-1}(W)$. We have $f\alpha(x) = \beta f'(x) = \beta(b')$. Since $b' \in$

$\beta^{-1}(V)$, $\beta(b') \in V$, which implies that $f\alpha(x) \in V$. Since $f(W) = V$ and f is monic, we have $\alpha(x) \in W$ or $x \in \alpha^{-1}(W)$. Therefore $\mathrm{Ker}g'_* \subseteq \mathrm{Im}f'_*$. The proof of converse is easy.

Step 2. $\mathrm{Im}g'_* = \mathrm{Ker}\omega$.

Suppose that $g'(b') + U' \in \mathrm{Im}g'_*$, where $b' \in \beta^{-1}(V)$. By definition of ω we have

$$\omega(g'(b') + U') = f^{-1}\beta g'^{-1}g'(b') + \mathrm{Im}\alpha = f^{-1}\beta(b') + \mathrm{Im}\alpha.$$

Since $b' \in \beta^{-1}(V)$, $\beta(b') \in V$. Since $f(W) = V$ and f is monic, $f^{-1}\beta(b') \in W$, which implies $f^{-1}\beta(b') \in \mathrm{Im}\alpha$. Hence $\omega(g'(b') + U') = \mathrm{Im}\alpha$, which implies $g'(b') + U' \in \mathrm{Ker}\omega$. Therefore $\mathrm{Im}g'_* \subseteq \mathrm{Ker}\omega$.

Conversely, let $c' + U' \in \mathrm{Ker}\omega$ with $c' \in \gamma^{-1}(U)$. Then $\omega(c' + U') = \mathrm{Im}\alpha$ and so $f^{-1}\beta g'^{-1}(c') + \mathrm{Im}\alpha = \mathrm{Im}\alpha$. Thus there exists $x \in A'$ with $f^{-1}\beta g'^{-1}(c') = \alpha(x)$, and so $\beta g'^{-1}(c') = f\alpha(x))$. We have $\beta g'^{-1}(c') = \beta f'(x)$, so $\beta(g'^{-1}(c') - f'(x)) = 0$. Therefore $g'^{-1}(c') - f'(x) \in \mathrm{Ker}\beta \subseteq \beta^{-1}(V)$, and hence

$$g'_*(g'^{-1}(c') - f'(x)) = g'g'^{-1}(c') - g'f'(x) + U' = c' + U',$$

which implies $c' + U' \in \mathrm{Im}g'_*$. Thus $\mathrm{Ker}\omega \subseteq \mathrm{Im}g'_*$.

Step 3. $\mathrm{Im}\omega = \mathrm{Ker}f^*$.

Consider $\omega(c' + U') \in \mathrm{Im}\omega$. Then $f^{-1}\beta g'^{-1}(c') + \mathrm{Im}\alpha \in \mathrm{Im}\omega$. Hence

$$f^*(f^{-1}\beta g'^{-1}(c') + \mathrm{Im}\alpha) = ff^{-1}\beta g'^{-1}(c') + \mathrm{Im}\beta = \beta g'^{-1}(c') + \mathrm{Im}\beta = \mathrm{Im}\beta.$$

Conversely, suppose $a + \mathrm{Im}\alpha \in \mathrm{Ker}f^*$. Then $f^*(a + \mathrm{Im}\alpha) = \mathrm{Im}\beta$, which implies $f(a) \in \mathrm{Im}\beta$ and there exists $b' \in B'$ with $\beta(b') = f(a)$. Since $gf(a) \in U$, $g\beta(b') \in U$. Clearly $\gamma g'(b') \in U$ and so $g'(b') \in \gamma^{-1}(U)$. Therefore

$$\omega(g'(b') + U') = f^{-1}\beta g'^{-1}g'(b') + \mathrm{Im}\alpha = f^{-1}\beta(b') + \mathrm{Im}\alpha = a + \mathrm{Im}\alpha.$$

Step 4. $\mathrm{Im}f^* = \mathrm{Ker}g^*$.

Suppose $f^*(a + \mathrm{Im}\alpha) \in \mathrm{Im}f^*$. Then $g^*f^*(a + \mathrm{Im}\alpha) = gf(a) + \mathrm{Im}\gamma$. Since $gf(a) \in U \subseteq \mathrm{Im}\gamma$, $g^*f^*(a + \mathrm{Im}\alpha) = \mathrm{Im}\gamma$ and so $f^*(a + \mathrm{Im}\alpha) \in \mathrm{Ker}g^*$.

Now, let $b + \mathrm{Im}\beta \in \mathrm{Ker}g^*$. Then $g(b) \in \mathrm{Im}\gamma$ and there exists $c' \in C'$ such that $g(b) = \gamma(c')$. Since g' is epic, there exists $b' \in B'$ with $g'(b') = c'$. Hence $g(b) = \gamma(g'(b'))$ and so $g(b) = g\beta(b')$, which implies $g(b - \beta(b')) = 0$. Then $b - \beta(b') \in \mathrm{Ker}g \subseteq g^{-1}(U) = \mathrm{Im}f$. Therefore there exists $a \in A$ such that $b - \beta(b') = f(a)$, so $b + \mathrm{Im}\beta = f(a) + \mathrm{Im}\beta = f^*(a + \mathrm{Im}\alpha)$, thus $b + \mathrm{Im}\beta \in \mathrm{Im}f^*$. Therefore the proof of lemma is complete. □

COROLLARY 3.10. *Let*

$$\begin{array}{ccccccc} & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow 0 \\ & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & \\ 0 \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \end{array}$$

be a commutative diagram such that

i) *The first row is U'-exact and the second row is U-exact*
ii) $U' \subseteq \gamma^{-1}(U)$ *and* $U \subseteq Im\gamma$.

Then the following sequence is exact:

$$\text{Ker}\alpha \xrightarrow{f'_*} \text{Ker}\beta \xrightarrow{g'_*} \frac{\gamma^{-1}(U)}{U'} \xrightarrow{\omega} \text{Coker}\alpha \xrightarrow{f^*} \text{Coker}\beta \xrightarrow{g^*} \text{Coker}\gamma.$$

4. Connection between chain U-complexes and U-exact sequences

The purpose of this section is to establish a few of the basic properties of the U-homological algebra.

DEFINITION 4.1. Let $\{(C^{(m)}, U^{(m)}, \partial^{(m)})\}$ be a sequence of chain $U^{(m)}$-complexes, and $\{F^{(m)} : (C^{(m)}, U^{(m)}, \partial^{(m)}) \longrightarrow (C^{(m+1)}, U^{(m+1)}, \partial^{(m+1)})\}$ a sequence of chain $(U^{(m)}, U^{(m+1)})$-maps. The sequence $\{(C^{(m)}, U^{(m)}, \partial^{(m)})\}$ is called $\widetilde{U}$-*exact* if for all p, the following sequence is U_p-exact, where $U_p = \{\cdots, U_p^{(m)}, U_p^{(m+1)}, U_p^{(m+2)}, \cdots\}$:

$$\cdots C_p^{(m)} \xrightarrow{F_p^{(m)}} C_p^{(m+1)} \xrightarrow{F_p^{(m+1)}} C_p^{(m+2)} \longrightarrow \cdots .$$

DEFINITION 4.2. Let (C, U, ∂), (C', U', ∂') and (C'', U'', ∂'') be chain U-complex, chain U'-complex and chain U''-complex, respectively. The sequence

$$0 \longrightarrow (C, U, \partial) \xrightarrow{F} (C', U', \partial') \xrightarrow{G} (C'', U'', \partial'') \longrightarrow 0$$

is called a *short* $\widetilde{U}$-*exact sequence* if for all p, the following sequence is a short U''_p-exact sequence:

$$0 \longrightarrow C_p \xrightarrow{F_p} C'_p \xrightarrow{G_p} C''_p \longrightarrow 0.$$

THEOREM 4.3 (A Generalization of Connecting Homomorphism). *Let*

$$0 \longrightarrow (C,U,\partial) \xrightarrow{F} (C',U',\partial') \xrightarrow{G} (C'',U'',\partial'') \longrightarrow 0$$

be a short $\widetilde{U}$*-exact sequence and* $F(U) = U'$*. For each* p*, there is a homomorphism,*

$$\gamma_p : \quad H_p(C'',U'',\partial'') \longrightarrow H_{p-1}(C,U,\partial)$$

defined by

$$z'' + B_p(C'',U'',\partial'') \longrightarrow F_{p-1}^{-1}\partial'_p\partial_p^{-1}(z'') + B_{p-1}(C,U,\partial).$$

Proof. Consider the commutative diagram with the first row is a short U''_p-exact sequence and the second row is a short U''_{p-1}-exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_p & \xrightarrow{F_p} & C'_p & \xrightarrow{G_p} & C''_p & \longrightarrow & 0 \\ & & \downarrow \partial_p & & \downarrow \partial'_p & & \downarrow \partial''_p & & \\ 0 & \longrightarrow & C_{p-1} & \xrightarrow{F_{p-1}} & C'_{p-1} & \xrightarrow{G_{p-1}} & C''_{p-1} & \longrightarrow & 0. \end{array}$$

Suppose $z'' \in Z_p(C'',U'',\partial'')$. Then $\partial''_p(z'') \in U''_{p-1}$. Since G_p is epic, we may lift z'' to $a'_p \in C'_p$ and then push down to $\partial'(a'_p) \in C'_{p-1}$. By commutativity,

$$G_{p-1}\partial'_p(a'_p) = \partial''G_p(a'_p) = \partial''_p(z'') \in U''_{p-1}.$$

It follows that $\partial'_p(a'_p) \in G_{p-1}^{-1}(U''_{p-1})$, which implies $\partial'_p(a_p) \in \mathrm{Im}F_{p-1}$, Then there is a unique (F_{p-1} is monic) $a_{p-1} \in C_{p-1}$ with $F_{p-1}(a_{p-1}) = \partial'_p(a'_p)$, hence $a_{p-1} = F_{p-1}^{-1}\partial'_p(a'_p)$. Suppose we had lifted z'' to $b'_p \in C'_p$. Then the construction above yields $b_{p-1} \in C_{p-1}$ with $F_{p-1}(b_{p-1}) = \partial'_p(b'_p)$. Therefore $G_p(a'_p - b'_p) = 0$ then $a'_p - b'_p \in \mathrm{Ker}G_p \subseteq G^{-1}(U''_p) = \mathrm{Im}F_p$, so there is $x \in C_p$ with $a'_p - b'_p = F_p(x)$. By commutativity, $\partial'_p(a'_p - b'_p) = F_{p-1}\partial_p(x)$. We obtain $F_{p-1}(a_{p-1} - b_{p-1}) = F_{p-1}\partial_p(x)$, and so $a_{p-1} - b_{p-1} = \partial_p(x)$. Hence $a_{p-1} + B_{p-1}(C,U,\partial) = b_{p-1} + B_{p-1}(C,U,\partial)$. There is thus a well- defined homomorphism

$$Z_p(C'',U'',\partial'') \longrightarrow \frac{C_{p-1}}{B_{p-1}(C,U,\partial)}.$$

It is easy to check that this map sends $B_p(C'',U'',\partial'')$ into 0 and also $a_{p-1} = F_{p-1}^{-1}\partial'_pG_p^{-1}(z'') \in Z_{p-1}(C,U,\partial)$. Therefore the formula does give a map

$$H_p(C'',U'',\partial'') \longrightarrow H_{p-1}(C,U,\partial),$$

as desired. □

THEOREM 4.4 (A Generalization of Exact Triangle). *If*

$$0 \longrightarrow (C, U, \partial) \xrightarrow{F} (C', U', \partial') \xrightarrow{G} (C'', U'', \partial'') \longrightarrow 0$$

is a short $\widetilde{U}$*-exact sequence with* $F(U) = U'$*, then there is an exact sequence of* R*-modules*

$$\cdots \longrightarrow H_p(C, U, \partial) \xrightarrow{F_p^*} H_p(C', U', \partial') \xrightarrow{G_p^*} H_p(C'', U'', \partial'')$$

$$\xrightarrow{\gamma_p} H_{p-1}(C, U, \partial) \longrightarrow \cdots .$$

In other words, the triangle

$$\begin{array}{ccc} H(C,U,\partial) & \xrightarrow{F^*} & H(C',U',\partial') \\ \gamma \nwarrow & & \swarrow G^* \\ & H(C'',U'',\partial'') & \end{array}$$

is exact.

Proof. The proof is similar to the proof of exact triangle theorem and omitted (see [4]). □

THEOREM 4.5 (Naturality). *Consider the commutative diagram with the first row is a short* U*-exact sequence with* $F(U) = U'$ *and the second row is a short* V*-exact sequence with* $\varphi(V) = V'$:

$$\begin{array}{ccccccccc} 0 \longrightarrow & (C,U,\partial) & \xrightarrow{F} & (C',U',\partial') & \xrightarrow{G} & (C'',U'',\partial'') & \longrightarrow 0 \\ & \downarrow f & & \downarrow g & & \downarrow h & \\ 0 \longrightarrow & (D,V,\sigma) & \xrightarrow{\varphi} & (D',V',\sigma') & \xrightarrow{\psi} & (D'',V'',\sigma'') & \longrightarrow 0. \end{array}$$

Then there is a commutative diagram of modules with exact rows:

$$\begin{array}{ccccccccc} \cdots \longrightarrow H_p(C,U,\partial) & \xrightarrow{F_p^*} & H_p(C',U',\partial') & \xrightarrow{G_p^*} & H_p(C'',U'',\partial'') & \xrightarrow{\gamma_p} & H_{p-1}(C,U,\partial) & \longrightarrow \cdots \\ \downarrow f_p^* & & \downarrow g_p^* & & \downarrow h_p^* & & \downarrow f_{p-1}^* & \\ \cdots \longrightarrow H_p(D,V,\sigma) & \xrightarrow{\varphi_p^*} & H_p(D',V',\sigma') & \xrightarrow{\psi_p^*} & H_p(D'',V'',\sigma'') & \xrightarrow{\gamma_p'} & H_{p-1}(D,V,\sigma) & \longrightarrow \cdots . \end{array}$$

Proof. Exactness of the rows is Theorem 4.4. The first two squares commutative because H_p is a functor. A routine but long chase gives commutativity of the square involving connecting homomorphism. □

5. U-exact functor and $\mathcal{U}$-functor

In this section we introduce the notions of left (right) U-exact functor and covariant $\mathcal{U}$-functor, and prove a few results concerning these concepts.

DEFINITION 5.1. Let R and S be commutative rings, and let T be a covariant additive functor from R-modules to S-modules. We say that T is *left U-exact* precisely when the following condition is satisfied: whenever

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

is a U-exact sequence of R-modules and R-homomorphisms, the induced sequence

$$0 \longrightarrow T(A) \xrightarrow{T(\varphi)} T(B) \xrightarrow{T(\psi)} T(C)$$

of S-modules and S-homomorphisms is $T(U)$-exact.

PROPOSITION 5.2. *$HOM_R(D,-)$ is a left U-exact functor.*

Proof. Suppose $0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ is U-exact. We show that the sequence $0 \longrightarrow HOM_R(D,A) \xrightarrow{\bar{\varphi}} HOM_R(D,B) \xrightarrow{\bar{\psi}} HOM_R(D,C)$ is $HOM_R(D,U)$-exact. We must prove:

i) $\text{Ker}\bar{\varphi} = 0$ (that is, $\bar{\varphi}$ is a monomorphism),
ii) $\text{Im}\bar{\varphi} \subseteq \bar{\psi}^{-1}(HOM_R(D,U))$,
iii) $\bar{\psi}^{-1}(HOM_R(D,U)) \subseteq \text{Im}\bar{\varphi}$.

i) Assume that $f \in \text{Ker}\bar{\varphi}$. Then $\bar{\varphi}(f) = 0$. By definition of $\bar{\varphi}$ we have $\varphi f = 0$, which implies $\varphi f(x) = 0$ for all $x \in D$. Since $0 \longrightarrow A \xrightarrow{\varphi} B$ is exact, φ is monomorphism and $f = 0$.

ii) Suppose $f \in \text{Im}\bar{\varphi}$. Then there exists $g \in HOM_R(D,A)$ such that $\bar{\varphi}(g) = \varphi g = f$. Therefore $f(x) \in \text{Im}\varphi$ for all $x \in D$. Since $\text{Im}\varphi = \psi^{-1}(U)$ by U-exactness, we have $f(x) \in \psi^{-1}(U)$ or $\psi(f(x)) \in U$ and so $\psi f \in HOM_R(D,U)$, or $\bar{\psi}(f) \in HOM_R(D,U)$, which implies that $f \in \bar{\psi}^{-1}(HOM_R(D,U))$. Therefore $\text{Im}\bar{\varphi} \subseteq \bar{\psi}^{-1}(HOM_R(D,U))$.

iii) Let $f \in \bar{\psi}^{-1}(HOM_R(D,U))$. Then there exists $g \in HOM_R(D,U)$ such that $\bar{\psi}(f) = \psi f = g$, and so $\psi f(x) \in U$ for all $x \in D$. Therefore $f(x) \in \psi^{-1}(U)$ and hence $f(x) \in \text{Im}\varphi$. Since φ is a monomorphism, $\varphi : A \longrightarrow \text{Im}\varphi$ is an isomorphism. If h is the composite $D \xrightarrow{f} \text{Im}f \subseteq \text{Im}\varphi \xrightarrow{\varphi^{-1}} A$, then $h \in HOM_R(D,A)$ and $f = \varphi h = \bar{\varphi}(h)$ and so $f \in \text{Im}\bar{\varphi}$. Therefore $\bar{\psi}^{-1}(HOM_R(D,U)) \subseteq \text{Im}\bar{\varphi}$. □

DEFINITION 5.3. We say a covariant additive functor T is *right U-exact* precisely when the following condition is satisfied: whenever

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

is a U-exact sequence of R-modules and R-homomorphisms, the induced sequence

$$T(A) \xrightarrow{T(\varphi)} T(B) \xrightarrow{T(\psi)} T(C) \longrightarrow 0$$

of S-modules and S-homomorphisms is $T(U)$-exact.

DEFINITION 5.4. We say a covariant additive functor T is *U-exact* if for every short U-exact sequence $0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$ the induced sequence $0 \longrightarrow T(A) \xrightarrow{T(\varphi)} T(B) \xrightarrow{T(\psi)} T(C) \longrightarrow 0$ is a short $T(U)$-exact sequence.

PROPOSITION 5.5. *Let S be a multiplicative closed subset of R and the sequence $0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$ be U-exact. Then the sequence*

$$0 \longrightarrow S^{-1}A \xrightarrow{S^{-1}\varphi} S^{-1}B \xrightarrow{S^{-1}\psi} S^{-1}C \longrightarrow 0$$

is a short $S^{-1}U$-exact sequence.

Proof. See Proposition 12 in [2]. Therefore S^{-1} is a U-exact functor. □

DEFINITION 5.6. Let T be a covariant functor. Then T is called a *covariant $\mathcal{U}$-functor* if for every homomorphism $\varphi : A \longrightarrow B$ and every submodule U of B with $U \subseteq \mathrm{Im}\varphi$ the following condition holds:

$$T(U) \subseteq \mathrm{Im}T(\varphi).$$

PROPOSITION 5.7. *$HOM_R(D,-)$ is a covariant additive $\mathcal{U}$-functor if D is a projective R-module.*

Proof. Assume that $\varphi : A \longrightarrow B$ is a homomorphism and U a submodule of B with $U \subseteq \varphi(A)$. We show that $HOM_R(D,U) \subseteq \bar{\varphi}(HOM_R(D,A))$. Let $\beta \in HOM_R(D,U)$. Then $\mathrm{Im}\beta \subseteq U$, and so $\mathrm{Im}\beta \subseteq U \subseteq \mathrm{Im}\varphi$. Hence we have the following diagram:

$$\begin{array}{ccc} & & D \\ & \swarrow & \downarrow \beta \\ A & \xrightarrow{\varphi} & \mathrm{Im}\varphi. \end{array}$$

Since D is a projective R-module, there is $\alpha \in HOM(D, A)$ such that $\varphi\alpha = \beta$, which implies $\bar{\varphi}(\alpha) = \beta$. Therefore $\beta \in \mathrm{Im}\bar{\varphi}$. and proof is completed. □

PROPOSITION 5.8. *Let T be a covariant $\mathcal{U}$-functor and (C, U, ∂) a chain U-complex. Then the sequence*

$$(T(C), T(U), T(\partial)) : \cdots \longrightarrow T(C_{p+1}) \overset{T(\partial_{p+1})}{\longrightarrow} T(C_p) \overset{T(\partial_p)}{\longrightarrow} T(C_{p-1}) \longrightarrow \cdots$$

is a chain $T(U)$-complex.

Proof. We must show that

i) $T(\partial_p)T(\partial_{p+1})(T(C_{p+1})) \subseteq T(U_{p-1})$,
ii) $\mathrm{Im}T(\partial_p) \supseteq T(U_{p-1})$.

i) Since (C, U, ∂) is a chain U-complex, we have $\partial_p\partial_{p+1}(C_{p+1}) \subseteq U_{p-1}$. Then the homomorphism $\partial_p\partial_{p+1} : C_{p+1} \longrightarrow U_{p-1}$ is defined. Hence $T(\partial_p\partial_{p+1}) : T(C_{p+1}) \longrightarrow T(U_{p-1})$. Since T is covariant, $T(\partial_p)\, T(\partial_{p+1}) : T(C_{p+1}) \longrightarrow T(U_{p-1})$. Therefore $T(\partial_p)T(\partial_{p+1})(T(C_{p+1})) \subseteq T(U_{p-1})$.

ii): Since (C, U, ∂) is a chain U-complex, we have $\mathrm{Im}\partial_p \supseteq U_{p-1}$. Since T is a $\mathcal{U}$-functor and $\partial_p : C_p \longrightarrow C_{p-1}$, we get

$$T(\partial_p)(T(C_p)) \supseteq T(U_{p-1}).$$

□

PROPOSITION 5.9. *Let (C, U, ∂) be a chain U-complex, (C', U', ∂') a chain U'-complex and $F = \{F_p\}$ a chain (U, U')-map. If T is a covariant $\mathcal{U}$-functor, then $TF = \{T(F_p)\}$ is a chain $(T(U), T(U'))$-map.*

Proof. Since $F_{p-1}\partial_p = \partial'_p F_p$ and T is covariant, $T(F_{p-1})T(\partial_p) = T(\partial'_p)T(F_p)$. We know $F_p(U_p) \subseteq U'_p$. Hence $F_p|_{U_p} : U_p \longrightarrow U'_p$ is defined. Therefore $T(F_p)(T(U_p)) \subseteq T(U'_p)$. □

PROPOSITION 5.10. *Let T be a covariant additive $\mathcal{U}$-functor. If $F, G : C \longrightarrow C'$ are two chain (U, U')-homotopic, then $TF, TG : T(C) \longrightarrow T(C')$ are two chain $(T(U), T(U'))$-homotopic.*

Proof. The proof follows from Definition 2.7. According to Theorem 2.10 and Proposition 5.9, we have $T^*(F_p) = T^*(G_p)$. □

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