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TOPOLOGICAL R^2-DIVISIBLE R^3-SPACES

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ABSTRACT. There are many models to study topological R^2-planes. Unlike topological R^2-planes, it is difficult to find models to study topological R^3-spaces. If an 4-dimensional affine plane intersects with R^3, we are able to get a geometrical structure on R^3 which is similar to R^3-space, and called R^2-divisible R^3-space. Such spatial geometric models is useful to study topological R^3-spaces. Hence, we introduce some classes of topological R^2-divisible R^3-spaces which are induced from 4-dimensional affine planes.

1. Introduction

In this paper we introduce a new class of topological space geometries, so-called *topological R^2-divisible R^3-spaces.* In particular we give lots of models of this space geometry. A topological projective plane $\mathcal{P}$ is a projective plane with point set P and line set $\mathcal{L}$, where both P and $\mathcal{L}$ carry topologies such that the operations of joining and intersecting are continuous in their domains of definition. A topological projective plane is called *n-dimensional* if P and $\mathcal{L}$ are n-dimensional, locally compact, connected topological spaces. As in the case of projective planes, we will call a locally compact, connected affine plane n-dimensional if its point set and line set are n-dimensional, locally compact, connected topological spaces. The lines in 2-(4-)dimensional affine planes are homeomorphic to R (R^2). For general information about topological planes the reader is referred to [16]. Since the fundamental papers of Salzmann [14, 15], Betten has tried to classify all 4-dimensional compact flexible projective planes. A topological projective plane is called *flexible* if the collineation group has an open orbit in the space of flags (flag=incident point-line pair). In a series of papers of Betten and Knarr many different types of 4-dimensional projective planes were found. These planes

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can be represented by 4-dimensional affine planes, and R^2-divisible R^3-spaces are derived from 4-dimensional affine planes. We can regard an R^2-divisible R^3-space as an intersection of a 4-dimensional affine plane and R^3. In order to explain of this geometrical structure, we consider the classical 4-dimensional affine plane $\mathcal{A}_2C$ over the complex field C and the induced R^2-divisible R^3-spaces. The affine plane $\mathcal{A}_2C$ consists of point set $C \times C$ and the following subsets of $C \times C$ are called lines: $L(s,t) = \{(x, sx+t)|x \in C\}$ for $s,t \in C$, $\{c\} \times C$ for $c \in C$. If we identify C^2 with $R^4 = \{(x,y,u,v)|x,y,u,v \in R\}$, then we can identify the lines with the following forms: $L(a,b,\xi,\eta) = \{(x,y,ax - by + \xi, ay + bx + \eta)|x,y \in R\}$ for $(a,b,\xi,\eta) \in R^4$, $\{(x,y)\} \times R^2$ for $(x,y) \in R^2$. Let $R^3_{x=0} := \{(0,y,u,v)|y,u,v \in R\}$ and let $l(a,b,\xi,\eta) := L(a,b,\xi,\eta) \cap R^3_{x=0} = \{(0,y,-by+\xi, ay+\eta)|y \in R\}$. $\mathcal{L}$ denote the set of all lines $l(a,b,\xi,\eta)$ with $(a,b,\xi,\eta) \in R^4$. If we identify $R^3_{x=0}$ with $R^3 = \{(y,u,v)|y,u,v \in R\}$, then we get a geometrical structure $(R^3, \mathcal{L}, \Lambda)$ on R^3, that is, for two points $(y_1,u_1,v_1),(y_2,u_2,v_2) \in R^3$ with $y_1 \neq y_2$ there exists a unique joining line $l(a,b,\xi,\eta)$, and $\Lambda = \{\{y\} \times R^2|y \in R\}$ is a partition of R^3. In the same way we get also a geometrical structure on $R^3_{y=0} := \{(x,0,u,v)|x,u,v \in R\}$. Hence we have an abstraction, so-called *R^2-divisible R^3-spaces.* The two geometrical structures are equal to the classical R^3-space without lines on the vertical planes $\{x\} \times R^2$ with $x \in R$. We call the classical R^3-space without lines on the vertical planes $\{x\} \times R^2$ with $x \in R$ *the real affine R^2-divisible R^3-space.* In the real affine R^2-divisible R^3-space on $R^3 = \{(x,y,z)|x,y,z \in R\}$, we can consider two projections on $< x,y >$-coordinate plane and $< x,z >$-coordinate plane, respectively. We get also two affine planes on $< x,y >$-coordinate plane and $< x,z >$-coordinate plane, respectively, where the line set is the set of all projections of lines in R^3 on $< x,y >$-coordinate plane and $< x,z >$-coordinate plane, respectively. In a series of papers of Betten and Knarr we have lots of examples of R^2-divisible R^3-spaces which are induced from 4-dimensional affine planes.

After inspection of all flexible 4-dimensional translation planes we see: the induced R^2-divisible R^3-spaces by translation planes are the real affine R^2-divisible R^3-spaces. The affine planes in [2] give rise to R^2-divisible R^3-spaces which are non-classical, that is, if we consider two projections on $< x,y >$-coordinate plane and $< x,z >$-coordinate plane, respectively, one of the projection is the real affine plane and the other is a Moulton plane. In [8] Knarr studied 4-dimensional shift planes. The shift planes give also rise to non-classical R^2-divisible R^3-spaces. In this case one of the projection is the classical affine plane and the other is a 2-dimensional shift plane. Conversely, with these observations we can reconstruct R^2-divisible R^3-spaces, so-called product spaces (see 2.1). In this viewpoint one of the induced R^2-divisible R^3-spaces in [2] are the product spaces of the real affine plane and a Moulton plane. The R^2-divisible R^3-spaces which are induced from 4-dimensional shift planes are the product spaces of the real affine plane and a 2-dimensional shift plane.

The main purpose of this paper is to give many examples of this geometry. It may give a motivation for studying topological R^2-divisible R^3-spaces continuously. First we introduce the class of product spaces and investigate some related topics. Using Theorems 2.5 and 2.6, we determine the collineation group of an (α, d)-space which is induced from a 4-dimensional shift plane. In section 3 we introduce different types of R^2-divisible R^3-spaces which seem to be a generalization of product spaces. These types of R^2-divisible R^3-spaces arise from [6, 8]. In section 4 we give more examples of topological R^2-divisible R^3-spaces. We are interested in topological R^2-divisible R^3-spaces, because the induced R^2-divisible R^3-spaces from 4-dimensional affine planes have already topological structure. We start with some basic definitions.

Let X be a topological space and $(A_n)_{n\in N}$ be a sequence of subsets of X. Denote by $\liminf A_n$ the set of all limit points of sequences $(a_n)_{n\in N}$ with $a_n \in A_n$, and denote by $\limsup A_n$ the set of all accumulation points of such sequences. The sequence $(A_n)_{n\in N}$ is *Hausdorff-convergent* to $A \subseteq X$ if and only if $\liminf A_n = \limsup A_n = A$ (written by $\lim A_n = A$ or $A_n \longrightarrow A$).

Hausdorff metric: Let $\mathcal{P}^n$ denote a topological space homeomorphic to R^n. Let $\mathcal{U}$ be the set of all non-empty closed subsets of $\mathcal{P}^3$. We define on $\mathcal{U}$ the following metric:

$$\delta : \mathcal{U} \times \mathcal{U} \longrightarrow R : (A, B) \longrightarrow \sup\{|d(x, A) - d(x, B)|e^{-d(p,x)} | x \in \mathcal{P}^3\},$$

where d is the metric on $\mathcal{P}^3$ and $p \in \mathcal{P}^3$. Then δ is a metric on $\mathcal{U}$. Let $(A_n)_{n\in N}$ be a sequence in $\mathcal{U}$ and $A \in \mathcal{U}$. Then $(A_n)_{n\in N}$ converges to A in $(\mathcal{U}, \delta)$ if and only if $\lim A_n \longrightarrow A$ (see [7, Chap. 1.3]).

Let $\mathcal{P}^n$ denote a topological space which is homeomorphic to R^n. A partition $\Lambda := \{S_i | i \in \mathcal{A}\}$ in $\mathcal{P}^n$ ($n \geq 2$) is *divisible* if each S_i is closed in $\mathcal{P}^n$ and homeomorphic to $\mathcal{P}^{n-1}$.

Definition 1.1. Let $\mathcal{L}$ be a system of subsets of $\mathcal{P}^3$, and let $\Lambda = \{S_i | i \in \mathcal{A}\}$ be a divisible partition in $\mathcal{P}^3$. The elements of $\mathcal{P}^3$ are called points, and the elements of $\mathcal{L}$ are called lines. We say that $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ is a topological R^2-divisible R^3-space if the following axioms hold:

(1) Each line $l \in \mathcal{L}$ is closed in the topological space $\mathcal{P}^3$ and homeomorphic to R.
(2) For all $(x, y) \in S_i \times S_j$ with $i \neq j$ there is a unique line $l \in \mathcal{L}$ with $x, y \in l$. For $i = j$ there are no lines $l \in \mathcal{L}$ with $x, y \in l$.
(3) The mapping

$$\vee : \mathcal{P}^3 \times \mathcal{P}^3 \setminus \cup_{i \in \mathcal{A}}(S_i \times S_i) \longrightarrow \mathcal{L}$$

is continuous, where $\mathcal{L}$ has the induced topology of Hausdorff-convergence.

The joining line in (2) is denoted by $l = x \vee y$. Let H denote the Hausdorff-convergence topology on $\mathcal{L}$. Without (3) $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ is called an R^2-divisible R^3-space. If $\Lambda = \{S_i | i \in \mathcal{A}\}$ is a divisible partition in $\mathcal{P}^2$, then we can similarly define an R-divisible R^2-plane $(\mathcal{P}^2, \mathcal{L}, \Lambda)$. If we think the partition as the added line set, we can regard an R-divisible R^2-plane as an R^2-plane.

DEFINITION 1.2. Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be an R^2-divisible R^3-space. A subset $E \subseteq \mathcal{P}^3$ is called a plane of $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ if the following conditions hold:
(1) E is closed in $\mathcal{P}^3$ and homeomorphic to R^2,
(2) $(E, \mathcal{L}_E, \Lambda_E)$ is an R-divisible R^2-plane, where $\mathcal{L}_E := \{l \in \mathcal{L} | l \subseteq E\}$ and $\Lambda_E = \{E \cap S_i | i \in \mathcal{A}\}$ is a divisible partition in E.

An R-divisible R^2-plane in $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ is obviously an R^2-plane. Let $\mathcal{E}$ denote the set of all planes of $(\mathcal{P}^3, \mathcal{L}, \Lambda)$. Since the plane set $\mathcal{E}$ is also a subset of $\mathcal{U}$ (see Hausdorff metric), we can take on $\mathcal{E}$ the induced topology of $\mathcal{U}$.

Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be an R^2-divisible R^3-space. Since lines are homeomorphic to R, there is a natural notion of intervals in lines. If $l \in \mathcal{L}$ is a line and $p, q \in l$ are two (not necessarily distinct) points on l, then we denote the *interval* which consists of all points on l between p and q by the symbol $[p, q]$. The *open interval* between p and q is defined as $(p, q) := [p, q] \setminus \{p, q\}$.

DEFINITION 1.3. Let $(R^2, \mathcal{L})$ be an R^2-plane. A subset $P \subseteq R^2$ is called a subgeometry of $(R^2, \mathcal{L})$ if $x, y \in P$ with $x \vee y \in \mathcal{L}$, then $x \vee y \subseteq P$.

LEMMA 1.4. *Let $E = (R^2, \mathcal{L})$ be an R^2-plane. Let P be a subgeometry of E which contains a non-empty open set of E, then $E = P$.*

PROOF. Let U be an open set in R^2 and $U \subseteq P$. Assume that there exists a point $q \in R^2 \setminus P$. Let $p \in U$. Since U is open in R^2, it is clear that $|U \cap (p \vee q)| \geq 2$. Therefore, $p \vee q \subseteq P$, so that $q \in P$, a contradiction. □

If $S_i \in \Lambda$, then $\mathcal{P}^3 \backslash S_i$ has precisely two components (denoted by S_i^+, S_i^-), of which S_i is the common (topological) boundary. If we choose more $S_j \in \Lambda$ with $i \neq j$, then one of the components of $\mathcal{P}^3 \setminus S_i$ (for example S_i^+) is also separated by S_j. We can choose one of the components of $S_i^+ \backslash S_j$ which contains S_i and S_j as the topological boundaries. Let $\overline{S_{ij}^{+-}}$ denote the union of the components which has S_i and S_j as topological boundaries, S_i and S_j. We identify $\overline{S_{ii}^{+-}}$ with simply S_i

DEFINITIONS 1.5. (1) The *final topology* F on $\mathcal{L}$ is the largest topology on $\mathcal{L}$ for which the mapping $\vee : \mathcal{P}^3 \times \mathcal{P}^3 \setminus \cup_{i \in \mathcal{A}}(S_i \times S_i) \longrightarrow \mathcal{L}$ is continuous.
(2) The *open join topology* OJ is generated by the subbasic elements $O_1 \vee O_2 = \{p \vee q \in \mathcal{L} | p \in O_1, q \in O_2\}$, where O_1, O_2 are disjoint open sets in $\mathcal{P}^3$.
(3) The *open meet topology* OM is defined by the subbasic sets $M_O = \{l \in \mathcal{L} | l \cap O \neq \emptyset\}$, where O is an open set in $\mathcal{P}^3$.
(4) The *open partition meet topology* OPM on $\mathcal{L}$ is generated by the subbasis elements $S_i^O = \{l \in \mathcal{L} | l \cap O \neq \emptyset\}$, where O is an open set in S_i and $S_i \in \Lambda$.
(5) The *compact open topology* COT on $\mathcal{L}$ is defined by the subbasis elements $S_{ij}^O = \{l \in \mathcal{L} | l \cap S_i = \{x\}, l \cap S_j = \{y\}, [x, y] \subseteq \overline{S_{ij}^{+-}} \cap O\}$, where S_i, $S_j \in \Lambda$, $\overline{S_{ij}^{+-}}$ is the union of the component which has S_i and S_j as topological boundaries, S_i and S_j, and O is open in $\mathcal{P}^3$.

LEMMA 1.6. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. Then:*

1. *The topologies H, F, OJ, OM for $\mathcal{L}$ coincide.*
2. *The join map $\vee : \mathcal{P}^3 \times \mathcal{P}^3 \setminus \cup_{i \in \mathcal{A}}(S_i \times S_i) \longrightarrow \mathcal{L}$ is open.*
3. *$H \subseteq OPM \subseteq COT$.*

PROOF. [9, 1.8, 1.10] □

DEFINITION 1.7. Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be an R^2-divisible R^3-space. Given two subsets $A, B \subseteq \mathcal{P}^3$, we define

$$[A, B] := \bigcup_{a \in A, b \in B} [a, b],$$

i.e., $[A, B]$ is the set of all points between A and B.

Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be an R^2-divisible R^3-space. Then we will consider the following additional axiom:

(B) (Bounded-axiom) If $A, B \subseteq \mathcal{P}^3$ are compact, then $\overline{[A,B]}$ is also compact.

THEOREM 1.8. *Let $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ be a topological R^2-divisible R^3-space. Then:*

(1) *(Order-condition) If the points sequences $(a_n)_{n\in N}$,$(b_n)_{n\in N}$,$(c_n)_{n\in N}$ have limits a, b, c. If $b_n \in [a_n, c_n]$ for all $n \in N$, then it is also that $b \in [a, c]$.*
(2) *If $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ satisfies the axiom (B), then $H = COT$.*
(3) *Let $H = COT$. If $A, B \subseteq \mathcal{P}^3$ are compact with $A \times S \subseteq \mathcal{P}^3 \times \mathcal{P}^3 \setminus \cup_{i\in\mathcal{A}}(S_i \times S_i)$, then $\overline{[A,B]}$ is also compact.*

PROOF. (1) [9, 2.5].
(2) Assume that $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ satisfies the axiom (B). Then we first show that $OPM \subseteq OJ = H$. Let $l \in S_i^U \in OPM$, $\{p\} = l \cap U$ and V be an open set in $\mathcal{P}^3$ such that $S_i \cap V = U$. Let $(V_n)_{n\in N}$ be decreasing sequence of neighborhoods of p such that $\{V_n | n \in N\}$ is a neighborhood basis at p. Let S_i^+ and S_i^- be the two connected components of $\mathcal{P}^3 \setminus S_i$, and let $V_n \cap (\mathcal{P}^3 \setminus S_i) = V_n^+ \cup V_n^-$ such that $V_n^+ \subseteq S_i^+$ and $V_n^- \subseteq S_i^-$. We will show that there exists a number $n \in N$ such that $l \in V_n^+ \vee V_n^- \subseteq S_i^U$. Suppose that it is not true; for each $n \in N$ we can choose $p_n \in V_n^+, q_n \in V_n^-$ such that $(p_n \vee q_n) \cap U = \emptyset$. Since (p_n) and (q_n) converge to p, and by the axiom (B) and order-condition, $(p_n \vee q_n) \cap S_i$ converges to p. Hence for sufficiently large $n \in N$ $(p_n \vee q_n) \cap S_i \subseteq V \cap S_i = U$, a contradiction.
We show that $COT \subseteq OPM$. Let $l \in S_{ij}^O \in COT$, and let $\{x\} = S_i \cap l, \{y\} = S_j \cap l$. If $i = j$, then it is clear that $S_{ij}^O \in OPM$. Hence let $i \neq j$. Let $(V_n(x))_{n\in N}$ and $(W_n(y))_{n\in N}$ be two decreasing sequences of neighborhoods of x and y such that $(V_n(x))_{n\in N}$ and $(W_n(y))_{n\in N}$ are neighborhood basis at x and y in S_i and S_j, respectively. Then we will show that there exists a number $n \in N$ such that $l \in S_i^{V_n(x)} \cap S_j^{W_n(x)} \subseteq S_{ij}^O$. If we assume that it is not true; for each $n \in N$ there exist $x_n \in V_n(x)$ and $y_n \in W_n(y)$ such that $[x_n, y_n] \not\subseteq S_{ij}^O$. For each $n \in N$ choose a point $p_n \in [x_n, y_n]$ such that $p_n \notin S_{ij}^O$. Then by the axiom (B) and order-condition, the sequence (p_n) has an accmulation point on $[x, y]$, a contradiction.
(3) Let $H = COT$. Assume that $\overline{[A,B]}$ is not compact. Then there exists a sequence $((a_n, b_n))$ in $A \times B$ and a sequence (p_n), $p_n \in [a_n, b_n] \setminus \{a_n, b_n\}$ such that (p_n) is unbounded. Since $A \times B$ is compact, there exists a convergent subsequence $((a_{n_k}, b_{n_k}))$ which converges to a point $(a, b) \in A \times B$. Let $a \neq b$. Let $a_1, b_1 \in a \vee b$ such that $a \in (a_1, b)$ and

$b \in (a, b_1)$. Choose a relative compact open set U which contains $[a_1, b_1]$. Since $H = COT$ and $a_{n_k} \vee b_{n_k} \longrightarrow a \vee b$, there exists N such that for all $k \geq N$ $\cup([a_{n_k}, b_{n_k}]) \cup [a, b] \subseteq U$. Consequently, $\cup([a_{n_k}, b_{n_k}]) \cup [a, b]$ is bounded, a contradiction. □

DEFINITION 1.9. An isomorphism between R^2-divisible R^3-spaces $(\mathcal{P}_1^3, \mathcal{L}_1, \Lambda_1)$ and $(\mathcal{P}_2^3, \mathcal{L}_2, \Lambda_2)$ is a bijection $\gamma : \mathcal{P}_1^3 \longrightarrow \mathcal{P}_2^3$ such that $l^\gamma \in \mathcal{L}_2$ and $S_i^\gamma \in \Lambda_2$ for each $l \in \mathcal{L}_1$ and $S_i \in \Lambda_1$. A collineation γ of an R^2-divisible R^3-space $(\mathcal{P}^3, \mathcal{L}, \Lambda)$ is a bijection of $\mathcal{P}^3$ such that $l^\gamma \in \mathcal{L}$ and $S_i^\gamma \in \Lambda$ for each $l \in \mathcal{L}$ and $S_i \in \Lambda$.

LEMMA 1.10. *(Reduction) Let $(R^3, \mathcal{L}, \Lambda)$ be an R^2-divisible R^3-space. Assume that the group $(R^3, +)$ is admissible as a transitive collineation group. If there exists a point $a \in R^3$ with $a \in S_i$ such that the restriction $\vee|\{a\} \times R^3 \setminus \{a\} \times S_i$ is continuous, then $\vee$ is continuous.*

PROOF. Let $(a, b) \in S_i \times S_j, i \neq j$ and $a_n \longrightarrow a, b_n \longrightarrow b$. Then we have to show that $a \vee b \subseteq \liminf a_n \vee b_n \subseteq \limsup a_n \vee b_n \subseteq a \vee b$. Let $x \in a \vee b$. Then $a_n - (a_n - a) = a$ and $b_n - (a_n - a) \longrightarrow b$. Since $\vee|\{a\} \times R^3 \setminus \{a\} \times S_i$ is continuous, there exists a sequence $(y_n)_{n \in N}, y_n \in a \vee (b_n - (a_n - a))$ such that $y_n \longrightarrow x$. Therefore, $x_n = y_n + (a_n - a) \in (a + a_n - a) \vee (b_n - (a_n - a) + (a_n - a)) = a_n \vee b_n$, and so $x_n \longrightarrow x$. Let $x \in \limsup a_n \vee b_n$. Assume that $x \notin a \vee b$. There exists a subsequence $(x_{n_k}), x_{n_k} \in a_{n_k} \vee b_{n_k}$ such that $x_{n_k} \longrightarrow x$. Then, $x_{n_k} - (a_{n_k} - a) \in (a_{n_k} - (a_{n_k} - a)) \vee (b_{n_k} - (a_{n_k} - a)) = a \vee (b_{n_k} - (a_{n_k} - a))$, and we have $x_{n_k} - (a_{n_k} - a) \longrightarrow x$. Since $b_{n_k} - (a_{n_k} - a) \longrightarrow b$ and $\vee|\{a\} \times R^3 \setminus \{a\} \times S_i$ is continuous, $x \in a \vee b$, a contradiction. □

DEFINITION 1.11. Let $\mathcal{B}$ be a system of subsets of $\mathcal{P}^3$. The element of $\mathcal{B}$ are called surfaces. The incidence structure $(\mathcal{P}^3, \mathcal{B})$ is called a (3,2,2)-geometry if the following axioms hold:

(1) Each surface $B \in \mathcal{B}$ is closed in the topological space $\mathcal{P}^3$ and homeomorphic to R^2.
(2) Each pair p, q of distinct points is contained in a unique surface B. Let $p \vee_B q$ denote the surface which contains two points p, q.

DEFINITION 1.12. A (3,2,2)-geometry $(\mathcal{P}^3, \mathcal{B})$ is topological if the join map $\vee_B : \mathcal{P}^3 \times \mathcal{P}^3 \setminus \triangle \longrightarrow \mathcal{B}$ is continuous, where $\triangle = \{(p, p) | p \in \mathcal{P}^3\}$ denotes the diagonal and $\mathcal{B}$ carries the topology of Hausdorff-convergence. The notation of (3,2,2)-geometries first appeared in [1].

Planar functions and shift planes

DEFINITION 1.13. Let G, H be two (additively written) abelian groups. A function $f : G \longrightarrow H$ is called *planar* if it has the following property: For all $d \in G \setminus \{0\}$ the mapping $f_d : G \longrightarrow H : x \longrightarrow f(x+d) - f(x)$ is bijective.

Let $l(c) = \{(c, y) | y \in H\}$ for $c \in G$ and $l(a, b) = \{(x, f(x-a)+b) | x \in G\}$ for $(a, b) \in G \times H$. Then $I(G, H; f)$ is the incidence structure with point set $\mathcal{P} = G \times H$ and line set $\mathcal{L} = \{l(c) | c \in G\} \cup \{l(a, b) | (a, b) \in G \times H\}$. This incidence structure turns out to be a special kind of affine plane, usually referred to as the shift plane generated by f. It is well known that if $f : R \longrightarrow R$ ($F : R^2 \longrightarrow R^2$) is a continuous planar function, then $I(R, R; f)$ ($I(R^2, R^2; F)$) is a 2-dimensional (4-dimensional) affine plane.

Examples of planar functions $R^2 \longrightarrow R^2$ and the induced R^2-divisible R^3-spaces.

(1) The differentiable planar function $C \longrightarrow C : z \longrightarrow z^2$ interpreted as a map from R^2 to R^2 is

$$R^2 \longrightarrow R^2 : (x, y) \longrightarrow (x^2 - y^2, 2xy).$$

If we set $x = 0$ ($y = 0$), then we get an R^2-divisible R^3-space which is interpreted as the product space of the usual shift plane and the real affine plane.

(2) Given planar functions f and g on R which are both convex. Polster [11] constructs a planar function $f * g$ on R^2, called the product of f and g, as follows:

$$(f * g) : R^2 \longrightarrow R^2; (x, y) \longrightarrow (f(x) - g(y), xy).$$

In this case, if we set $x = 0$ ($y = 0$), then we get an R^2-divisible R^3-space which is interpreted as the product space of a 2-dimensional shift plane and the real affine plane

(3) Two further differentiable planar functions in [8] are

$$R^2 \longrightarrow R^2 : (x, y) \longrightarrow (xy - \frac{1}{3}x^3, \frac{1}{2}y^2 - \frac{1}{12}x^4)$$

and

$$R^2 \longrightarrow R^2 : (x, y) \longrightarrow (xy - \frac{1}{3}x^3 - x, \frac{1}{2}(y^2 - x^2) - \frac{1}{12}x^4).$$

In two cases, if we set $x = 0$ ($y = 0$), then we get an R^2-divisible R^3-space which is interpreted as an R^2-divisible R^3-space induced by a planar function (see section 3).

The 4-dimensional shift planes have played a significant role in the classification of all flexible 4-dimensional compact projective planes.

2. Product spaces of two standard R^2-planes

An R^2-plan $(R^2, \mathcal{L})$ is called *standard* if all vertical lines $\{x\} \times R$ are in $\mathcal{L}$ and the other lines $l \in \mathcal{L}$ can be written as the graph(f) of a continuous mapping $f : R \longrightarrow R$. Let $E_1 = (R^2, \mathcal{L})$ and $E_2 = (R^2, \Im)$ be two standard R^2-planes. We identify E_1 with the horizontal plane $z = 0$ and E_2 with the vertical plane $y = 0$ in $R^3 = \{(x, y, z)|x, y, z \in R\}$, respectively. We define on R^3 the following curves as lines: $f \times g := \{(x, f(x), g(x))|x \in R\}$, where f and g are non-vertical lines of E_1 and E_2, respectively. Then we can construct on R^3 a topological R^2-divisible R^3-space.

DEFINITION 2.1. Let $E_1 = (R^2, \mathcal{L})$ and $E_2 = (R^2, \Im)$ be standard R^2-planes. Let $\mathcal{L} \times \Im = \{f \times g | f \in \mathcal{L}, g \in \Im\}$ and let $\Lambda = \{\{x\} \times R^2 | x \in R\}$. The incidence structure $(R^3, \mathcal{L} \times \Im, \Lambda)$ is called the product space of two standard R^2-planes E_1 and E_2 and written by $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$. In a product space $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$ there exist always the planes on the lines of E_1 and the planes on the lines of E_2. A plane on a line of E_1 is called a *vertical plane*, and a plane on a line of E_2 is called a *horizontal plane*. We note that a vertical plane is the set $\{(x, f(x), z)|x, z \in R\}$ with $f \in \mathcal{L}$ and a horizontal plane is the set $\{(x, y, g(x))|x, y \in R\}$ with $g \in \Im$.

THEOREM 2.2. *Let $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$ be the product space of two standard R^2-planes E_1 and E_2. Then $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$ is a topological R^2-divisible R^3-space.*

PROOF. It is clear that each line $f \times g \in \mathcal{L} \times \Im$ is homeomorphic to R and closed in R^3. We first show that $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$ is an R^2-divisible R^3-space. Let $x = (x_1, y_1, z_1)$ and $y = (x_2, y_2, z_2)$ with $x_1 \neq x_2$. Since for the pair of points (x_1, y_1) and (x_2, y_2) in E_1, there exists a unique line f in $\mathcal{L}$, and since for the pair of points (x_1, z_1) and (x_2, z_2) in E_2, there exists a unique line g in $\Im$, hence $f \times g$ is the unique join line of two points (x_1, y_1, z_1) and (x_2, y_2, z_2). We next show that $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$ is topological. Let $(a_n)_{n \in N}$ and $(b_n)_{n \in N}$ be two sequences with limits $a = (x_1, y_1, z_1)$ and $b = (x_2, y_2, z_2)$, $x_1 \neq x_2$, respectively. Then we have to show that $a \vee b \subseteq \liminf a_n \vee b_n \subseteq \limsup a_n \vee b_n \subseteq a \vee b$. Let $c \in a \vee b = f \times g = \{(x, f(x), g(x))|x \in R\}$, i.e., $c = (x_0, f(x_0), g(x_0))$. Let $a_n \vee b_n = f_n \times g_n = \{(x, f_n(x), g_n(x))|x \in R\}$. Since E_1 is topological, the sequence $((x_0, f_n(x_0))_{n\in N}$ converges to $(x_0, f(x_0))$, and since E_2 is also topological, the sequence $((x_0, g_n(x_0))_{n\in N}$ converges to $(x_0, g(x_0))$. It implies that $c \in \liminf a_n \vee b_n$. Let $c = (x_0, y_0, z_0) \in \limsup a_n \vee b_n$. Since E_1 and E_2 are topological, it follows that $(x_0, y_0) \in f$ and $(x_0, z_0) \in g$, therefore $c \in f \times g$. □

We note that the topologies H, OPM and COT on $\mathcal{L} \times \Im$ coincide, because of existence of vertical and horizontal planes, a product space satisfies the bounded axiom. Let $\mathcal{V}$ be the space of all vertical planes and let $\mathcal{H}$ be the space of all horizontal planes. Furthermore, let $\overline{\mathcal{V}} = \mathcal{V} \cup \Lambda$ and $\overline{\mathcal{H}} = \mathcal{H} \cup \Lambda$. We define an incidence relation $(R^3, \overline{\mathcal{V}})$ (resp. $(R^3, \overline{\mathcal{B}})$). Let $p = (x_1, y_1, z_1)$ and $q = (x_2, y_2, z_2)$ be two distinct points. If $x_1 = x_2$, then we set $p \vee_B q = \{x_1\} \times R^2$. If $x_1 \neq x_2$, then there exists a unique line $p \vee q = f \times g$, hence we can determine a unique vertical plane V (resp. a unique horizontal plane H) such that $p, q \in f \times g \subseteq V$ (resp. $\subseteq H$). Therefore, we set $p \vee_B q = V$ (resp. =H).

LEMMA 2.3. *The defined* $(3,2,2)$*-geometry* $(R^3, \overline{\mathcal{V}})$ $((R^3, \overline{\mathcal{H}}))$ *is topological.*

PROOF. Since R^2-planes are topological, it is easy to check that the defined (3,2,2)-geometries are topological. □

Let Λ be a divisible partition in $\mathcal{P}^3$ and let $\mathcal{A}$ and $\mathcal{B}$ be two systems of subsets in $\mathcal{P}^3$. Furthermore, let $\overline{\mathcal{A}} = \mathcal{A} \cup \Lambda$ and $\overline{\mathcal{B}} = \mathcal{B} \cup \Lambda$. Let $(\mathcal{P}^3, \overline{\mathcal{A}})$ and $(\mathcal{P}^3, \overline{\mathcal{B}})$ be two (3,2,2)-geometries such that if for each $(A, B) \in \mathcal{A} \times \mathcal{B}$ with $A \cap B \neq \emptyset$, then $A \cap B$ is closed in $\mathcal{P}^3$ and homeomorphic to R. It can be easily shown that $(\mathcal{P}^3, \mathcal{AB}, \Lambda)$ is an R^2-divisible R^3-space, where $\mathcal{AB} = \{A \cap B | (A, B) \in \mathcal{A} \times \mathcal{B}, A \cap B \neq \emptyset\}$. If the mapping $\varphi : \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{AB} : (A, B) \longrightarrow A \cap B$ is continuous, then $(\mathcal{P}^3, \mathcal{AB}, \Lambda)$ is also topological.

LEMMA 2.4. *Let* $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$ *be the product space of two standard* R^2*-planes* E_1 *and* E_2*. Then:*

(1) *The mapping* $\alpha : \overline{\mathcal{V}} \longrightarrow \mathcal{L} : \{(x, f(x), z) | x, z \in R\} \longrightarrow \{(x, f(x)) | x \in R\}$ *and* $\{x\} \times R^2 \longrightarrow \{x\} \times R$ *is a homeomorphism, where* $f \in \mathcal{L}$.
(2) *For all* $f \times g \in \mathcal{L} \times \Im$ *let* $\gamma(f \times g) = \{(x, f(x), z) | x, z \in R\} \in \mathcal{V}$. *Then the mapping* $\gamma : \mathcal{L} \times \Im \longrightarrow \mathcal{V}$ *is continuous.*
(3) *The mapping* $\beta : \overline{\mathcal{H}} \longrightarrow \Im : \{(x, y, g(x)) | x, y \in R\} \longrightarrow \{(x, g(x)) | x \in R\}$ *and* $\{x\} \times R^2 \longrightarrow \{x\} \times R$ *is a homeomorphism, where* $g \in \Im$.
(4) *For all* $f \times g \in \mathcal{L} \times \Im$ *let* $\delta(f \times g) = \{(x, y, g(x)) | x, y \in R\} \in \mathcal{H}$. *Then the mapping* $\gamma : \mathcal{L} \times \Im \longrightarrow \mathcal{H}$ *is continuous.*

(5) *The mapping* $\Phi : \mathcal{V} \times \mathcal{H} \longrightarrow \mathcal{L} \times \Im : (V, H) \longrightarrow V \wedge H$ *is a homeomorphism.*
(6) *The space* $\mathcal{L} \times \Im$ *is homeomorphic to* R^4.

PROOF. (1) By definition of Hausdorff-convergence, it is easy to check that α is a homeomorphism.
(2) Let $l_n, l \in \mathcal{L} \times \Im$ such that $l_n \longrightarrow l$. Let $a, b \in l$ with $a \neq b$. Then there exist $a_n, b_n \in l_n$ such that $a_n \neq b_n$, $a_n \longrightarrow a$ and $b_n \longrightarrow b$. Since the projection $P : R^3 \longrightarrow R^2 : (x, y, z) \longrightarrow (x, y)$ is continuous, it follows that $P(a_n) \longrightarrow P(a)$, $P(b_n) \longrightarrow P(b)$ and $P(a) \neq P(b)$. Since $(P(R^3), P(\mathcal{L} \times \Im)) = (R^2, \mathcal{L})$, where $P(\mathcal{L} \times \Im) = \{P(l) | l \in \mathcal{L} \times \Im\}$, it implies that $P(a_n) \vee P(b_n) \longrightarrow P(a) \vee P(b)$. Since $P(a_n) \vee P(b_n) = \alpha(\gamma(l_n))$ and $P(a) \vee P(b) = \alpha(\gamma(l))$, it follows that $\alpha\gamma$ is continuous. By (1), γ is also continuous.
Proof of (3), (4) is similar to the proof of (1), (2).
(5) By definition of Hausdorff-convergence, the following mappings are continuous: $\Phi : \mathcal{V} \times \mathcal{H} \longrightarrow \mathcal{L} \times \Im : (V, H) \longrightarrow V \wedge H$, $\Psi : \mathcal{L} \times \Im \longrightarrow \mathcal{V} \times \mathcal{H} : f \times g \longrightarrow (f \subseteq V, g \subseteq H)$. It is clear that $\Psi \circ \Phi = id$ and $\Phi \circ \Psi = id$, i.e., Φ is a homeomorphism.
(6) See [9, Th. 2.10]. □

A pair of points (x_1, y_1, z_1) and (x_2, y_2, z_2) is called vertical if $x_1 = x_2, y_1 = y_2$. A pair of points (x_1, y_1, z_1) and (x_2, y_2, z_2) is called horizontal if $x_1 = x_2, z_1 = z_2$.

LEMMA 2.5. *Let* $E \subseteq R^3$ *be a plane of* $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$. *Then:*
(1) *If* E *contains two vertical points, i.e.,* $(x, y, z_1), (x, y, z_2) \in E$ *with* $z_1 \neq z_2$, *then* E *is a vertical plane.*
(2) *If* E *contains two horizontal points, i.e.,* $(x, y_1, z), (x, y_2, z) \in E$ *with* $y_1 \neq y_2$, *then* E *is a horizontal plane.*

PROOF. (1) Let p, q be two vertical points with $p, q \in E$. By [9, lemma 2.2], the joining line $l := p \vee q = \{x\} \times \{y\} \times R$ is contained in E. Let $a \in E \setminus \{x\} \times R^2$, and let V be a vertical plane with $a \in V$ and $l \subseteq V$. Then $a \vee p$ and $a \vee q$ lie on E. Consequently, $E = V$ is a vertical plane.
(2) The assertion can be proved as (1). □

THEOREM 2.6. *Let* $E_1 = (R^2, \mathcal{L})$ *and* $E_2 = (R^2, \Im)$ *be two standard* R^2*-planes, and let* $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$ *be the product space of* E_1 *and* E_2. *If there exists a plane which is neither vertical nor horizontal, then* $E_1 = (R^2, \mathcal{L})$ *is isomorphic to* $E_2 = (R^2, \Im)$.

PROOF. Suppose that E is a plane which is neither vertical nor horizontal. The projection $P_1 : E \longrightarrow R^2 : (x, y, z) \longrightarrow (x, y)$ is continuous, and since E contains no two vertical points, P_1 is injective. By theorem on the invariance of domain (see for example [10, III. 6]), $P_1(E) \subseteq R^2$ is open and $P_1 : E \longrightarrow P(E)$ is a homeomorphism. Let $x, y \in P_1(E)$ with $x \neq y$. Let $a, b \in E$ with $x = P_1(a)$ and $y = P_1(b)$, then $a \vee b \subseteq E$. Since $x \vee y = P_1(a \vee b) \subseteq P_1(E)$ and $x \vee y = P_1(a \vee b) \in \mathcal{L}$, hence $P_1(E)$ is a subgeometry of $E_1 = (R^2, \mathcal{L})$ which is open. By lemma 1.4, $P_1(E) = R^2$, and for each line $f \times g \subseteq E$, $P_1(f \times g) = f \in \mathcal{L}$. It is clear that for each $S_i \in \Lambda$ $P_1(E \cap S_i)$ is a vertical line. Hence $P_1 : E \longrightarrow (R^2, \mathcal{L})$ is an isomorphism. Similarly, we consider the projection $P_2 : E \longrightarrow R^2 : (x, y, z) \longrightarrow (x, z)$. Then $P_2 : E \longrightarrow (R^2, \Im)$ is also an isomorphism. Therefore, $(R^2, \mathcal{L})$ is isomorphic to $(R^2, \Im)$. □

Let $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$ be the product space of E_1 and E_2. Let Σ_i be the collineation group of E_i, $i = 1, 2$, which fixes each line x-const. (hence $\delta_i : R^2 \longrightarrow R^2 : (x, y) \longrightarrow (x, h_i(x, y))$. If $\delta_i \in \Sigma_i, i = 1, 2$, then $\delta := \delta_1 \times \delta_2 : R^3 \longrightarrow R^3 : (x, y, z) \longrightarrow (x, h_1(x, y), h_2(x, z))$ is a collineation of $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$. Let $\Sigma_1 \times \Sigma_2$ denote the set of all collineations $\delta = \delta_1 \times \delta_2$ with $\delta_1 \in \Sigma_1$ and $\delta_2 \in \Sigma_2$. Let S be the common induced action on the x-coordinate. Then $< \Sigma_1 \times \Sigma_2, S >$ is a collineation group of $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$.

THEOREM 2.7. *Let $E_1 = (R^2, \mathcal{L})$ and $E_2 = (R^2, \Im)$ be two standard R^2-planes which are not isomorphic, and let $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$ be the product space of E_1 and E_2. Let Σ be the collineation group of $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$. Then Σ is the group $< \Sigma_1 \times \Sigma_2, S >$.*

PROOF. Let $\gamma \in \Sigma$ and let $R^2_{<x,y>}$ (resp. $R^2_{<x,z>}$) be the $< x, y >$-coordinate plane (resp. $< x, z >$-coordinate plane). Since γ maps the sets $\{x\} \times R^2$ onto itself, hence γ is the form $\gamma(x, y, z) = (f(x), g(x, y, z), h(x, y, z))$. By Theorem 2.5, there exist no further planes, hence γ is an isomorphism in the vertical planes (resp. in the horizontal planes). Therefore, γ must be the form $\gamma(x, y, z) = (f(x), g(x, y), h(x, z))$. It follows that $\gamma | R^2_{<x,y>}$ (resp. $\gamma | R^2_{<x,z>}$) is a collineation of E_1 (resp. E_2), which maps vertical lines onto itself, hence $\gamma \in < \Sigma_1 \times \Sigma_2, S >$. □

THEOREM 2.8. *Let $E_1 = (R^2, \mathcal{L}_1)$ and $E_2 = (R^2, \mathcal{L}_2)$ be two standard R^2-planes isomorphic to the real affine plane $(R^2, \mathcal{L})$, respectively. Let $\tau_t : (x, y) \longrightarrow (x + t, y)$, $t \in R$ be collineations of E_1 (resp. E_2). Then $(R^3, \mathcal{L}_1 \times \mathcal{L}_2, \Lambda)_{E_1 \times E_2}$ is isomorphic to the real affine R^2-divisible R^3-space.*

PROOF. Let $\alpha : (R^2, \mathcal{L}) \longrightarrow (R^2, \mathcal{L}_1)$ be an isomorphism. Let β be a collineation of $(R^2, \mathcal{L})$. Then the composition $\gamma := \alpha\beta$ is an isomorphism from $(R^2, \mathcal{L})$ to $(R^2, \mathcal{L}_1)$. Since all vertical lines are in $\mathcal{L}$ (resp. in $\mathcal{L}_1$) and the full collineation group of $(R^2, \mathcal{L})$ is transitive on R^2 (resp. $\mathcal{L}$), we can choose β such that $\gamma := \alpha\beta$ maps vertical lines onto itself. Hence γ is the form $\gamma(x, y) = (f(x), g(x, y))$, and we may assume that $f(0) = 0$. By assumption, $\tau_t, t \in R$ are collineations of $(R^2, \mathcal{L}_1)$. For all $t \in R$ $\tau_t' = \gamma^{-1}\tau_t\gamma$ are collineations of $(R^2, \mathcal{L})$. Since τ_t' maps vertical lines onto itself and $\{\tau_t'\}$ is transitive on the vertical lines, Hence τ_t' has the form $\tau_t'(x, y) = (x+h(t), *)$, where h is continuous and for all $s, t \in R$ $h(s+t) = h(s) + h(t)$. Since a continuous additive function is linear, it follows that h is linear. Since $\gamma\tau_t' = \tau_t\gamma$, we have $f(x + h(t)) = f(x) + t$. From $f(0) = 0$, putting $x = 0$, we get $f(h(t)) = t$. Hence f is linear and we may assume that $\gamma(x, y) = (x, g(x, y))$. Let $\gamma_1(x, y) = (x, g_1(x, y))$ (resp. $\gamma_2(x, z) = (x, g_2(x, z))$) be an isomorphism from $(R^2, \mathcal{L})$ to $(R^2, \mathcal{L}_1)$ (resp. $(R^2, \mathcal{L}_2)$). We define

$$\gamma_1 * \gamma_2 : (x, y, z) \longrightarrow (x, g_1(x, y), g_2(x, z)).$$

Hence $\gamma_1 * \gamma_2$ is an isomorphism between the real affine R^2-divisible R^3-space and $(R^3, \mathcal{L}_1 \times \mathcal{L}_2, \Lambda)_{E_1 \times E_2}$. □

2.1. (α, d)-space

Let $E_1 = (R^2, \mathcal{L})$ be the real affine plane. Let $E_2 = (R^2, \Im)$ with $\Im = \{\{(x, g(x + n) + \eta) | x \in R\} | n, \eta \in R\} \cup \{\{c\} \times R | c \in R\}$, where

$$g(x) = \begin{cases} |x|^d & : \ x \geq 0 \\ \alpha |x|^d & : \ x \leq 0 \end{cases} \quad \text{with } 0 < \alpha \leq 1 < d$$

is a planar function.

The product space of E_1 and E_2 is called an (α, d)-space. We note that if $(\alpha, d) = (1, 2)$, then by Theorem 2.8, the space is isomorphic to the real affine R^2-divisible R^3-space. An (α, d)-space is induced from a 4-dimensional shift plane.

LEMMA 2.9. *Let $(\alpha, d) \neq (1, 2)$. Then the full collineation group Σ of $E_2 = (R^2, \Im)$ has dimension 3 and is the group*

$$\{(x, y) \longrightarrow (ax + \xi, a^d y + \eta) : a > 0,\ \xi,\ \eta \in R\}.$$

Furthermore $\Sigma = \Sigma^1$ for $\alpha \neq 1$, and $\Sigma = \Sigma^1 < (x, y) \longrightarrow (-x, y) >$ for $\alpha = 1$.

PROOF. [12]. □

THEOREM 2.10. *Let $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$ be an (α, d)-space with $(\alpha, d) \neq (1,2)$. Then full collineation group Σ of $(R^3, \mathcal{L} \times \Im, \Lambda)_{E_1 \times E_2}$ has dimension 6 and is the group*

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} a & & \\ b & c & \\ & & a^d \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \right\},$$

$a > 0,\ c \neq 0, b, t_i \in R, i = 1,2,3.$

If $\alpha = 1$, then the reflection to the vertical plane $R \times \{0\} \times R$ is a collineation.

PROOF. Let Σ_i be the collineation groups of E_i, $i = 1,2$, which fixes each line x =const., and let S be the common induced action on x-coordinate. Hence $\Sigma_1 = \{(x,y) \longrightarrow (x, bx + cy + \xi) | c \neq 0, b, \xi \in R\}$, $\Sigma_2 = \{(x,z) \longrightarrow (x, z + \eta) | \eta \in R\}$, and S is the group $S = \{x \longrightarrow ax + t | a > 0, t \in R\}$. By Theorem 2.7, the proof is complete. □

3. R^2-divisible R^3-spaces induced by planar functions

From now on we consider always the divisible partition $\Lambda = \{\{x\} \times R^2 | x \in R\}$. In sections 3,4 we will introduce a method, construction of topological R^2-divisible R^3-spaces. The affine plane in Lemma 3.1 is a generalized type of the real affine plane.

LEMMA 3.1. *Let $g : R \longrightarrow R$ be a continuous function and $\alpha : R \longrightarrow R$ a continuous bijective function. We define an incidence structure $(R^2, \mathcal{L}^A_{g,\alpha})$ with the following lines:*

(1) *All vertical lines $\{x\} \times R$ with $x \in R$ are in $\mathcal{L}^A_{g,\alpha}$.*
(2) *The sets $l(t,\eta) = \{(x, g(x) + t\alpha(x) + \eta) | x \in R\}$ with $t, \eta \in R$ are in $\mathcal{L}^A_{g,\alpha}$.*

Then $(R^2, \mathcal{L}^A_{g,\alpha})$ is an affine plane.

PROOF. We first show that each pair p, q of distinct points is contained in a unique line $p \vee q \in \mathcal{L}^A_{g,\alpha}$. Since all verticals are in $\mathcal{L}^A_{g,\alpha}$, we will show that for each $(x_1, y_1), (x_2, y_2) \in R^2$ with $x_1 \neq x_2$ there exists a unique join line $l \in \mathcal{L}^A_{g,\alpha}$ such that $l = (x_1, y_1) \vee (x_2, y_2)$. Hence we have the following equations:

$$g(x_1) + t\alpha(x_1) + \eta = y_1,$$

$$g(x_2) + t\alpha(x_2) + \eta = y_2.$$

Therefore, $t(\alpha(x_1) - \alpha(x_2)) = y_1 - y_2 - g(x_1) + g(x_2)$. Since $\alpha(x)$ is bijective, it follows that $\alpha(x_1) - \alpha(x_2) \neq 0$. Hence there exists a unique $t \in R$ which satisfies the given equations. Thereby the corresponding $\eta \in R$ is uniquely determined. We next show that $(R^2, \mathcal{L}^A_{g,\alpha})$ holds the parallel axiom, i.e., for each line l and each point $p = (u, v)$, there is a unique line which passes through p and is parallel to l.
Case 1: Let the given line l be vertical. Then there exists obviously a unique line h with $l \cap h = \emptyset, p \in h$.
Case 2: Let the given line l be not vertical. Hence there exist $t_0, \eta_0 \in R$ with $l = \{(x, g(x) + t_0\alpha(x) + \eta_0)|x \in R\}$. Let $p = (u, v)$ with $p \notin l$, i.e., $g(u) + t_0\alpha(u) + \eta_0 \neq v$. We can calculate the pencil of p:

$$\mathcal{L}^A_{g,\alpha_p} = \{l(t, v - g(u) - t\alpha(u))|t \in R\} \cup \{u\} \times R.$$

Let $t := t_0$. Then we get a line

$$h := \{(x, g(x) + t_0\alpha(x) + v - g(u) - t_0\alpha(u))|x \in R\} \in \mathcal{L}^A_{g,\alpha_p}.$$

Since $p \notin l$, it implies that $l \cap h = \emptyset$ with $p \in h$. We have to show that h is uniquely determined. Let $t \neq t_0$. Then we have the following equations:

$$g(x) + t_0\alpha(x) + \eta_0 = g(x) + t\alpha(x) + v - g(u) - t\alpha(u).$$

Therefore, $\alpha(x)(t_0 - t) = v - g(u) - t\alpha(u) - \eta_0$, and so

$$\alpha(x) = (v - g(u) - t\alpha(u) - \eta_0)/(t_0 - t).$$

Since α is bijective, it follows that $l \cap k \neq \emptyset$ for all $k(\neq h) \in \mathcal{L}^A_{g,\alpha_p}$. Hence h is uniquely determined. □

LEMMA 3.2. *Let $(R^2, \mathcal{L}^A_{g,\alpha})$ be an affine plane as in lemma 3.1. Then $(R^2, \mathcal{L}^A_{g,\alpha})$ is isomorphic to $(R^2, \mathcal{L}^A_{0,\alpha})$, i.e., $g(x) = 0$.*

PROOF. Let $(R^2, \mathcal{L}^A_{g,\alpha})$ and $(R^2, \mathcal{L}^A_{0,\alpha})$ be two affine planes. Define

$$\varphi : (R^2, \mathcal{L}^A_{g,\alpha}) \longrightarrow (R^2, \mathcal{L}^A_{0,\alpha}) : (x, y) \longrightarrow (x, -g(x) + y).$$

Then φ is a homeomorphism. Furthermore, $(x, g(x) + t\alpha(x) + \eta) \longrightarrow (x, t\alpha(x) + \eta)$. Hence the proof is complete. □

THEOREM 3.3. *Let $f : R \longrightarrow R$ be a continuous planar function, let $g : R \longrightarrow R$ be a continuous function and let $\alpha : R \longrightarrow R$ be a continuous bijective function. We define an incidence structure $(R^3, \mathcal{L}_I, \Lambda)_{f,g,\alpha}$-I with the following line set*

$$\mathcal{L}_I := \{\{(x, f(x - k) + \xi, g(x - k) + t\alpha(x) + \eta)|x \in R\}| \\ k, \xi, \eta, t \in R\}.$$

Then:

(1) $(R^3, \mathcal{L}_I, \Lambda)_{f,g,\alpha}$-*I is a topological* R^2-*divisible* R^3-*space.*
(2) *The sets*

$$E_{k,\xi} = \{(x, f(x-k)+\xi, z) | x, z \in R\}$$

are planes for $k, \xi \in R$.
(3) *If* $g := 0$, *then* $(R^3, \mathcal{L}_I, \Lambda)_{f,g,\alpha}$-*I is a product space.*

PROOF. (1) We first show that $(R^3, \mathcal{L}_I, \Lambda)_{f,g,\alpha}$-I is an R^2-divisible R^3-space. Let (x_1, y_1, z_1), (x_2, y_2, z_2) in R^3 with $x_1 \neq x_2$. Hence we consider the following equations:

$$f(x_1 - k) + \xi = y_1, g(x_1 - k) + t\alpha(x_1) + \eta = z_1,$$

$$f(x_2 - k) + \xi = y_2, g(x_2 - k) + t\alpha(x_2) + \eta = z_2.$$

Since f is planar, there exist k and ξ uniquely. Since α is bijective, there exist t and η uniquely. Hence $(R^3, \mathcal{L}_I, \Lambda)_{f,g,\alpha}$-I is an R^2-divisible R^3-space. Next we have to show that $(R^3, \mathcal{L}_I, \Lambda)_{f,g,\alpha}$-I is topological. Let $(a_n = (x_n, y_n, z_n))_{n \in N}$ and $(b_n = (u_n, v_n, w_n))_{n \in N}$ be two convergent sequences in R^3 with the limits $a = (x_0, y_0, z_0)$, $b = (u_0, v_0, w_0), x_0 \neq u_0$. We denote the join lines $a_n \vee b_n$ and $a \vee b$ as the forms:

$$a_n \vee b_n = \{(x, f(x - k_n) + \xi_n, g(x - k_n) + t_n\alpha(x) + \eta_n) | x \in R\},$$

$$a \vee b = \{(x, f(x - k_0) + \xi_0, g(x - k_0) + t_0\alpha(x) + \eta_0) | x \in R\},$$

$k_n, t_n, \xi_n, \eta_n \in R, k_0, t_0, \xi_0, \eta_0 \in R$.
Let $f_n(k) := f(x_n - k) - f(u_n - k)$ and $f_0(k) := f(x_0 - k) - f(u_0 - k)$. Then k_n and k_0 are the solutions of the equations $f_n(k) = y_n - v_n$ and $f_0(k) = y_0 - v_0$, i.e., $k_n = f_n^{-1}(y_n - v_n)$ and $k_0 = f_0^{-1}(y_0 - v_0)$. Since $\lim_{n\to\infty} f_n(k) = f_0(k)$ and $\lim_{n\to\infty}(y_n - v_n) = y_0 - v_0$, it follows that $k_n \longrightarrow k_0$. Since $x_n \longrightarrow x_0$ and $k_n \longrightarrow k_0$, therefore, $\xi_n \longrightarrow \xi_0$. Since $a_n \longrightarrow a$ and $b_n \longrightarrow b$,

$$\lim_{n\to\infty} [g(x_n - k_n) - g(u_n - k_n) + t_n(\alpha(x_n) - \alpha(u_n))]$$

$$= g(x_0 - k_0) - g(u_0 - k_0) + t_0(\alpha(x_0) - \alpha(u_0)).$$

Since $x_n \longrightarrow x_0$, $u_n \longrightarrow u_0$ and $k_n \longrightarrow k_0$, it also implies that $t_n \longrightarrow t_0$, and so $\eta_n \longrightarrow \eta_0$. This implies also that $a_n \vee b_n \longrightarrow a \vee b$. Hence $(R^3, \mathcal{L}_I, \Lambda)_{f,g,\alpha}$-I is topological.
The assertions (2) and (3) are clear. □

We note that the method of construction in Theorem 3.3 can be generalized in the following way. Let $(R^2, \mathcal{L})$ be a standard R^2-plane and let $g : R \longrightarrow R$ be a continuous function. For each line $\alpha : R \longrightarrow R$ in $\mathcal{L}$ we take a new line $g+\alpha : R \longrightarrow R$. Let $\mathcal{L}_g$ denote the set of all lines $g + \alpha$ with $\alpha \in \mathcal{L}$ and all verticals. Then we get an R^2-plane $(R^2, \mathcal{L}_g)$ which is isomorphic to $(R^2, \mathcal{L})$. We apply the method in Theorem 3.3 and get a topological R^2-divisible R^3-space. In next theorem we give variations of Theorem 3.3.

THEOREM 3.4. *Let $f : R \longrightarrow R$ be a continuous planar function, let $g : R \longrightarrow R$ be a continuous function and let $\alpha : R \longrightarrow R$ be a continuous bijective function. We define the following R^2-divisible R^3-spaces:*

(i) $(R^3, \mathcal{L}_{II}, \Lambda)_{f,g,\alpha}$-*II with*

$$\mathcal{L}_{II} := \{\{(x, f(x-k)+\xi, g(x)+t\alpha(x-k)+\eta) | x \in R\} | k, \xi, \eta, t \in R\}.$$

(ii) $(R^3, \mathcal{L}_{III}, \Lambda)_{f,g,\alpha}$-*III with*

$$\mathcal{L}_{III} := \{\{(x, f(x-k)+\xi, g(x-k)+t\alpha(x-k)+\eta) | x \in R\} | k, \xi, \eta, t \in R\}.$$

(iii) $(R^3, \mathcal{L}_{I'}, \Lambda)_{f,g,\alpha}$-*I′ with*

$$\mathcal{L}_{I'} := \{\{(x, f(x-k)+\xi, g(x-\xi)+t\alpha(x)+\eta) | x \in R\} | k, \xi, \eta, t \in R\}.$$

(iv) $(R^3, \mathcal{L}_{II'}, \Lambda)_{f,g,\alpha}$-*II′ with*

$$\mathcal{L}_{II'} := \{\{(x, f(x-k)+\xi, g(x)+t\alpha(x-\xi)+\eta) | x \in R\} | k, \xi, \eta, t \in R\}.$$

(v) $(R^3, \mathcal{L}_{III'}, \Lambda)_{f,g,\alpha}$-*III′ with*

$$\mathcal{L}_{III'} := \{\{(x, f(x-k)+\xi, g(x-\xi)+t\alpha(x-\xi)+\eta) | x \in R\} | k, \xi, \eta, t \in R\}.$$

(vi) $(R^3, \mathcal{L}_{IV}, \Lambda)_{f,g,\alpha}$-*IV with*

$$\mathcal{L}_{IV} := \{\{(x, f(x-k)+\xi, g(x-k)+t\alpha(x-\xi)+\eta) | x \in R\} | k, \xi, \eta, t \in R\}.$$

(vii) $(R^3, \mathcal{L}_V, \Lambda)_{f,g,\alpha}$-*V with*

$$\mathcal{L}_V := \{\{(x, f(x-k)+\xi, g(x-\xi)+t\alpha(x-k)+\eta) | x \in R\} | k, \xi, \eta, t \in R\}.$$

Then:

(1) *Each defined space is a topological R^2-divisible R^3-space*
(2) *The sets are planes*

$$E_{k,\xi} = \{(x, f(x-k)+\xi, z) | x, z \in R\}$$

for $k, \xi \in R$

PROOF. Proof is similar to the proof of Theorem 3.3. □

LEMMA 3.5. *Let $\alpha : R \longrightarrow R$ be a continuous bijective function with $\alpha(-x) = -\alpha(x), x \in R$. We define an incidence structure $(R^2, \mathcal{L}_\alpha^B)$ with the following lines:*

(1) *All vertical lines $\{x\} \times R$ with $x \in R$ are in $\mathcal{L}_\alpha^B$.*
(2) *All horizontal lines $R \times \{y\}$ with $y \in R$ are in $\mathcal{L}_\alpha^B$.*
(3) *The sets $\{(x, e^t\alpha(x)+\eta) | x \in R\}$ and $\{(x, e^t\alpha(-x)+\eta) | x \in R\}$ with $t, \eta \in R$ are in $\mathcal{L}_\alpha^B$.*

Then $(R^2, \mathcal{L}_\alpha^B)$ is an affine plane.

PROOF. We may assume that α is strictly monotonic. We first show that for each pair of distinct points there exists a unique line $l \in \mathcal{L}_\alpha^B$ which contains the given two points. Since all vertical and horizontal lines are in $\mathcal{L}_\alpha^B$, we only show that for $(x_1, y_1), (x_2, y_2) \in R^2$ with $x_1 < x_2$ and $y_1 \neq y_2$ there exists a unique join line $l \in \mathcal{L}_\alpha^B$ with $l = (x_1, y_1) \vee (x_2, y_2)$. Hence we consider the following equations:

$$e^t\alpha(x_1) + \eta = y_1, \; e^t\alpha(x_2) + \eta = y_2 \text{ or}$$

$$e^t\alpha(-x_1) + \eta = y_1, \; e^t\alpha(-x_2) + \eta = y_2.$$

Hence $e^t = (y_2 - y_1)/(\alpha(\pm x_2) - \alpha(\pm x_1))$. Therefore we can choose $t, \eta \in R$ uniquely. It implies that there exists a unique line $l \in \mathcal{L}_\alpha^B$ with $l = (x_1, y_1) \vee (x_2, y_2)$. We next show that $(R^2, \mathcal{L}_\alpha^B)$ holds the parallel axiom, i.e., for each line l and each point $p = (u, v)$, there is a unique line which passes through p and parallel to l.
If l is a vertical or a horizontal line, then there exists obviously a unique join line $p \in h$ with $l \cap h = \emptyset$. Assume that l is neither vertical nor horizontal. We have the following two cases.
Case 1: There exist $t_0, \eta_0 \in R$ with $l := \{(x, e^{t_0}\alpha(x) + \eta_0) | x \in R\}$. Let $p = (u, v)$ with $p \notin l$, i.e., $e^{t_0}\alpha(u) + \eta_0 \neq v$. We can calculate the pencil of p:

$$\mathcal{L}_{\alpha_p}^B = \{\{(x, e^t\alpha(\pm x) + v - e^t\alpha(\pm u)) | x \in R\} | t \in R\} \cup \{u\} \times R \cup R \times \{v\}.$$

Let $t := t_0$. Then there exists a line $h := \{(x, e^{t_0}\alpha(x) + v - e^{t_0}\alpha(u)) | x \in R\} \in \mathcal{L}_{\alpha_p}^B$. Since $p \notin l$, it follows that $l \cap h = \emptyset$ with $p \in h$. Next we show that h is uniquely determined. Let $t \neq t_0$. Then we have the following equation

$$e^{t_0}\alpha(x) + \eta_0 = e^t\alpha(\pm x) + v - e^t\alpha(\pm u).$$

Therefore, $\alpha(x) = (v - e^t\alpha(u) \pm \eta_0)/(e^{t_0} \mp e^t)$. Since α is bijective and $e^{t_0} \mp e^t \neq 0$, it follows that $l \cap k \neq \emptyset$, for $k(\neq h) \in \mathcal{L}^B_{\alpha_p}$. Hence h is uniquely determined.

Case 2: There exist $t_0, \eta_0 \in R$ with $l := \{(x, e^{t_0}\alpha(-x) + \eta_0) | x \in R\}$. This case can be proved as the first case. □

THEOREM 3.6. *Let $\alpha : R \longrightarrow R$ be a continuous bijective function with $\alpha(-x) = -\alpha(x)$, $x \in R$. We define an incidence structure $(R^3, \mathcal{L}_\alpha, \Lambda)$ with the following line set*

$$\mathcal{L}_\alpha := \{\{(x, mx + \xi, \pm e^t\alpha(x - m) + \eta) | x \in R\} | m, \xi, \eta, t \in R\}.$$
$$\cup \{(x, mx + \xi, \eta) | x \in R\} | m, \xi, \eta \in R\}. \text{ Then:}$$

(1) *$(R^3, \mathcal{L}_\alpha, \Lambda)$ is a topological R^2-divisible R^3-space.*
(2) *The sets are planes*

$$E_{m,\xi} = \{(x, mx + \xi, v) | x, v \in R\},$$
$$E_\eta = \{(x, u, \eta) | x, u \in R\}$$

for $m, \xi, \eta \in R$.

PROOF. Proof is similar to the proof of Theorem 3.3. □

4. H-spaces and spiral spaces

4.1. H-spaces

DEFINITION 4.1. An R^2-plane $(R^2, \Im)$ is called h-admissible if the following conditions hold:

(1) All verticals $\{x\} \times R$ with $x \in R$ are in $\Im$.
(2) All translations $(x, y) \longrightarrow (x + \xi, y + \eta)(\xi, \eta \in R)$ are collineations of $(R^2, \Im)$.
(3) The reflection $\gamma : (x, y) \longrightarrow (x, -y)$ is a collineation of $(R^2, \Im)$.

We note that all the horizontals $R \times \{y\}$ with $y \in R$ are in $\Im$, because the reflection γ is a collineation of $(R^2, \Im)$.

Let $(R^2, \Im)$ be h-admissible. We identify $(R^2, \Im)$ with the horizontal plane $R^2 \times \{0\}$ in $R^3 = \{(x, y, z) | x, y, z \in R\}$. We apply all translations of R^3 and all rotations with a horizontal axis and get from $\Im$ a line set $\mathcal{L}$ in R^3. Formally we have the following definition: Let

$$D_\alpha = \left\{ \begin{pmatrix} 1 & & \\ & \cos\alpha & \sin\alpha \\ & -\sin\alpha & \cos\alpha \end{pmatrix} : \alpha \in R \right\}.$$

DEFINITION 4.2. Let $(R^2, \Im)$ be a h-admissible R^2-plane. We define a line set $\mathcal{L} := \{D_\alpha(l) + (0,\xi,\eta) | l \in \Im \text{ (not vertical) }, \alpha \in R, \xi, \eta \in R\}$. $(R^3, \mathcal{L}, \Lambda)_R$ is called a H-space (generated by the h-admissible plane $(R^2, \Im)$), where $\Lambda = \{\{x\} \times R^2 | x \in R\}$.

THEOREM 4.3. *Let $E = (R^2, \Im)$ be h-admissible, and let $(R^3, \mathcal{L}, \Lambda)_R$ be the H-space generated by $(R^2, \Im)$. Then:*

1. *$(R^3, \mathcal{L}, \Lambda)_R$ is a topological R^2-divisible R^3-space.*
2. *For given $\alpha, \xi, \eta \in R$, $D_\alpha(E) + (0, \xi, \eta)$ is a plane of $(R^3, \mathcal{L}, \Lambda)_R$.*
3. *$(R^3, \mathcal{B})$ is a topological (3,2,2)-geometry, where $\mathcal{B} = \{D_\alpha(E) + (0, \xi, \eta) | \alpha, \xi, \eta \in R\}$.*

PROOF. It is clear that each line $l \in \mathcal{L}$ is closed in R^3 and homeomorphic to R. We first show that $(R^3, \mathcal{L}, \Lambda)_S$ is an R^2-divisible R^3-space. Let $x, y \in R^3$ with $x_1 \neq y_1$.
Case 1: $(x_2, x_3) = (y_2, y_3)$. Let $E = R^2 \times \{0\} \subseteq R^3$, and let $\alpha \in R$ with $(0, x_2, x_3) \in D_\alpha(E)$. Hence let $u \in R$ with $D_\alpha(0, u, 0) = (0, x_2, x_3)$. Then $l = D_\alpha(R \times \{(u, 0)\}) \in \mathcal{L}$ and $x \vee y = l$. Next we have to show that l is uniquely determined. Let $\beta, \xi, \eta \in R$ and $h \in \Im$ with $x, y \in D_\beta(h) + (0, \xi, \eta)$. Let $x', y' \in h$ with $x = D_\beta(x') + (0, \xi, \eta)$ and $y = D_\beta(y') + (0, \xi, \eta)$. Then it follows that

$$x = (x'_1, x'_2 \cos\beta + \xi, -x'_2 \sin\beta + \eta),$$

$$y = (y'_1, y'_2 \cos\beta + \xi, -y'_2 \sin\beta + \eta).$$

Therefore, $x'_1 = x_1$, $y'_1 = y_1$, $x'_2 \cos\beta = y'_2 \cos\beta$ and $x'_2 \sin\beta = y'_2 \sin\beta$, so that $x'_2 = y'_2$, i.e., h is a horizontal line, and $D_\beta(h) + (0, \xi, \eta) = R \times \{(x_2, x_3)\} = l$.
Case 2: $(x_2, x_3) \neq (y_2, y_3)$. Let $\alpha \in R$ with $(0, y_2 - x_2, y_3 - x_3) \in D_\alpha(E)$. Let $u \in R$ with $D_\alpha(0, u, 0) = (0, y_2 - x_2, y_3 - x_3)$ and let $g := (x_1, 0, 0) \vee (y_1, u, 0) \in \Im$. Then $l := D_\alpha(g) + (0, x_2, x_3) \in \mathcal{L}$ with $x \vee y = l$. Next we show that l is uniquely determined. Let now $\beta, \xi, \eta \in R, h \in \Im$ with $x, y \in D_\beta(h) + (0, \xi, \eta)$. Let $x', y' \in h$ with $x = D_\beta(x') + (0, \xi, \eta)$ and

$y = D_\beta(y') + (0, \xi, \eta)$. Then it follows that

$$x = (x_1', x_2' \cos\beta + \xi, -x_2' \sin\beta + \eta) \quad \text{and}$$
$$y = (y_1', y_2' \cos\beta + \xi, -y_2' \sin\beta + \eta)$$
$$= (y_1, u\cos\alpha + x_2, -u\sin\alpha + x_3).$$

This implies $x_1 = x_1', y_1 = y_1'$, and

$$(y_2, y_3) = (y_2' \cos\beta + \xi, -y_2' \sin\beta + \eta) = (u\cos\alpha + x_2, -u\sin\alpha + x_3).$$

Since $x_2 = x_2' \cos\beta + \xi$ and $x_3 = -x_2' \sin\beta + \eta$,

$(u\cos\alpha + x_2, -u\sin\alpha + x_3) = (u\cos\alpha + x_2' \cos\beta + \xi, -u\sin\alpha - x_2' \sin\beta + \eta)$ $= (y_2, y_3)$. This follows that $u\cos\alpha = (y_2' - x_2')\cos\beta$, $u\sin\alpha = (y_2' - x_2')\sin\beta$, therefore, $| y_2' - x_2' | = | u |$. There exists also a $\delta \in \{-1, 1\}$ with $y_2' - x_2' = \delta u (\neq 0)$. Therefore, $\cos\alpha = \delta\cos\beta$ and $\sin\alpha = \delta\sin\beta$. We consider the following two cases: $\delta = 1$, $\delta = -1$.

$\delta = 1$. Then $y_2' - x_2' = u$, $\beta = \alpha + 2\pi n$ for a $n \in Z$, so that $D_\alpha = D_\beta$. Since $(x_1, 0, 0) = (x_1', x_2', 0) - (0, x_2', 0) = x' - (0, x_2', 0)$ and $y' - (0, x_2', 0) = (y_1', y_2' - x_2', 0) = (y_1', u, 0)$, This implies $h - (0, x_2', 0) = g$, therefore, $h = g + (0, x_2', 0)$. Furthermore it implies that

$$D_\beta(0, x_2', 0) = (0, x_2' \cos\beta, -x_2' \sin\beta) = (0, x_2 - \xi, x_3 - \eta),$$

therefore,

$$D_\beta(h) + (0, \xi, \eta) = D_\alpha(g) + (0, x_2 - \xi, x_3 - \eta) + (0, \xi, \eta)$$
$$= D_\alpha(g) + (0, x_2, x_3) = l.$$

$\delta = -1$. Then $y_2' - x_2' = -u$ and $\beta = \alpha + (2n+1)\pi$ for a $n \in Z$. Let now σ be the mapping $(x, y, z) \longrightarrow (x, -y, z)$. Then $\sigma | E$ is a collineation of $(R^2, \Im)$. Since $x' - (0, x_2', 0) = (x_1, 0, 0) = (x_1, 0, 0)^\sigma$,

$$y' - (0, x_2', 0) = (y_1', y_2' - x_2', 0) = (y_1', -u, 0) = (y_1', u, 0)^\sigma,$$

therefore $h - (0, x_2', 0) = g^\sigma$, i.e., $h = g^\sigma + (0, x_2', 0)$. For all $p \in E$ it follows that $D_\beta(\sigma(p)) = D_\alpha(p)$. Then it follows that

$$D_\beta(h) + (0, \xi, \eta) = D_\beta(\sigma(g)) + (0, x_2 - \xi, x_3 - \eta) + (0, \xi, \eta)$$
$$= D_\alpha(g) + (0, x_2, x_3) = l.$$

We have shown that for $x, y \in R^3$ with $x_1 \neq y_1$ there exists a unique line, i.e., $(R^3, \mathcal{L}, \Lambda)_R$ is an R^2-divisible R^3-space.

We have to show that $(R^3, \mathcal{L}, \Lambda)_R$ is topological. Let $(b_n)_{n \in N}$ be a sequence in $R^3, b \in R^3$ and $0 \neq b_n \longrightarrow b \neq 0$. Let l be the horizontal line passing through 0 and l' passing through b. We separate two cases:

Case 1: $l \neq l'$. Let $E = R^2 \times \{0\}$ and $\alpha \in R$ with $b \in F := D_\alpha(E)$. Since $b_n \longrightarrow b \in l'$, it follows that for sufficiently large n $b_n \notin l$. Since $b_n \longrightarrow b$, there exist $\alpha_n \in R$ with $\alpha_n \longrightarrow 0$ and $b_n \in D_{\alpha_n}(F)$. Then $D_{-\alpha_n}(b_n) \in F$ and $D_{-\alpha_n}(b_n) \longrightarrow b \neq 0$. Since F is an R^2-plane, it implies that $0 \vee D_{-\alpha_n}(b_n) \longrightarrow 0 \vee b$. This implies also

$$0 \vee b_n = D_{\alpha_n}(0 \vee D_{-\alpha_n}(b_n)) \longrightarrow 0 \vee b.$$

Case : 2. $l = l'$. Let $E = R^2 \times \{0\}$. Then $l \subseteq E$. It is also $b_2 = b_3 = 0$. We may assume that $b_n \notin l$ for all $n \in N$. Choose $0 \leq \alpha_n < \pi$ with $b'_n := D_{\alpha_n}(b_n) \in E$. Since $b_n \longrightarrow b, b_2 = b_3 = 0$, it is also $b'_n \longrightarrow b$. Since E is an R^2-plane, it implies that $0 \vee b'_n \longrightarrow 0 \vee b = l$. We will show that $0 \vee b_n \longrightarrow l$. Let $x \in l$. Then there exist $x_n \in 0 \vee b'_n$ with $x_n \longrightarrow x$. Let $y_n := D_{-\alpha_n}(x_n) \in 0 \vee b_n$, and since $x_2 = x_3 = 0$, it follows that $y_n \longrightarrow x$. Now let $x \in \lim_{n\to\infty} \sup(0 \vee b_n)$. Then there exists a sequence n_k of N and $x_{n_k} \in 0 \vee b_{n_k}$ with $x_{n_k} \longrightarrow x$. If $x \notin l$, by case 1, it implies that $0 \vee x_{n_k} \longrightarrow 0 \vee x$, but $0 \vee x_{n_k} = 0 \vee b_{n_k}$, therefore,

$$l \subseteq \lim_{n\to\infty} \inf(0 \vee b_n) \subseteq \lim_{k\to\infty} \inf(0 \vee b_{n_k}) = 0 \vee x,$$

a contradiction, hence $x \in l$. By reduction lemma, $(R^3, \mathcal{L}, \Lambda)_R$ is topological. The assertion (2) is clear.

(3) We define an incidence relation $(R^3, \mathcal{B})$. Let $P : R^3 \longrightarrow R^2_{<y,z>}$ be the projection on the $< y, z >$-coordinate plane. Then $(P(R^3), P(\mathcal{B}))$ is the real affine plane with $P(\mathcal{B}) = \{P(B) | B \in \mathcal{B}\}$. Let $p = (x_1, x_2, x_3)$ and $q = (x_2, y_2, z_2)$ be two distinct points. If $(y_1, z_1) \neq (y_2, z_2)$, then there exists a unique join line $P(p) \vee P(q) = P(B)$, hence we set $p \vee_B q = B$. If $(y_1, z_1) = (y_2, z_2)$, then we set $p \vee_B = R^2 \times \{z_1\}$. Since $P : \mathcal{B} \longrightarrow P(\mathcal{B}) : B \longrightarrow P(B)$ is a homeomorphism, the defined (3,2,2)-geometry is topological. □

EXAMPLE 1. Let $\varphi, \varphi' : R \longrightarrow (0, \infty)$ be strictly monotonic functions. Let $l_+ := \{(x, \varphi(x)) | x \in R\}$ and $l_- := \{(x, -\varphi(x)) | x \in R\}$. We define an incidence structure $(R^2, \Im_\varphi)$ on R^2 with the following lines:

(1) All verticals $\{x\} \times R$ with $x \in R$ are in $\Im_\varphi$.
(2) All horizontals $R \times \{y\}$ with $y \in R$ are in $\Im_\varphi$.
(3) All translations of l_+ and l_- are in $\Im_\varphi$.

Then $(R^2, \Im_\varphi)$ is a h-admissible R^2-plane.

PROOF. We have to show that for a pair of distinct points (x_1, y_1) and (x_2, y_2) there exists a unique line in $\Im_\varphi$. Since $(R^2, +)$ is as a collineation group admissible, we will show that for two points $(0, 0) \neq (x, y)$ there exists a unique line in $\Im_\varphi$. Since two sets $R \times \{0\}$ and $\{0\} \times R$ are lines, we may assume that $x \neq 0$ and $y \neq 0$. Then we have the following equations: $\varphi(x+\xi) - \varphi(\xi) = y$ or $-\varphi(x+\eta) + \varphi(\eta) = y$, $\xi, \eta \in R$. By the mean value theorem, we have $\varphi'(c_1 + \xi) = y/x$ or $-\varphi'(c_2 + \eta) = -y/x$ for some $c_1, c_2 \in R$. Since φ' is also bijective, we can determine a unique ξ or η. □

4.2. Spiral spaces

LEMMA 4.4. *Let $f : R \longrightarrow R^2 : x \longrightarrow (u(x), v(x))$ be a mapping such that for each $d \in R \setminus \{0\}$ the mapping $f_d : R \longrightarrow R^2 : x \longrightarrow (u(x+d) - u(x), v(x+d) - v(x))$ is injective. Let $l(k, \xi, \eta) := \{(x, u(x+k)+\xi, v(x+k)+\eta) | x \in R\} \subseteq R^3$ with $k, \xi, \eta \in R$. Then for $(k_1, \xi_1, \eta_1) \neq (k_2, \xi_2, \eta_2)$ $|l(k_1, \xi_1, \eta_1) \cap l(k_2, \xi_2, \eta_2)| = 0$ or 1.*

PROOF. Let $f_1 : R \longrightarrow R^2 : f_1(x) = (u(x + k_1) + \xi_1, v(x + k_1) + \eta_1)$ and $f_2 : R \longrightarrow R^2 : f_2(x) = (u(x + k_2) + \xi_2, v(x + k_2) + \eta_2)$.
Case 1: $k_1 = k_2 = k$. Then $(\xi_1, \eta_1) \neq (\xi_2, \eta_2)$, hence $(\xi_1 - \xi_2, \eta_1 - \eta_2) \neq (0, 0)$. Since the mapping $f_1 - f_2 : R \longrightarrow R^2 : x \longrightarrow (\xi_1 - \xi_2, \eta_1 - \eta_2) \neq (0, 0)$ is constant, it follows that $(f_1 - f_2)(x) \neq (0, 0)$ for all $x \in R$. Therefore, $|l(k_1, \xi_1, \eta_1) \cap l(k_2, \xi_2, \eta_2)| = 0$.
Case 2: $k_1 \neq k_2$. Then the mapping $f_1 - f_2 : R \longrightarrow R^2 : (f_1 - f_2)(x) = (u(x+k_1) - u(x+k_2) + \xi_1 - \xi_2, v(x+k_1) - v(x+k_2) + \eta_1 - \eta_2)$ is injective, because f_d is injective. In this case $|l(k_1, \xi_1, \eta_1) \cap l(k_2, \xi_2, \eta_2)| = 0$ or 1. □

LEMMA 4.5. *Let $\varphi, \varphi' : R \longrightarrow (0, \infty)$ be strictly monotonic functions. For each $d \in R \setminus \{0\}$ we define the function*

$$g : R \longrightarrow (0, \infty) : x \longrightarrow \varphi(x + d)^2 + \varphi^2(x) - 2\varphi(x + d)\varphi(x) \cos d$$
$$= (\varphi(x + d) - \varphi(x))^2 + 2\varphi(x + d)\varphi(x)(1 - \cos d).$$

Then g is bijective.

PROOF. Since
$g'(x) = 2(\varphi(x + d) - \varphi(x))(\varphi'(x + d) - \varphi'(x))$
$+2(\varphi'(x+d)\varphi(x) + \varphi(x+d)\varphi'(x))(1 - \cos d) > 0$, hence g is injective.
Since

$$\begin{aligned} g(x) &= (\varphi(x + d) - \varphi(x))^2 + 2\varphi(x + d)\varphi(x)(1 - \cos d) \\ &= [\int_0^d \varphi'(t + x)dt]^2 + 2\varphi(x + d)\varphi(x)(1 - \cos d) \\ &= [d\varphi'(c + x)]^2 + 2\varphi(x + d)\varphi(x)(1 - \cos d) \text{ for } 0 < c < d. \end{aligned}$$

Therefore, $\lim_{x\to-\infty} g(x) = 0, \lim_{x\to\infty} g(x) = \infty$, hence g is surjective. □

We construct an R^2-divisible R^3-space which is induced from the mapping

$$f : R \longrightarrow R^2 : x \longrightarrow (\varphi(x)\cos x, \varphi(x)\sin x),$$

where $\varphi, \varphi' : R \longrightarrow (0,\infty)$ are strictly monotonic functions.

LEMMA 4.6. *For each* $d \in R \setminus \{0\}$ *the mapping*
$f_d : R \longrightarrow R^2$:
$x \longrightarrow (\varphi(x+d)\cos(x+d) - \varphi(x)\cos x, \varphi(x+d)\sin(x+d) - \varphi(x)\sin x)$
is injective.

PROOF. For $x_1, x_2 \in R$ let $f_d(x_1) = f_d(x_2)$. Hence

$(\varphi(x_1+d)\cos(x_1+d) - \varphi(x_1)\cos x_1, \varphi(x_1+d)\sin(x_1+d) - \varphi(x_1)\sin x_1)$
$= (\varphi(x_2+d)\cos(x_2+d) - \varphi(x_2)\cos x_2, \varphi(x_2+d)\sin(x_2+d) - \varphi(x_2)\sin x_2)$.
Then

(1) $\varphi(x_1+d)\cos(x_1+d) - \varphi(x_1)\cos x_1 = \varphi(x_2+d)\cos(x_2+d) - \varphi(x_2)\cos x_2$,

(2) $\varphi(x_1+d)\sin(x_1+d) - \varphi(x_1)\sin x_1 = \varphi(x_2+d)\sin(x_2+d) - \varphi(x_2)\sin x_2$.

We calculate $(1)^2 + (2)^2$:

$\varphi(x_1+d)^2 + \varphi(x_1)^2 - 2\varphi(x_1+d)\varphi(x_1)(\cos(x_1+d)\cos x_1 + \sin(x_1+d)\sin x_1)$
$= \varphi(x_2+d)^2 + \varphi(x_2)^2 - 2\varphi(x_2+d)\varphi(x_2)(\cos(x_2+d)\cos x_2 + \sin(x_2+d)\sin x_2)$.
Also

$$\varphi(x_1+d)^2 + \varphi(x_1)^2 - 2\varphi(x_1+d)\varphi(x_1)\cos d$$
$$= \varphi(x_2+d)^2 + \varphi(x_2)^2 - 2\varphi(x_2+d)\varphi(x_2)\cos d.$$

By lemma 4.5, g is injective, hence $x_1 = x_2$. □

Let $l(k,\xi,\eta) := \{(x, \varphi(x+k)\cos(x+k)+\xi, \varphi(x+k)\sin(x+k)+\eta) | x \in R\}$. By lemma 4.4,4.5, $|l(k_1,\xi_1,\eta_1) \wedge l(k_2,\xi_2,\eta_2)| = 0$ or 1 for $(k_1,\xi_1,\eta_1) \neq (k_2,\xi_2,\eta_2)$. Next we consider the pencil of $0 = (0,0,0)$, i.e., $\mathcal{L}_0' := \{\{(x, \varphi(x+k)\cos(x+k) - \varphi(k)\cos k, \varphi(x+k)\sin(x+k) - \varphi(k)\sin k) | x \in R\} | k \in R\}$. We rotate the pencil $\mathcal{L}_0'$ with the x-axis, in order to get the full pencil of 0, i.e.,$\alpha \in R$

$$(D_\alpha =) \begin{pmatrix} 1 & & \\ & \cos\alpha & \sin\alpha \\ & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ \varphi(x+k)\cos(x+k) - \varphi(k)\cos k \\ \varphi(x+k)\sin(x+k) - \varphi(k)\sin k \end{pmatrix} =$$

$$\begin{pmatrix} x \\ (\varphi(x+k)\cos(x+k) - \varphi(k)\cos k)\cos\alpha + (\varphi(x+k)\sin(x+k) - \varphi(k)\sin k)\sin\alpha \\ (\varphi(x+k)\cos(x+k) - \varphi(k)\cos k)(-\sin\alpha) + (\varphi(x+k)\sin(x+k) - \varphi(k)\sin(k))\cos\alpha \end{pmatrix}.$$

Then we have the full pencil of $(0,0,0)$:

$$\begin{aligned} \mathcal{L}_0 = \{\{(x, (\varphi(x+k)\cos(x+k) - \varphi(k)\cos k)\cos\alpha + (\varphi(x+k) \\ \sin(x+k) - \varphi(k)\sin k)\sin\alpha, (\varphi(x+k)\cos(x+k) - \varphi(k)\cos k) \\ (-\sin\alpha) + (\varphi(x+k)\sin(x+k) - \varphi(k)\sin(k))\cos\alpha)|x \in R\} \\ |k,\alpha \in R\} \cup \{(x,0,0)|x \in R\}. \end{aligned}$$

LEMMA 4.7. *For $(0,0,0), (x,y,z) \in R^3, x \neq 0$ there exists a unique join line $l \in \mathcal{L}_0$.*

PROOF. Case 1: Let $x \neq 0, (y,z) = (0,0)$. Then $l := \{(x,0,0)|x \in R\}$ is the unique join line.
Case 2: Let $x \neq 0, (y,z) \neq (0,0)$. Then we have the following equation

$$\begin{aligned} ((\varphi(x+k)\cos(x+k) - \varphi(k)\cos k)\cos\alpha + (\varphi(x+k)\sin(x+k) \\ - \varphi(k)\sin k)\sin\alpha, (\varphi(x+k)\cos(x+k) - \varphi(k)\cos k)(-\sin\alpha) \\ + (\varphi(x+k)\sin(x+k) - \varphi(k)\sin(k))\cos\alpha) = (y,z). \end{aligned}$$

Next we will show that there exists a unique $k \in R$. Through calculation we get the following equation

$$\varphi(x+k)^2 + \varphi(k)^2 - 2\varphi(x+k)\varphi(k)\cos x = y^2 + z^2, x \neq 0, (y,z) \neq 0.$$

Hence $g(k) := \varphi(x+k)^2 + \varphi(k)^2 - 2\varphi(x+k)\varphi(k)\cos x = y^2 + z^2$.
By lemma 4.5, g is bijective. Therefore there exists a unique $k \in R$. It follows that the rotation D_α is also uniquely determined. □

THEOREM 4.8. *Let $f : R \longrightarrow R^2 : x \longrightarrow (\varphi(x)\cos x, \varphi(x)\sin x)$, where $\varphi, \varphi' : R \longrightarrow (0,\infty)$ are strictly monotonic functions. Then there exists a topological R^2-divisible R^3-space which is induced from the graph(f) with the following line set*
$\mathcal{L} = \{\{(x, \varphi(x+k)\cos(x+k-\alpha)+\xi, \varphi(x+k)\sin(x+k-\alpha)+\eta)|x \in R\}| k,\alpha,\xi,\eta \in R\} \cup \{(x,\xi,\eta) \mid x \in R\}|\xi,\eta \in R\}$. This R^2-divisible R^3-space is called a spiral space generated by f.

PROOF. By lemma 4.7, and since $(R^3,+)$ is a collineation group, for (x_1,y_1,z_1), $(x_2,y_2,z_2) \in R^3$ with $x_1 \neq x_2$ there exists a unique join line. We have to show that this space is topological.
Let $(b_n = (x_n,y_n,z_n))_{n\in N}$ be a sequence in R^3, $b = (x_0,y_0,z_0) \in R^3$ and $0 \neq b_n \longrightarrow b \neq 0$. We have the following two cases:
Case 1: $b \notin \{(x,0,0)|x \in R\}$, i.e., $(y_0,z_0) \neq (0,0)$. Since $b_n \longrightarrow b$, it

follows that for sufficiently large $n \in N$ $b_n \notin \{(x,0,0)|x \in R\}$. For all $n \in N$ we may assume that $b_n \notin \{(x,0,0)|x \in R\}$. We consider the join lines $0 \vee b_n$ and $0 \vee b$ as the forms $0 \vee b_n :=$

$$\left\{ \begin{pmatrix} 1 & & \\ & \cos\alpha_n & \sin\alpha_n \\ & -\sin\alpha_n & \cos\alpha_n \end{pmatrix} \begin{pmatrix} x \\ \varphi(x+k_n)\cos(x+k_n) - \varphi(k_n)\cos k_n \\ \varphi(x+k_n)\sin(x+k_n) - \varphi(k_n)\sin k_n \end{pmatrix} \right|$$

$$\left. x \in R \right\}, \alpha_n, k_n \in R,$$

$$0 \vee b := \left\{ \begin{pmatrix} 1 & & \\ & \cos\alpha & \sin\alpha \\ & -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ \varphi(x+k)\cos(x+k) - \varphi(k)\cos k \\ \varphi(x+k)\sin(x+k) - \varphi(k)\sin k \end{pmatrix} \right|$$

$$\left. x \in R \right\}, \alpha, k \in R.$$

We write simply $0 \vee b_n = \{(x, h_n(x), k_n(x))|x \in R\}$ and $0 \vee b = \{(x, h(x), k(x))|x \in R\}$. Set

$$g_n(k) := \varphi(x_n + k)^2 + \varphi(k)^2 - 2\varphi(x_n + k)\varphi(k)\cos x_n,$$

$$g(k) := \varphi(x_0 + k)^2 + \varphi(k)^2 - 2\varphi(x_0 + k)\varphi(k)\cos x_0.$$

By lemma 4.5, g_n and g are homeomorphisms, and k_n , k are solutions of the equations $g_n(k) = y_n^2 + z_n^2$ and $g(k) = y_0^2 + z_0^2$, i.e., $k_n = g_n^{-1}(y_n^2 + z_n^2)$ and $k = g^{-1}(y_0^2 + z_0^2)$. Since $\lim_{n\to\infty} g_n(k) = g(k)$ and $\lim_{n\to\infty}(y_n^2 + z_n^2) = y^2 + z^2$, it follows that $k_n \longrightarrow k$. Since $b_n \longrightarrow b$, hence $\lim_{n\to\infty}\cos\alpha_n = \cos\alpha$ and $\lim_{n\to\infty}\sin\alpha_n = \sin\alpha$. Let $(x_1, y_1, z_1) = (x_1, h(x_1), k(x_1)) \in 0 \vee b$. Then $(x_1, h_n(x_1), k_n(x_1)) \in 0 \vee b_n$ and $\lim_{n\to\infty}(x_1, h_n(x_1), k_n(x_1)) = (x_1, h(x_1), k(x_1))$. This implies that $0 \vee b \subseteq \lim_{n\to\infty}\inf(0 \vee b_n)$. Since $k_n \longrightarrow k$, $\lim_{n\to\infty}\cos\alpha_n = \cos\alpha$ and $\lim_{n\to\infty}\sin\alpha_n = \sin\alpha$, it follows that $\lim_{n\to\infty}\sup(0 \vee b_n) \subseteq 0 \vee b$. It implies also that $0 \vee b_n \longrightarrow 0 \vee b$.
Case 2: $b \in \{(x,0,0)|x \in R\}$, i.e., $(y_0, z_0) = (0,0)$. We may assume that $b_n \notin \{(x,0,0)|x \in R\}$. Since $k_n = g_n^{-1}(y_n^2 + z_n^2)$ and $\lim_{n\to\infty}(y_n^2 + z_n^2) = 0$, This follows that $\lim_{n\to\infty} k_n = -\infty$. Hence we have $0 \vee b_n \longrightarrow 0 \vee b$. By reduction lemma, this space is topological. □

In [1, 3, 4, 5] Betten studied topological R^3-spaces. An incidence structure $(\mathcal{P}^3, \mathcal{L})$ is called a topological R^3-space if (1) each line $l \in \mathcal{L}$ is closed in $\mathcal{P}^3$ and homeomorphic to R, (2) each pair p, q of distinct points is contained in a unique line $p \vee q \in \mathcal{L}$ and (3) the mapping $\vee : \mathcal{P}^3 \times \mathcal{P}^3 \setminus \triangle \longrightarrow \mathcal{L}$ is continuous, where $\triangle = \{(p,p)|p \in \mathcal{P}^3\}$ denotes the diagonal and $\mathcal{L}$ carries the topology of Hausdorff-convergence. Naturally one can ask for extension of topological R^2-divisible R^3-spaces to topological R^3-spaces. For example H-spaces can be extended as topological R^3-spaces if we regard each vertical plane as the real affine plane. Conversely, if topological R^3-spaces contain suitable planes which become a divisible partition in $\mathcal{P}^3$, then we get from these spaces topological R^2-divisible R^3-spaces.

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