

REIDEMEISTER CLASSES FOR COINCIDENCE

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ABSTRACT. We generalize algebraic results of Nielsen fixed point theory to Nielsen coincidence theory. We use the algebraic methods of D. Ferrario in Nielsen fixed point theory.

1. Introduction

Let X be a finite CW -complex and let $f : X \rightarrow X$ be a given map. The generalized Lefschetz number $L(f)$ is the alternating sum of the Reidemeister traces of f . The Nielsen number $N(f)$ is the minimum number of nonzero summands in the sum representing $L(f)$ and give a lower bound for the number of fixed points of maps homotopic to f (see [1, 8, 9]). X is defined to be of *Jiang type* if it satisfies the following property; if $L(f) \neq 0$ then $N(f) = R(f)$, where $R(f)$ is the Reidemeister number, and if $L(f) = 0$ then $N(f) = 0$. It has been recently proved in [12] that a wide class of homogeneous spaces are of Jiang type. For such spaces Ferrario computes the Reidemeister number of the map [2]. For the coincidence theory, let $f, g : X \rightarrow Y$ be maps between compact connected closed oriented topological manifolds with $\dim X = \dim Y$. Suppose that Y is a Jiang space; if $L(f, g) \neq 0$ then $N(f, g) = R(f, g)$ where $N(f, g), R(f, g)$ are defined in Section 3, and if $L(f, g) = 0$ then $N(f, g) = 0$ (see [5]). Thus we need to compute the Reidemeister number $R(f, g)$ of the maps.

In Section 2, let $f, g : G' \rightarrow G$ be homomorphisms between groups, H', H normal subgroups of G', G , respectively such that $f(H') \subset H$ and $g(H') \subset H$. Assume that $g|_{H'} : H' \rightarrow H$ is surjective. Then we can generalize to coincidence theory some algebraic results which allow one to distinguish Reidemeister classes due to Ferrario [2]. In fact, if g is the identity endomorphism, then the conclusions are the same as in [2]. In

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Section 3, we summarize the coincidence theory which we need in this paper. In the last section we obtain some algebraic results of the Nielsen numbers of fibre maps.

2. The Reidemeister classes of homomorphisms

Let G', G be groups and $f, g : G' \rightarrow G$ homomorphisms. There is an equivalence relation on G defined by the rule that $\alpha \simeq \beta$ if and only if there is a $\gamma \in G'$ with $\alpha = g(\gamma)\beta f(\gamma^{-1})$. The class containing α will be denoted by $[\alpha]$, and $\mathcal{R}(f, g)$ the set of equivalence classes. Then there is an exact sequence (with the obvious base points)

$$1 \rightarrow \text{Coin}(f, g) \rightarrow G' \xrightarrow{gf^{-1}} G \xrightarrow{j} \mathcal{R}(f, g) \rightarrow 1$$

of groups and based sets, where $\text{Coin}(f, g) = \{\gamma \in G' : f(\gamma) = g(\gamma)\}$, $(gf^{-1})(\gamma) = g(\gamma)f(\gamma^{-1})$ and $j : G \rightarrow \mathcal{R}(f, g)$ is the quotient function. The order $\#\mathcal{R}(f, g)$ of the equivalence classes is called the *Reidemeister number of (f, g)* on G and is written $R(f, g)$. If G is abelian, gf^{-1} is a homomorphism and $\mathcal{R}(f, g)$ has a canonical group structure in which j is a homomorphism. Note that for all $\gamma \in G', \alpha \in G$, $[g(\gamma)\alpha] = [\alpha f(\gamma)]$. In particular, $[g(\gamma)] = [f(\gamma)]$ for all $\gamma \in G'$.

Next we consider the naturality of Reidemeister operations of pairs. Suppose we are given a commutative diagram of groups and homomorphisms

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H' & \xrightarrow{i'} & G' & \xrightarrow{q'} & \bar{G}' & \longrightarrow & 1 \\ & & f|_{H'}, g|_{H'} \downarrow & & f, g \downarrow & & \bar{f}, \bar{g} \downarrow & & \\ 1 & \longrightarrow & H & \xrightarrow{i} & G & \xrightarrow{q} & \bar{G} & \longrightarrow & 1 \end{array}$$

in which the rows are exact and $f|_{H'}, g|_{H'}$ are the restrictions of f, g to H' . Then q' restricts to a homomorphism $q' : \text{Coin}(f, g) \rightarrow \text{Coin}(\bar{f}, \bar{g})$ also denoted by q' : further q induces a function $q_* : \mathcal{R}(f, g) \rightarrow \mathcal{R}(\bar{f}, \bar{g})$ in the obvious way. Thus by some modifications of [3; Theorem 1.8] we have

THEOREM 2.1([10]). *In the above situation there is an exact sequence*

$$\begin{aligned} 1 \rightarrow \text{Coin}(f|_{H'}, g|_{H'}) \rightarrow \text{Coin}(f, g) \xrightarrow{q'} \text{Coin}(\bar{f}, \bar{g}) \\ \xrightarrow{\delta} \mathcal{R}(f|_{H'}, g|_{H'}) \xrightarrow{i_*} \mathcal{R}(f, g) \xrightarrow{q_*} \mathcal{R}(\bar{f}, \bar{g}) \rightarrow 1 \end{aligned}$$

of groups and based sets in which δ is given by

$$\delta(\bar{\gamma}) = [g(\gamma)f^{-1}(\gamma)] \text{ where } q'(\gamma) = \bar{\gamma}.$$

Furthermore, if G is abelian, then the sequence can be regarded as an exact sequence of groups.

The function $q_* : \mathcal{R}(f, g) \rightarrow \mathcal{R}(\bar{f}, \bar{g})$ in Theorem 2.1 is surjective so

$$R(\bar{f}, \bar{g}) \leq R(f, g).$$

Also, we can consider the following commutative diagram

$$\begin{array}{ccc} G' & \xrightarrow{f, g} & G \\ q' \downarrow & & \downarrow q \\ G'/H' & \xrightarrow{\bar{f}, \bar{g}} & G/H. \end{array}$$

Then we have equivalence relation on G/H given by the rule that $H\alpha \simeq H\beta$ if and only if there exists $\gamma \in G'$ with $H\alpha = Hg(\gamma)\beta f(\gamma^{-1})$. The class containing $H\alpha$ will be denoted by $[H\alpha]$, and the set of all classes by $\mathcal{R}(f, g; H)$. We call $\mathcal{R}(f, g; H)$ the set of H -Reidemeister classes of f and g . The number $\#\mathcal{R}(f, g; H)$ of the equivalence classes is called the mod H Reidemeister number of (f, g) on G and is written $R(f, g; H)$. Note that in Theorem 2.1 $\mathcal{R}(\bar{f}, \bar{g}) = \mathcal{R}(f, g; H)$. We see that $\mathcal{R}(f, g; H)$ is independent of the choice of H' . If H is trivial then $\mathcal{R}(f, g; H) = \mathcal{R}(f, g)$; if $H = G$ then $\mathcal{R}(f, g; H) = \{[H]\}$. More generally we have the following result.

LEMMA 2.2. *If H_1, H_2 are normal subgroups of G with $H_1 \subset H_2$, then there is a natural surjective function*

$$i : \mathcal{R}(f, g; H_1) \rightarrow \mathcal{R}(f, g; H_2)$$

defined by $i([H_1\alpha]) = [H_2\alpha]$.

EXAMPLE 2.3. Let $G' = G = Z$ be the additive group of integers; let $l, m, n \in Z$ be given with $l|m, n$; let $f, g : G' \rightarrow G$ be the endomorphisms given by $f(x) = mx, g(x) = nx$ for all $x \in G'$, let $H = H' = lZ \subset G'$ be a normal subgroup of multiples of l . Then $\mathcal{R}(f, g) = \mathcal{R}(f|_{H'}, g|_{H'}) =$

$Z/(n - m)Z$ and $\mathcal{R}(f, g; H) = Z/dZ$ where $d = \gcd(n - m, l)$ is the greatest common divisor of $n - m$ and l ; the sequence

$$Z/(n - m)Z \xrightarrow{l} Z/(n - m)Z \rightarrow Z/dZ \rightarrow 0$$

is exact where the inclusion $l = i_*$ is the homomorphism induced by the inclusion $i : H \rightarrow G$. Note that $i_* : \mathcal{R}(f|_{H'}, g|_{H'}) \rightarrow \mathcal{R}(f, g)$ is not injective unless $d = 1$.

In this section, we try to find a normal subgroup T of H that the map $i'_* : \mathcal{R}(f|_{H'}, g|_{H'}; T) \rightarrow q_*^{-1}([H])$ induced by i_* in Theorem 2.1 is a bijection. If this happens, the sequence of pointed sets

$$1 \rightarrow \mathcal{R}(f|_{H'}, g|_{H'}; T) \xrightarrow{i'_*} \mathcal{R}(f, g) \xrightarrow{q_*} \mathcal{R}(f, g; H) \rightarrow 1$$

would be exact.

Throughout this section we assume that $g|_{H'} : H' \rightarrow H$ is surjective. Since $q'^{-1}\text{Coin}(\bar{f}, \bar{g})$ is a subgroup of G' and H' is a subgroup of $q'^{-1}\text{Coin}(\bar{f}, \bar{g})$, there exists at least one subgroup K' of $q'^{-1}\text{Coin}(\bar{f}, \bar{g})$ such that $K'H' = q'^{-1}\text{Coin}(\bar{f}, \bar{g})$. Let $[K', H']$ denote the subgroup of G' generated by all $khk^{-1}h^{-1}$ such that $k \in K'$ and $h \in H'$. If $K'^{G'}$ is defined as the smallest normal subgroup of G' containing K' , we see that $[K'^{G'}, H'] = [K', H']^{G'}$ is normal. Let the set $O_{(f,g)}K'$ be defined by $O_{(f,g)}K' := \{g(k)f(k^{-1}) : k \in K'\}$. Since $g : H' \rightarrow H$ is surjective, $g([K'^{G'}, H'])$ is normal in H . For any such subgroup K' let the subgroup $T_{(f,g)}(K')$ be defined as

$$T := T_{(f,g)}(K') = g([K'^{G'}, H']) \cup O_{(f,g)}K'$$

the smallest subgroup of G containing both $g([K'^{G'}, H'])$ and $O_{(f,g)}K'$.

PROPOSITION 2.4. *The subgroup T is normal in H and the equality*

$$T = \{g(x)g(k)f(k^{-1}) | x \in [K'^{G'}, H'], k \in K'\}$$

holds true.

PROOF. See [2; Proposition 3.1]. □

LEMMA 2.5. For any subgroup K' of G' such that

$$K'H' = q'^{-1} \text{Coin}(\bar{f}, \bar{g})$$

there exists a surjection

$$A : q_*^{-1}([H]) = i_* \mathcal{R}(f|_{H'}, g|_{H'}) \rightarrow \mathcal{R}(f|_{H'}, g|_{H'}; T)$$

defined by $A([h]) = [p(h)]$ where p is the projection $p : H \rightarrow H/T$; A is injective whenever $\mathcal{R}(f, g) = \mathcal{R}(f, g; g([K'^{G'}, H']))$.

PROOF. We prove the Lemma by the method of [2; Lemma 3.2]. Consider the natural projections

$$i_*(\mathcal{R}(f|_{H'}, g|_{H'})) \xleftarrow{i_*} \mathcal{R}(f|_{H'}, g|_{H'}) \xrightarrow{p_*} \mathcal{R}(f|_{H'}, g|_{H'}; T).$$

We will show that $p_* i_*^{-1}([h])$ is a single element in $\mathcal{R}(f|_{H'}, g|_{H'}; T)$ for all $h \in H$. If $h' = g(\gamma)hf(\gamma^{-1})$ with $h, h' \in H$ and $\gamma \in G'$, then $q(g(\gamma)) = q(f(\gamma))$, and hence $\gamma \in q'^{-1} \text{Coin}(\bar{f}, \bar{g}) = K'H'$; then $\gamma = k_1 h_1$ with $k_1 \in K'$ and $h_1 \in H'$. Therefore, $h' = g(k_1)g(h_1)hf(h_1^{-1})f(k_1^{-1})$ and the equality

$$p(h') = p(g(h_1)hf(h_1^{-1}))p(g(k_1)f(k_1^{-1})) = p(g(h_1))p(h)p(f(h_1^{-1}))$$

shows that A is well defined and surjective. Because $g : H' \rightarrow H$ is surjective, $p(hg(k)) = p(g(k)h)$ for all $h \in H$ and $k \in K'$.

Now assume that $\mathcal{R}(f, g) = \mathcal{R}(f, g; g([K'^{G'}, H']))$. If $A([h_1]) = A([h_2])$ then there exists $h \in H'$ such that

$$p(h_2) = p(g(h))p(h_1)p(f(h^{-1})).$$

Thus, from Proposition 2.4, we can find $h \in H', k \in K'$ and $x \in [K'^{G'}, H']$ such that $h_2 = g(h)h_1f(h^{-1})g(x)g(k)f(k^{-1})$; as a consequence, there exist $h \in H', k \in K'$ and $x' \in [K'^{G'}, H']$ such that $h_2 = g(k)g(h)h_1f(h^{-1})f(k^{-1})g(x')$, i.e., $[h_1] = [h_2] \in \mathcal{R}(f, g; g([K'^{G'}, H']))$; by the assumption we have $[h_1] = [h_2] \in \mathcal{R}(f, g)$. Therefore, A is injective. \square

For all $\alpha \in G$, let $f^\alpha : G' \rightarrow G$ be the homomorphism defined by $f^\alpha(\gamma) := \alpha^{-1}f(\gamma)\alpha$. Then there is a canonical bijection $\alpha_* : \mathcal{R}(f, g) \rightarrow \mathcal{R}(f^\alpha, g)$ defined by $\alpha_*([\beta]) = [\beta\alpha]$.

Let S_1 and S_2 be sets. Then we write $S_1 \geq S_2$ if there exists a surjection $S_1 \rightarrow S_2$. If there is a bijection between S and the disjoint union $\bigsqcup_{j \in Z} S_j$ then we write $S = \sum_{j \in Z} S_j$.

THEOREM 2.6. For every $[H\beta] \in \mathcal{R}(f, g; H)$, let $\alpha \in G$ be such that $[q(\alpha^{-1})] = [H\beta]$. Let K'_α be a subgroup of G' such that

$$q'^{-1} \text{Coin}(\bar{f}^{q(\alpha)}, \bar{g}) = K'_\alpha H'.$$

Then

$$\mathcal{R}(f, g) \geq \sum_{[H\beta] \in \mathcal{R}(f, g; H)} \mathcal{R}(f^\alpha|_{H'}, g|_{H'}; T_\alpha),$$

where $T_\alpha = T_{(f^\alpha, g)}(K'_\alpha)$ and the equality holds if $\mathcal{R}(f, g) = \mathcal{R}(f, g; g([K'^{G'}_\alpha, H']))$ for all $[H\beta]$.

PROOF. Applying Lemma 2.5 to the homomorphisms f^α, g . Let $q_{*\alpha} : \mathcal{R}(f^\alpha, g) \rightarrow \mathcal{R}(f^{q(\alpha)}, g; H)$ denote the function induced by q on $\mathcal{R}(f^\alpha, g)$. Since the following diagram

$$\begin{array}{ccc} G' & \xrightarrow{f^\alpha, g} & G \\ q' \downarrow & & \downarrow q \\ G'/H' & \xrightarrow{\bar{f}^{q(\alpha)}, \bar{g}} & G/H \end{array}$$

commutes, by hypothesis if we choose a subgroup K'_α such that $K'_\alpha H = q'^{-1} \text{Coin}(\bar{f}^{q(\alpha)}, \bar{g})$ then there exists a surjection

$$A_\alpha : q_{*\alpha}^{-1}([H]) \rightarrow \mathcal{R}(f^\alpha|_{H'}, g|_{H'}; T_\alpha)$$

defined by $A_\alpha([h]) = [p_\alpha(h)]$ where $p_\alpha : H \rightarrow H/T_\alpha$ is a natural projection. Moreover, A_α is injective if $\mathcal{R}(f^\alpha, g) = \mathcal{R}(f^\alpha, g; g([K'^{G'}_\alpha, H']))$ for all $[H\beta]$. By Theorem 2.1, $\mathcal{R}(f, g)$ is the disjoint union of $q_*^{-1}([H\beta])$ for all $[H\beta] \in \mathcal{R}(f, g; H)$. The bijection $\alpha_* : \mathcal{R}(f, g) \rightarrow \mathcal{R}(f^\alpha, g)$ induces a bijection $\alpha'_* : q_*^{-1}([H\beta]) \rightarrow q_{*\alpha}^{-1}([H])$ because the following diagram

$$\begin{array}{ccc} \mathcal{R}(f, g) & \xrightarrow{\alpha_*} & \mathcal{R}(f^\alpha, g) \\ q_* \downarrow & & \downarrow q_{*\alpha} \\ \mathcal{R}(f, g; H) & \xrightarrow{q^{(\alpha)}_*} & \mathcal{R}(f^{q(\alpha)}, g; H) \end{array}$$

commutes. Thus we have a surjection

$$A_\alpha \circ \alpha'_* : q_*^{-1}([H\beta]) \rightarrow \mathcal{R}(f^\alpha|_{H'}, g|_{H'}; T_\alpha)$$

for all $[H\beta]$ which gives the desired inequality.

Since $\mathcal{R}(f^\alpha, g) = \mathcal{R}(f^\alpha, g; g([K'^{G'}_\alpha, H']))$ for all $[H\beta]$ if and only if $\mathcal{R}(f, g) = \mathcal{R}(f, g; g([K'^{G'}_\alpha, H']))$ for all $[H\beta]$, the proof is complete using again Lemma 2.5. □

COROLLARY 2.7. If $\text{Coin}(\bar{f}^{q(\alpha)}, \bar{g}) = \{1\}$ for all $[H\beta] \in \mathcal{R}(f, g; H)$, then

$$\mathcal{R}(f, g) = \sum_{[H\beta] \in \mathcal{R}(f, g; H)} \mathcal{R}(f^\alpha|_{H'}, g|_{H'}).$$

PROOF. It suffices to define $K'_\alpha = \{1\}$ for all $[H\beta] \in \mathcal{R}(f, g; H)$. \square

COROLLARY 2.8. For every $[H\beta] \in \mathcal{R}(f, g; H)$, let $\alpha \in G$ be such that $[q(\alpha^{-1})] = [H\beta]$. Put $K'_\alpha = q'^{-1} \text{Coin}(\bar{f}^{q(\alpha)}, \bar{g})$. Then

$$\mathcal{R}(f, g) \geq \sum_{[H\beta] \in \mathcal{R}(f, g; H)} H/T_\alpha$$

and equality holds whenever $\mathcal{R}(f, g) = \mathcal{R}(f, g; g([K'^{G'}_\alpha, H']))$ for every $[H\beta]$.

PROOF. Consider the following commutative diagram

$$\begin{array}{ccc} H' & \xrightarrow{f^\alpha|_{H'}, g|_{H'}} & H \\ \downarrow & & \downarrow \\ H'/T'_\alpha & \xrightarrow{f^\alpha|_{H'^*}, g|_{H'^*}} & H/T_\alpha, \end{array}$$

where $T'_\alpha = f^{-1}(T_\alpha) \cap g^{-1}(T_\alpha)$. Since $H' \subset K'_\alpha$, $f|_{H'^*} = g|_{H'^*}$. But H/T_α is an abelian group because $[H, H] = g([H', H']) \subset g([K'_\alpha, H']) \subset T_\alpha$ for all $[H\beta]$, and hence

$$\mathcal{R}(f|_{H'}, g|_{H'}; T_\alpha) = \mathcal{R}(f^\alpha|_{H'^*}, g|_{H'^*}) = H/T_\alpha$$

which completes the proof. \square

COROLLARY 2.9. If $\mathcal{R}(f, g) = \mathcal{R}(f, g; g([G', H']))$, then

$$\mathcal{R}(f, g) = \sum_{[H\beta] \in \mathcal{R}(f, g; H)} H/T_{(f^\alpha, g)}(q'^{-1} \text{Coin}(\bar{f}^{q(\alpha)}, \bar{g})),$$

where as before $[q(\alpha^{-1})] = [H\beta]$. In particular, if $\mathcal{R}(f, g) = \mathcal{R}(f, g; g([G', G']))$ then

$$\mathcal{R}(f, g) = G/T_{(f, g)}(G').$$

PROOF. By Theorem 2.1 and Lemma 2.2, the following diagram

$$\begin{array}{ccc} \mathcal{R}(f, g) & \longrightarrow & \mathcal{R}(f, g; g([G', H'])) \\ \parallel & & \uparrow_i \\ \mathcal{R}(f, g) & \longrightarrow & \mathcal{R}(f, g; g([K'_\alpha, H'])) \end{array}$$

commutes. The first assumption implies that $\mathcal{R}(f, g) = \mathcal{R}(f, g; g([K'_\alpha, H']))$ for each $[H\beta]$, where $K'_\alpha = q'^{-1}\text{Coin}(\bar{f}^{q(\alpha)}, \bar{g})$. Applying Corollary 2.8 we obtain the stated formula. If we take $H' = G'$ we have

$$\mathcal{R}(f, g) = \sum_{[H\beta] \in \mathcal{R}(f, g; G) = \{1\}} G/T_{(f, g)}(G') = G/T_{(f, g)}(G').$$

□

3. Nielsen classes of continuous maps

Let X and Y be compact connected oriented manifolds of the same dimension. Suppose that $f, g : X \rightarrow Y$ are maps. Let

$$\Phi(f, g) = \{x \in X : f(x) = g(x)\}$$

denote the set of geometric coincidences of f and g on X . There are two sides to the theory: the algebraic and the geometric. For the geometric side we say that $x, y \in \Phi(f, g)$ are Nielsen equivalent provided that there is a path c from x to y so that $f(c) \simeq g(c)$ rel end points. The set of equivalence classes thus generated will be denoted by $\Phi'(f, g)$. An element $[x] \in \Phi'(f, g)$ is said to be a (geometric) Nielsen class of f and g .

Let M, N be compact connected oriented n -dimensional topological manifolds without boundary. Let $U \subset M$ be an open subset and let $f, g : U \rightarrow N$ be a pair of maps such that $\Phi(f, g)$ is compact. Now recall the definition of index of a Nielsen class of such a pair [7, 11]. Let $z_M \in H_n M$ be the fundamental class and let $\mu_N \in H^n(N \times N, N \times N - \Delta N)$ be the Thom class corresponding to the chosen orientations. By the normality of M there exists an open set V in M with $\Phi(f, g) \subset V \subset \bar{V} \subset U$. All homology and cohomology groups are taken with rational coefficients.

We define the coincidence index of f and g as the image of z_M in the sequence of homomorphisms

$$H_n M \rightarrow H_n(M, M - V) \xrightarrow{exc} H_n(U, U - V) \\ \xrightarrow{\langle f \times g \rangle_*} H_n(N \times N, N \times N - \Delta N) \xrightarrow{\langle \mu_N, \cdot \rangle} Q$$

(exc denotes the excision isomorphism, \langle, \rangle the Kronecker index and Q the field of rational numbers). We denote it by $\text{ind}(f, g : U)$. We recall only that it is an integer and that $\text{ind}(f, g : U) \neq 0$ implies the existence of coincidence points of f and g in U . If $U = M$ then $\text{ind}(f, g : M)$ equal the coincidence Lefschetz number $L(f, g)$ [11].

For the map $f, g : X \rightarrow Y$ and $[x] \in \Phi'(f, g)$, we define the index of the class $[x]$ as

$$\text{ind}(f, g : [x]) = \text{ind}(f, g : U),$$

where U is an arbitrary open subset of X containing $[x]$ and disjoint from the other classes of f and g . A compact class $[x]$ on X is said to be essential if its index is nonzero. The number $N(f, g)$ denotes the number of essential coincidence classes of f and g on X . Furthermore $N(f, g)$ is a homotopy invariant (see [5]).

In what follows we shall not distinguish between a path and its class in the fundamental groupoid $\pi(X)$ (or $\pi(Y)$). To come to the algebraic side of the theory we choose base points $x_0 \in X, y_0 \in Y$, but we do not assume that either f or g is base point preserving. So we choose a path ω from y_0 to $f(x_0)$ and μ from y_0 to $g(x_0)$. Using the paths ω and μ we define homomorphisms

$$f_*^\omega, g_*^\mu : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$$

by $f_*^\omega(\gamma) = \omega f(\gamma) \omega^{-1}$ and $g_*^\mu(\gamma) = \mu g(\gamma) \mu^{-1}$. The homomorphisms f_*^ω and g_*^μ determine an equivalence relation $\pi_1(Y, y_0)$ defined by the rule that $\alpha \simeq \beta$ in $\pi_1(Y, y_0)$ if and only if there exists $\gamma \in \pi_1(X, x_0)$ with $\alpha = g_*^\mu(\gamma) \beta f_*^\omega(\gamma^{-1})$. The class containing α will be denoted by $[\alpha]$, and the set of all classes by $\mathcal{R}(f_*^\omega, g_*^\mu)$. We call $\mathcal{R}(f_*^\omega, g_*^\mu)$ the set of (algebraic) Reidemeister classes of f and g . The symbol $\mathcal{R}(f, g)$ denotes the Reidemeister number $\sharp \mathcal{R}(f_*^\omega, g_*^\mu)$ of f and g where \sharp denotes cardinality. From [5, 7] the number $\mathcal{R}(f, g)$ is independent of the choice of x_0 and y_0 , and of the paths ω and μ . Thus if $y_0 = f(x_0)$, and ω is the constant path, then we shall omit the ω and write f_* etc. In what

follows, $j : \pi_1(Y, y_0) \rightarrow \mathcal{R}(f_*, g_*^\mu)$ denotes the quotient function defined by $j(\alpha) = [\alpha] \in \mathcal{R}(f_*, g_*^\mu)$. Let

$$\text{Coin}(f_*^\omega, g_*^\mu) = \{\gamma \in \pi_1(X, x_0) : f_*^\omega(\gamma) = g_*^\mu(\gamma)\}$$

be the set of coincidences of f_*^ω and g_*^μ . The sequence of bases sets and base-point preserving functions

$$1 \rightarrow \text{Coin}(f_*^\omega, g_*^\mu) \rightarrow \pi_1(X, x_0) \xrightarrow{g_*^\mu f_*^{-\omega}} \pi_1(Y, y_0) \xrightarrow{j} \mathcal{R}(f_*^\omega, g_*^\mu) \rightarrow 1$$

is exact, where $(g_*^\mu f_*^{-\omega})(\alpha) = g_*^\mu(\alpha) f_*^\omega(\alpha^{-1})$. Furthermore, by [8; p.27 Theorem 2.1] we have a commutative diagram as follows:

$$\begin{array}{ccccc} \pi_1(X, x_0) & \xrightarrow{g_*^\mu f_*^{-\omega}} & \pi_1(Y, y_0) & \xrightarrow{j} & \mathcal{R}(f_*^\omega, g_*^\mu) \\ \theta_X \downarrow & & \downarrow \theta_Y & & \downarrow \tilde{\theta}_Y \\ H_1(X) & \xrightarrow{g_* - f_*} & H_1(X) & \xrightarrow{\eta_Y} & \text{Coker}(g_* - f_*), \end{array}$$

where θ_X, θ_Y are Hurewicz homomorphisms, and $\tilde{\theta}_Y$ the induced function. If $\pi_1(Y, y_0)$ is abelian there is a canonical group structure on $\mathcal{R}(f_*^\omega, g_*^\mu)$.

The algebraic and geometric theory are related by an injective function

$$\rho := \rho_{\omega, \mu} : \Phi'(f, g) \rightarrow \mathcal{R}(f_*^\omega, g_*^\mu)$$

defined by $\rho([x]) = [\mu g(c) f(c^{-1}) \omega^{-1}]$, where c is any path in X with $c(0) = x_0, c(1) = x$. The $\text{Ind}([\alpha])$ of a class $[\alpha] \in \mathcal{R}(f_*^\omega, g_*^\mu)$ is defined as follows

$$\text{Ind}([\alpha]) = \begin{cases} \text{ind}(f, g : [x]) & \text{if } [\alpha] = \rho([x]) \\ 0 & \text{otherwise,} \end{cases}$$

where $\text{ind}(f, g : [x])$ is the index defined as above. $[\alpha] \in \mathcal{R}(f_*^\omega, g_*^\mu)$ is essential if it has nonzero index. From this definition of index of a Reidemeister class, we can define the Nielsen number $N(f, g)$ of f and g either as the number of essential (geometric) Nielsen classes or as the number of essential (algebraic) Reidemeister classes of f and g . Thus we have $N(f, g) \leq R(f, g)$.

Let H be a normal subgroups of $\pi_1(Y, y)$ (this means that for each $y \in Y$ a normal subgroup $H(y) \subset \pi_1(Y, y)$ is given such that for any path u from y to y' and any loop α based at y representing an element of $H(y)$

we have $u^{-1}\alpha u \in H(y')$. Two paths u, v are said to be H -homotopic if $u(0) = v(0), u(1) = v(1)$ and $uv^{-1} \in H(u(0))$. We then write $u \stackrel{H}{\simeq} v$. For the geometric side we say that $x, x' \in \Phi(f, g)$ are H -Nielsen equivalent if there is a path c in X from x to x' such that $f(c) \stackrel{H}{\simeq} g(c)$ in Y . The set of equivalence classes thus generated will be denoted by $\Phi'_H(f, g)$. An element $[Hx] \in \Phi'_H(f, g)$ is said to be a (geometric) H -Nielsen class of f and g . We define the index of the class $[Hx]$ as

$$\text{ind}(f, g : [Hx]) = \text{ind}(f, g : U),$$

where U is an arbitrary open subset of X containing $[Hx]$ and disjoint from the other classes of f and g as in Section 2. A compact class $[Hx]$ on X is said to be essential if its index is nonzero. The number $N(f, g; H)$ denotes the number of essential coincidence classes of f and g on X . Furthermore $N(f, g; H)$ is a homotopy invariant (see [7]).

The group homomorphisms f_*^ω, g_*^μ define an equivalence relation on $\pi_1(Y, y_0)/H$ by setting $H\alpha \simeq H\beta$ if and only if there exists a $\gamma \in \pi_1(X, x_0)$ such that $H\alpha = Hg_*^\mu(\gamma)\beta f_*^\omega(\gamma^{-1})$. The class containing $H\alpha$ will be denoted by $[H\alpha]$, and the set of all classes by $\mathcal{R}(f_*^\omega, g_*^\mu; H)$. We call $\mathcal{R}(f_*^\omega, g_*^\mu; H)$ the set of (algebraic) mod H Reidemeister classes of f and g . The symbol $R(f, g; H)$ denotes the Reidemeister number $\#\mathcal{R}(f_*^\omega, g_*^\mu; H)$ of f and g . From [7] the number $\mathcal{R}(f_*^\omega, g_*^\mu; H)$ is independent of the choice of x_0 and y_0 , and of the paths ω and μ .

The algebraic and geometric theory are related by an injective function

$$H\rho := H\rho_{\omega, \mu} : \Phi'_H(f, g) \rightarrow \mathcal{R}(f_*^\omega, g_*^\mu; H)$$

defined by $\rho([Hx]) = [H\mu g(c)f(c^{-1})\omega^{-1}]$, where c is any path in X with $c(0) = x_0, c(1) = x$. The $\text{Ind}([H\alpha])$ of a class $[H\alpha] \in \mathcal{R}(f_*^\omega, g_*^\mu; H)$ is defined as follows

$$\text{Ind}([H\alpha]) = \begin{cases} \text{ind}(f, g : [Hx]) & \text{if } [H\alpha] = H\rho([Hx]) \\ 0 & \text{otherwise,} \end{cases}$$

where $\text{ind}(f, g : [Hx])$ is the index defined as above. $[H\alpha] \in \mathcal{R}(f_*^\omega, g_*^\mu; H)$ is essential if it has nonzero index. From this definition of index of a Reidemeister class, we can define the Nielsen number $N(f, g; H)$ of f and g either as the number of essential (geometric) H -Nielsen classes or as the number of essential (algebraic) mod H Reidemeister classes of f and g . Thus we have $N(f, g; H) \leq R(f, g; H)$.

Moreover, we have the following relationship between Reidemeister classes and mod H Reidemeister classes;

PROPOSITION 3.1. *There is a commutative diagram*

$$\begin{CD} \Phi'(f, g) @>\rho>> \mathcal{R}(f_*^\omega, g_*^\mu) \\ @VqVV @VVq_*V \\ \Phi'_H(f, g) @>H\rho>> \mathcal{R}(f_*^\omega, g_*^\mu; H), \end{CD}$$

where q is defined by $q([x]) = [Hx]$ and q_* is the induced function by $q_* : \pi_1(Y, y_0) \rightarrow \pi_1(Y, y_0)/H$.

From this we see that q_* maps essential Reidemeister class of f and g into essential mod H Reidemeister classes of f and g .

4. Nielsen numbers of fibre maps

Let $p_1 : E_1 \rightarrow B_1$ and $p_2 : E_2 \rightarrow B_2$ be locally trivial bundles such that the total spaces, base spaces and fibres are compact connected closed oriented topological manifolds. We assume $\dim E_1 = \dim E_2$, $\dim B_1 = \dim B_2$ (hence the dimensions of all fibres are equal) and the orientations of the base spaces, total spaces and fibres are compatible. We consider the coincidence indices of $(f, g), (\bar{f}, \bar{g}), (f_b, g_b)$ with respect to these orientations. Suppose we are given two fibre maps ,i.e., a commutative diagram

$$\begin{CD} E_1 @>f, g>> E_2 \\ @Vp_1VV @VVp_2V \\ B_1 @>\bar{f}, \bar{g}>> B_2. \end{CD}$$

Let $x \in E_1, y \in E_2, p_1(x) = b, F_{1b} = p_1^{-1}(b), F_{2b} = p_2^{-1}(p_2(y)), f_b = f|_{F_{1b}}$ and $g_b = g|_{F_{1b}}$. We choose paths ω from y to $f(x)$ and μ from y to $g(x)$ in F_{2b} ; then there is a commutative diagram of groups and homomorphisms

$$\begin{CD} 1 @>>> \pi_1(F_{1b}, x)/K_1 @>j_{1*}>> \pi_1(E_1, x) @>p_{1*}>> \pi_1(B_1, p_1(x)) @>>> 1 \\ @. @V\bar{f}_b^\omega, \bar{g}_b^\mu VV @VVf_*^\omega, g_*^\mu V @VV\bar{f}_*^{p_2(\omega)}, \bar{g}_*^{p_2(\mu)} V \\ 1 @>>> \pi_1(F_{2b}, y)/K_2 @>j_{2*}>> \pi_1(E_2, y) @>p_{2*}>> \pi_1(B_2, p_2(y)) @>>> 1 \end{CD}$$

in which the rows are exact where $K_1 := \text{Ker } i_{1*}$ is the kernel of the inclusion induced homomorphism $i_{1*} : \pi_1(F_{1b}) \rightarrow \pi_1(E_1)$ and $K_2 :=$

$\text{Ker } i_{2*}$ is the kernel of the inclusion induced homomorphism $i_{2*} : \pi_1(F_{2b}) \rightarrow \pi_1(E_2)$; j_{1*}, j_{2*} are induced homomorphisms by i_{1*}, i_{2*} respectively, and $\bar{f}_{b*}^\omega, \bar{g}_{b*}^\mu$ are induced homomorphisms by $f_{b*}^\omega, g_{b*}^\mu$ respectively. By Theorem 2.1 we have an exact sequence

$$1 \rightarrow \text{Coin}(\bar{f}_{b*}^\omega, \bar{g}_{b*}^\mu) \xrightarrow{j_{1*}} \text{Coin}(f_*^\omega, g_*^\mu) \xrightarrow{p_{1*}} \text{Coin}(\bar{f}_*^{p_2(\omega)}, \bar{g}_*^{p_2(\mu)}) \\ \xrightarrow{\delta} \mathcal{R}(f_{b*}^\omega, g_{b*}^\mu; K_2) \xrightarrow{j_{2*}} \mathcal{R}(f_*^\omega, g_*^\mu) \xrightarrow{p_{2*}} \mathcal{R}(\bar{f}_*^{p_2(\omega)}, \bar{g}_*^{p_2(\mu)}) \rightarrow 1.$$

Let $\mathcal{ER}(f_*^\omega, g_*^\mu) = \{[\tilde{\alpha}] \in \mathcal{R}(f_*^\omega, g_*^\mu) : \text{Ind}([\tilde{\alpha}]) \neq 0\}$ is the set of essential Reidemeister classes of f and g , that is

$$N(f, g) = \#\mathcal{ER}(f_*^\omega, g_*^\mu).$$

By Theorem 2.1 and Proposition 3.1, we have the following result (see [4] for the Nielsen fixed point theory).

THEOREM 4.1. *In the above situation, for any x in an essential Nielsen class of f and g , the sequence*

$$1 \rightarrow \text{Coin}(\bar{f}_{b*}^\omega, \bar{g}_{b*}^\omega) \xrightarrow{j_{1*}} \text{Coin}(f_*^\omega, g_*^\omega) \xrightarrow{p_{1*}} \text{Coin}(\bar{f}_*^{p_2(\omega)}, \bar{g}_*^{p_2(\omega)}) \\ \xrightarrow{\delta} \mathcal{ER}(f_{b*}^\omega, g_{b*}^\omega; K_2) \xrightarrow{j_{2*}} \mathcal{ER}(f_*^\omega, g_*^\omega) \xrightarrow{p_{2*}} \mathcal{ER}(\bar{f}_*^{p_2(\omega)}, \bar{g}_*^{p_2(\omega)}) \rightarrow 1$$

is an exact sequence of groups and based sets.

PROOF. Since x is in an essential Nielsen class of f and g , we assume that $\omega = \mu$. The following diagram

$$\begin{array}{ccc} \Phi'(f, g) & \xrightarrow{\rho} & \mathcal{R}(f_*^\omega, g_*^\omega) \\ p_1 \downarrow & & \downarrow p_{2*} \\ \Phi'(\bar{f}, \bar{g}) & \xrightarrow{\rho} & \mathcal{R}(\bar{f}_*^{p_2(\omega)}, \bar{g}_*^{p_2(\omega)}) \end{array}$$

is commutative by Proposition 3.1. Consider a function $j_1 : \Phi'(f_b, g_b; K_2) \rightarrow \Phi'(f, g)$ defined by $j_1([K_2x]) = [x]$. If $[K_2x] = [K_2x']$, then there exists a path c in F_{1b} such that $f(c) \stackrel{K_2}{\simeq} g(c)$. Thus $f(i_1(c)) \simeq g(i_1(c))$ for a path $i_1(c)$ in E_1 ; so $[x] = [x']$. This shows that j_1 is well-defined. Also the following diagram

$$\begin{array}{ccc} \Phi'(f_b, g_b; K_2) & \xrightarrow{K_2\rho_b} & \mathcal{R}(f_{b*}^\omega, g_{b*}^\omega; K_2) \\ j_1 \downarrow & & \downarrow j_{2*} \\ \Phi'(f, g) & \xrightarrow{\rho} & \mathcal{R}(f_*^\omega, g_*^\omega) \end{array}$$

is commutative. This gives the desired result. □

It is clear from Theorem 4.1 that

$$\mathcal{ER}(f_*^\omega, g_*^\omega) = \bigsqcup_{[\alpha] \in \mathcal{ER}(\bar{f}_*^{p_2(\omega)}, \bar{g}_*^{p_2(\omega)})} p_{2*}^{-1}([\alpha]).$$

For every $[\alpha] \in \mathcal{ER}(\bar{f}_*^{p_2(\omega)}, \bar{g}_*^{p_2(\omega)})$ there is an exact sequence

$$\pi_1(F_{2b_\alpha}, y) \xrightarrow{i_{2*}} \pi_1(E_2, y) \xrightarrow{p_{2*}} \pi_1(B_2, b_\alpha) \rightarrow 1,$$

where $b_\alpha \in \bar{\rho}^{-1}([\alpha])$ and $p_2(y) = b_\alpha$. Let $k : \pi_1(F_{2b_\alpha}, y) \rightarrow \pi_1(F_{2b_\alpha}, y) / K_2$ be the quotient homomorphism. Now we apply Theorem 2.6 to these fibre maps. Throughout this section we assume that $g_{b_\alpha} : F_{1b_\alpha} \rightarrow F_{2b_\alpha}$ is surjective for every α .

THEOREM 4.2. *In the above situation, for all $[\alpha] \in \mathcal{ER}(\bar{f}_*^{p_2(\omega)}, \bar{g}_*^{p_2(\omega)})$ let $\tilde{\alpha} \in \pi_1(E_2, y)$ be such that $[p_2(\tilde{\alpha}^{-1})] = [\alpha]$. Suppose there exists a subgroup $K'_{\tilde{\alpha}}$ of $\pi_1(E_1, x)$ such that*

$$p_{1*}^{-1} \text{Coin}(\bar{f}^{p_2(\tilde{\alpha})}, \bar{g}^{p_2(\omega)}) = K'_{\tilde{\alpha}} \text{Ker } p_{1*}.$$

Then

$$N(f, g) \geq \sum_{[\alpha] \in \mathcal{ER}(\bar{f}_*^{p_2(\omega)}, \bar{g}_*^{p_2(\omega)})} N(f_{b_\alpha}, g_{b_\alpha}; H_{\tilde{\alpha}}),$$

where $H_{\tilde{\alpha}} = k^{-1}(T_{(f_*^\alpha, g_*^\alpha)}(K'_{\tilde{\alpha}}))$. If

$$\mathcal{R}(f_*^\omega, g_*^\omega) = \mathcal{R}(f_*^\omega, g_*^\omega; g_*^\omega([K'_{\tilde{\alpha}}, \text{Ker } p_{1*}]^{\pi_1(E_1)}))$$

for every $[\alpha]$, then

$$N(f, g) = \sum_{[\alpha] \in \mathcal{ER}(\bar{f}_*^{p_2(\omega)}, \bar{g}_*^{p_2(\omega)})} N(f_{b_\alpha}, g_{b_\alpha}; H_{\tilde{\alpha}}).$$

PROOF. The inequality suffices to show that $\sharp p_{2*}^{-1}([\alpha]) \geq N(f_{b_\alpha}, g_{b_\alpha}; H_{\tilde{\alpha}})$. Let

$$1 \rightarrow \pi_1(F_{2b_\alpha}, y) / \text{Ker } i_{2*} \xrightarrow{j_{2*}} \pi_1(E_2, y) \xrightarrow{p_{2*}} \pi_1(B_2, b_\alpha) \rightarrow 1$$

be the exact sequence where $b_\alpha \in \bar{p}^{-1}([\alpha])$ and $p_2(y) = b_\alpha$. Since $p_2 : E_2 \rightarrow B_2$ is a fibration, $\pi_1(F_{2b_\alpha}, y)/\text{Ker } i_{2*} \cong \text{Ker } p_{2*} = i_{2*}(\pi_1(F_{2b_\alpha}))$. Thus we have the commutative diagram

$$\begin{array}{ccc} \pi_1(F_{2b_\alpha}) & \xrightarrow{k} & i_{2*}(\pi_1(F_{2b_\alpha})) \subset \pi_1(E_2) \\ \downarrow & & \downarrow \\ \pi_1(F_{2b_\alpha})/H_{\bar{\alpha}} & \xrightarrow{k} & i_{2*}(\pi_1(F_{2b_\alpha}))/T_\alpha, \end{array}$$

where $T_\alpha = T_{(f_*^\alpha, g_*^\omega)}(K'_{\bar{\alpha}})$. Now using Lemma 2.5, we obtain a surjection

$$A_\alpha : i_{2*}(\mathcal{R}(f_{b_\alpha}^\omega, g_{b_\alpha}^\omega)) \rightarrow \mathcal{R}(f_{b_\alpha}^\omega, g_{b_\alpha}^\omega; H_{\bar{\alpha}}).$$

Thus $\#p_{2*}^{-1}([\alpha]) \geq N(f_{b_\alpha}, g_{b_\alpha}; H_{\bar{\alpha}})$.

Also if $\mathcal{R}(f_*^\omega, g_*^\omega) = \mathcal{R}(f_*^\omega, g_*^\omega; g_*^\omega([K'_{\bar{\alpha}}, \text{Ker } p_{1*}]^{\pi_1(E_1)}))$ for every $[\alpha]$, then A_α is a bijection. Hence $\#p_{2*}^{-1}([\alpha]) = N(f_{b_\alpha}, g_{b_\alpha}; H_{\bar{\alpha}})$. \square

COROLLARY 4.3. *If $\text{Coin}(\bar{f}_*^{p_2(\bar{\alpha})}, \bar{g}_*^{p_2(\omega)}) = \{1\}$ for every $[\alpha] \in \mathcal{ER}(\bar{f}_*^{p_2(\omega)}, \bar{g}_*^{p_2(\omega)})$, then*

$$N(f, g) = \sum_{[\alpha] \in \mathcal{ER}(\bar{f}_*^{p_2(\omega)}, \bar{g}_*^{p_2(\omega)})} N(f_{b_\alpha}, g_{b_\alpha}; K_2).$$

PROOF. Apply Corollary 2.7. \square

COROLLARY 4.4. *If there exists $b \in \Phi(\bar{f}, \bar{g})$ such that*

$$\mathcal{R}(f_*^\omega, g_*^\omega; g_*([\pi_1(E_1), i_*(\pi_1(F_{1b}))]])) = \mathcal{R}(f_*^\omega, g_*^\omega)$$

then

$$N(f, g) = \sum_{[\alpha] \in \mathcal{ER}(\bar{f}_*^{p_2(\omega)}, \bar{g}_*^{p_2(\omega)})} N(f_{b_\alpha}, g_{b_\alpha}; H'_{\bar{\alpha}}),$$

where $H'_{\bar{\alpha}} = k^{-1}(T_{(f_*^\alpha, g_*^\omega)}(p_{1*}^{-1} \text{Coin}(\bar{f}_*^{p_2(\bar{\alpha})}, \bar{g}_*^{p_2(\omega)})))$; we can distinguish the Reidemeister classes by just computing the quotient groups

$$\text{Ker } p_{2*}/T_{(f_*^\alpha, g_*^\omega)}(p_{1*}^{-1} \text{Coin}(\bar{f}_*^{p_2(\bar{\alpha})}, \bar{g}_*^{p_2(\omega)})).$$

PROOF. Note that if the hypothesis holds true for a $b \in \Phi(\bar{f}, \bar{g})$ then it is satisfied for every $b \in \Phi(\bar{f}, \bar{g})$. Then apply Corollary 2.9. \square

References

- [1] R. F. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Glenview, Ill, 1971.
- [2] D. Ferrario, *Computing Reidemeister classes*, *Fund. Math.* **158** (1998), 1–18.
- [3] P. R. Heath, *Product formula for Nielsen numbers of fibre maps*, *Pacific J. Math.* **117** (1985), no. 2, 267–289.
- [4] P. R. Heath, E. Keppelmann and P. N. S. Wong, *Addition formulae for Nielsen numbers and Nielsen-type numbers of fibre preserving maps*, *Topology Appl.* **67** (1995), 133–157.
- [5] J. Guo and P. R. Heath, *Coincidence theory on the complement*, *Topology Appl.* **95** (1999), 229–250.
- [6] S. T. Hu, *Homotopy Theory*, Academic Press Inc., 1959.
- [7] J. Jezierski, *The Nielsen number product formula for coincidences*, *Fund. Math.* **134** (1989), 183–212.
- [8] B. J. Jiang, *Lectures on Nielsen Fixed Point Theory*, *Contemp. Math.* 14, Amer. Math. Soc., Providence, 1959.
- [9] T. H. Kiang, *The Theory of Fixed Point Classes*, Scientific Press. Peking, 1979.
- [10] S. H. Lee, *A study on Reidemeister operation*, *Bull. Korean Math. Soc.* **36** (1999), no. 4, 661–669.
- [11] J. Vick, *Homology Theory*, Academic Press, Inc., New York, 1973.
- [12] P. Wong, *Fixed-point theory for homogeneous spaces*, *Amer. J. Math.* **120** (1998), 23–42.
- [13] C. Y. You, *Fixed point classes of a fiber map*, *Pacific J. Math.* **100** (1982), 217–241.

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