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ON AUTOMORPHISM GROUPS OF AN ϵ-FRAMED MANIFOLD

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Abstract. Two examples of ϵ-framed manifolds are constructed. It is proved that an ϵ-framed structure on a manifold is not unique. Automorphism groups of ϵ-framed manifolds are studied. Lastly we prove that a connected Lie group G admits a left invariant normal ϵ-framed structure if and only if the Lie algebra of all left invariant vector fields on G is an ϵ-framed Lie algebra.

1. Introduction

In 1963 K. Yano ([14]) introduced the notion of a f-structure on a manifold, which is defined by a $(1,1)$ tensor field f satisfying $f^3 + f = 0$. The concept of f-structure includes the notions of almost complex and almost contact structures and it is well known that it is really a more general structure. For instance, hypersurfaces of almost contact manifolds are not in general almost complex manifolds, but they have always f-structures associated to them.

Almost product structure is another type of structure widely studied by several authors (see [15], [5]). Analogously to the situation for almost complex and almost contact structures, almost paracontact structures ([6]) are closely related to almost product structures. Moreover, concept of structure defined by a $(1,1)$ tensor field f satisfying $f^3 - f = 0$ ([9], [10]) includes the notions of almost product and almost paracontact structures. The purpose of including in a general notion all the mentioned structures and others (r -contact, r-paracontact, etc.) leads to introduce the notion of $f(3,\epsilon)$-structure ([12]) which is defined by a $(1,1)$ tensor field f satisfying $f^3 - \epsilon f = 0$, $(\epsilon = \pm 1)$. It turns out that

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f is of constant rank and there are two complementary distributions associated with the $f(3,\epsilon)$-structure, as it happens with f-structures and some other known cases.

In [4], A. Morimoto introduced the notion of isomorphism and automorphism of almost contact structures. He also treated the left invariant normal almost contact structure on a Lie group and showed that the problem can be reduced to a purely algebraic one in Lie algebras.

Thus motivated sufficiently, in the present paper, we study automorphism groups of ϵ-framed structure manifolds. We begin with giving preliminary definitions and related concepts which we need to introduce the notion of ϵ-framed structure. This is a general structure which includes almost complex structures, almost product structures, almost contact structures, almost paracontact structures, etc. In section 3, two examples of ϵ-framed manifolds are constructed. It is proved that an ϵ-framed structure on a manifold is not unique. Automorphism groups of ϵ-framed manifolds are studied in section 4. In the last section, we show that the problem to find a left invariant normal ϵ-framed structure on a group manifold is equivalent to a purely algebraic problem in Lie algebra. In fact, we prove that a connected Lie group G admits a left invariant normal ϵ-framed structure if and only if the Lie algebra of all left invariant vector fields on G is an ϵ-framed Lie algebra.

2. Preliminaries

Let M be an n-dimensional differentiable manifold and let there be given a non-null $(1,1)$ tensor field φ satisfying

$$\varphi^3 - \epsilon\varphi = 0. \tag{1}$$

We call such a structure an $\varphi\,(3,\epsilon)$-*structure* ([7]). Following [11], we know that rank of φ is constant. Let $rank\,(\varphi) = k$. If we put

$$l = \epsilon\varphi^2, \qquad m = I - \epsilon\varphi^2,$$

then the tensors l and m acting in the tangent space at each point of M are complementary projection operators which define complementary distributions $\mathcal{L}$ and $\mathcal{M}$, respectively. Then the dimension of the distribution $\mathcal{L}$ is k and the dimension of the distribution $\mathcal{M}$ is $(n-k)$. If $\epsilon = -1$, $\varphi\,(3,\epsilon)$-structure becomes a f-structure ([14]) and in this case $rank\,(\varphi)$ is even.

Let $n - k = r$. If M admits r linearly independent vector fields $\xi_1, \ldots, \xi_r$ spanning the distribution $\mathcal{M}$ at each point of M and if in addition, there are r 1-forms $\eta^1, \ldots, \eta^r$ such that

$$\varphi(\xi_\alpha) = 0, \tag{2}$$

$$\varphi^2 = \epsilon(I - \eta^\alpha \otimes \xi_\alpha), \tag{3}$$

then the structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha)$ is called an *ϵ-framed structure* on M, and the pair $(M, \sum)$ or simply M is called an *ϵ-framed manifold.*

From the above two equations it follows that

$$\eta^\alpha \circ \varphi = 0, \tag{4}$$

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta. \tag{5}$$

The ϵ-framed structure is a generalized structure which in special cases reduces to several known structures shown below which have been widely studied in the past.

ϵ	r	**Structure**
	0	$f(2,\epsilon)$-structure ([7])
-1		framed structure ([15])
-1		almost r-contact structure ([13])
-1	1	almost contact structure ([2])
-1	0	almost complex structure ([15])
1		almost r-paracontact structure ([3])
1	1	almost paracontact structure ([6])
1	0	almost product structure ([15])

If $(M, \sum)$ and $(\overline{M}, \overline{\sum}) \equiv (\overline{M}, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha)$ be two ϵ-framed manifolds, then a $(1,1)$ tensor field F on the product manifold $M \times \overline{M}$ defined by [8]

$$F(X, \overline{X}) = (\varphi X + \epsilon\bar{\eta}^\alpha(\overline{X})\xi_\alpha, \bar{\varphi}\overline{X} + \eta^\alpha(X)\bar{\xi}_\alpha), \tag{6}$$

where X and $\overline{X}$ are any vector fields on M and $\overline{M}$ respectively, satisfies

$$F^2 = \epsilon I,$$

which is a *$F(2,\epsilon)$-structure* on $M \times \overline{M}$ ([7]).

An ϵ-framed manifold $(M, \sum)$ is *normal* ([8]) if it satisfies

$$\overset{1}{N} = [\varphi, \varphi] - \epsilon d\eta^\alpha \otimes \xi_\alpha = 0. \tag{7}$$

3. Non-uniqueness of an ϵ-framed structure

First, we construct two examples of ϵ-framed manifolds.

EXAMPLE 3.1. Let $\theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Define r functions $\theta^\alpha : \mathbb{R}^{2n+r} \to \mathbb{R}$, $\alpha \in \{1, \dots, r\}$, by

$$\theta^\alpha\left(x^1, \dots, x^{2n+r}\right) \equiv \theta\left(x^1, \dots, x^{2n}\right) + x^{\alpha'}, \qquad \alpha' = 2n + \alpha.$$

We define r 1-forms η^α, r vector fields ξ_α, and a $(1,1)$ tensor field φ on $\mathbb{R}^{2n+r}$ as follows:

$$\eta^\alpha \equiv \operatorname{grad}(\theta^\alpha),$$

$$\xi_\alpha \equiv \frac{\partial}{\partial x^{\alpha'}},$$

$$\begin{aligned} \varphi X &\equiv \varphi\left(X^a \frac{\partial}{\partial x^a} + X^{a'} \frac{\partial}{\partial x^{a'}} + X^{\alpha'} \frac{\partial}{\partial x^{\alpha'}}\right), \\ &\equiv X^{a'} \frac{\partial}{\partial x^a} + \epsilon X^a \frac{\partial}{\partial x^{a'}} - \left(\theta_a X^{a'} + \epsilon \theta_{a'} X^a\right) \sum_{\alpha'} \frac{\partial}{\partial x^{\alpha'}}, \end{aligned}$$

where $\epsilon = \pm 1$, $a \in \{1, \dots, n\}$, $a' = n + a$, $\theta_i = \dfrac{\partial \theta}{\partial x^i}$, $i \in \{1, \dots, 2n\}$, and

$$P^a Q_{a'} \equiv P^1 Q_{1'} + \dots + P^n Q_{n'}, \qquad P^{a'} Q_a \equiv P^{1'} Q_1 + \dots + P^{n'} Q_n.$$

Then $(\varphi, \xi_\alpha, \eta^\alpha)$ is an ϵ-framed structure on $\mathbb{R}^{2n+r}$.

EXAMPLE 3.2. We construct another example of an ϵ -framed structure in the Euclidean space $\mathbb{R}^6$. We define $(\varphi, \xi_1, \xi_2, \eta^1, \eta^2)$ in $\mathbb{R}^6$ by their matrices as follows:

$$\varphi = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon & 0 & 0 \end{bmatrix}, \quad \xi_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \xi_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$\eta^1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \eta^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The above set provides the required structure on $\mathbb{R}^6$.

In view of relations (2)–(5), we are able to state the following theorem.

THEOREM 3.3. *Let* $(\varphi, \xi_\alpha, \eta^\alpha)$ *and* $(\varphi, \xi_\alpha, \bar{\eta}^\alpha)$ *(resp.* $(\varphi, \bar{\xi}_\alpha, \eta^\alpha)$*) be two* ϵ*-framed structures on a manifold* M*, then we have* $\eta^\alpha = \bar{\eta}^\alpha$ *(resp.* $\xi_\alpha = \bar{\xi}_\alpha$*).*

Thus we see that two ϵ-framed structures having same φ and same ξ_α (resp. η^α) on a manifold are always identical. However, an ϵ-framed structure on a manifold M always induces another ϵ-framed structure on M. This is stated in the following theorem.

THEOREM 3.4. *An ϵ-framed structure on a manifold is not unique.*

PROOF. Let $(\varphi, \xi_\alpha, \eta^\alpha)$ be an ϵ-framed structure on a manifold M. Let ψ be a non-singular $(1,1)$ tensor field on M. Defining

$$\bar{\varphi} = \psi^{-1}\varphi\psi, \quad \bar{\xi}_\alpha = \psi^{-1}\xi_\alpha, \quad \bar{\eta}^\alpha = \eta^\alpha \circ \psi, \tag{8}$$

one can show that $(\bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha)$ is also an ϵ-framed structure on a manifold M. □

4. Automorphism groups of ϵ-framed manifolds

We begin this section with the definition of the ϵ-framed structure isomorphisms and automorphisms of ϵ-framed manifolds, analogous to that of A. Morimoto ([4]) as follows:

DEFINITION 4.1. Let $(M, \sum)$ and $(\overline{M}, \overline{\sum})$ be two ϵ-framed manifolds. A diffeomorphism f of M onto $\overline{M}$ is called an *ϵ-framed structure isomorphism* of M onto $\overline{M}$ if the following conditions are satisfied:

$$f_* \circ \varphi = \bar{\varphi} \circ f_*, \tag{9}$$

$$f_*(\xi_\alpha) = \bar{\xi}_\alpha, \tag{10}$$

$$f^*(\bar{\eta}^\alpha) = \eta^\alpha, \tag{11}$$

where f_* is the differential of f. In particular, if $(M, \sum) = (\overline{M}, \overline{\sum})$, then f is called an *ϵ-framed structure automorphism* of M. The set of all ϵ-framed structure automorphisms of $(M, \sum)$ forms a group of transformations of M. This group is denoted by $A_{\sum}(M)$.

LEMMA 4.2. *A diffeomorphism f of $(M, \sum)$ onto $(\overline{M}, \overline{\sum})$ is an ϵ-framed structure isomorphism if and only if $f_* \circ \varphi = \bar{\varphi} \circ f_*$ and $f_*(\xi_\alpha) = \bar{\xi}_\alpha$ (or $f^*(\bar{\eta}^\alpha) = \eta^\alpha$).*

LEMMA 4.3. *A diffeomorphism f of $(M, \sum)$ onto itself is an ϵ-framed structure automorphism if and only if $f_* \circ \varphi = \varphi \circ f_*$ and $f_*(\xi_\alpha) = \xi_\alpha$ (or $f^*(\eta^\alpha) = \eta^\alpha$).*

The proofs of above two Lemmas are straightforward and hence omitted.

Definition 4.4. Let $\widetilde{M}$ be an $F(2,\epsilon)$-structure manifold equipped with an $F(2,\epsilon)$-structure given by $F^2 = \epsilon I$, $\epsilon = \pm 1$. We define an *$F(2,\epsilon)$-structure automorphism* f of $\widetilde{M}$ by a diffeomorphism f of $\widetilde{M}$ onto itself which leaves invariant the $F(2,\epsilon)$-structure of $\widetilde{M}$, that is, $f_* \circ F = F \circ f_*$. We denote by $A(\widetilde{M})$ the group of all $F(2,\epsilon)$-structure automorphisms of $\widetilde{M}$.

Definition 4.5. Let $D(M)$ denote the group of all diffeomorphisms of a differentiable manifold M onto itself. For any two differentiable manifolds M and $\overline{M}$, a homomorphism H of $D(M) \times D(\overline{M})$ into $D(M \times \overline{M})$ is defined by

$$H(f,g) = f \times g$$

for $f \in D(M)$ and $g \in D(\overline{M})$, where

$$(f \times g)(p,q) = (f(p), g(q)), \qquad (p,q) \in M \times \overline{M}.$$

Theorem 4.6. *Let $(M, \sum)$ and $(\overline{M}, \overline{\sum})$ be two ϵ-framed manifolds. Then*

$$H\left(A_{\sum}(M) \times A_{\overline{\sum}}(\overline{M})\right) \subset A(M \times \overline{M}),$$

where $M \times \overline{M}$ is considered as an $F(2,\epsilon)$-structure manifold with the induced $F(2,\epsilon)$-structure by $\sum$ and $\overline{\sum}$ defined by equation (6).

Proof. For $(f,g) \in A_{\sum}(M) \times A_{\overline{\sum}}(\overline{M})$, we put $H = H(f,g)$. Let H_*, f_*, g_* denote the differentials of H, f, g respectively. Then for any tangent vectors $X_p \in T_p(M)$ and $\overline{X}_q \in T_q(\overline{M})$, we have

$$\begin{aligned}
& FH_*(X_p, \overline{X}_q) \\
= \;& F_{(f(p),g(q))}(f_*(X_p), g_*(\overline{X}_q)) \\
= \;& \big(\varphi(f_*(X_p)) + \epsilon\bar{\eta}^\alpha(g_*(\overline{X}_q))\xi_{\alpha(f(p))}, \bar{\varphi}(g_*(\overline{X}_q)) \\
& \quad + \eta^\alpha(f_*(X_p))\bar{\xi}_{\alpha(g(q))}\big) \\
= \;& \left(f_*\varphi(X_p) + \epsilon g^*\bar{\eta}^\alpha(\overline{X}_q) f_*(\xi_\alpha)_p, g_*\bar{\varphi}(\overline{X}_q) + f^*\eta^\alpha(X_p) g_*(\bar{\xi}_\alpha)_q\right) \\
= \;& \left(f_*\varphi(X_p) + \epsilon\bar{\eta}^\alpha(\overline{X}_q) f_*(\xi_\alpha)_p, g_*\bar{\varphi}(\overline{X}_q) + \eta^\alpha(X_p) g_*(\bar{\xi}_\alpha)_q\right) \\
= \;& \left(f_*\left(\varphi(X_p) + \epsilon\bar{\eta}^\alpha(\overline{X}_q)\xi_{\alpha(p)}\right), g_*\left(\bar{\varphi}(\overline{X}_q) + \eta^\alpha(X_p)\bar{\xi}_{\alpha(q)}\right)\right) \\
= \;& H_*F(X_p, \overline{X}_q).
\end{aligned}$$

Hence, $FH_* = H_*F$, which proves that $H \in A(M \times \overline{M})$. □

Let $(M, \sum)$ and $(\overline{M}, \overline{\sum})$ be two ϵ-framed manifolds, and let $M \times \overline{M}$ admit an $F(2,\epsilon)$-structure defined by equation (6). We denote by

$\mathcal{A}(M \times \overline{M})$ the Lie algebra of all infinitesimal $F(2,\epsilon)$-structure automorphisms of $M \times \overline{M}$. The homomorphism H induces a homomorphism of $\mathcal{X}(M) \times \mathcal{X}(\overline{M})$ into $\mathcal{X}(M \times \overline{M})$, where $\mathcal{X}(M)$ is the Lie algebra of all vector fields on M, such that

$$H(X, \overline{X}) = X + \overline{X}$$

for $X \in \mathcal{X}(M)$ and $\overline{X} \in \mathcal{X}(\overline{M})$.

In the following theorem we determine the inverse image of $\mathcal{A}(M \times \overline{M})$ by the homomorphism H.

THEOREM 4.7. *Let $\mathcal{A}(M \times \overline{M})$ be the Lie algebra of all infinitesimal $F(2,\epsilon)$-structure automorphisms of $M \times \overline{M}$. Then $X+\overline{X} \in \mathcal{A}(M \times \overline{M})$ if and only if the following equations are satisfied:*

$$\mathfrak{L}_X \varphi = 0, \tag{12}$$

$$\mathfrak{L}_{\overline{X}} \bar{\varphi} = 0, \tag{13}$$

and

$$(\mathfrak{L}_X \eta^\alpha)\, \xi_\beta = (\mathfrak{L}_{\overline{X}} \bar{\eta}^\alpha)\, \bar{\xi}_\beta, \tag{14}$$

where $\mathfrak{L}$ is the operator of Lie derivative.

PROOF. Let $X + \overline{X} \in \mathcal{A}(M \times \overline{M})$. Since $X + \overline{X}$ is an infinitesimal transformation, we have $\mathfrak{L}_{(X+\overline{X})} F = 0$, that is, the Lie bracket satisfies

$$[X + \overline{X}, F(Y + \overline{Y})] = F[X + \overline{X}, Y + \overline{Y}] \tag{15}$$

for all $Y \in \mathcal{X}(M)$ and $\overline{Y} \in \mathcal{X}(\overline{M})$. Using equation (6), we get the left hand side of (15) equal to

$$\begin{aligned} &[X, \varphi Y] + \epsilon \bar{\eta}^\alpha (\overline{Y})\, [X, \xi_\alpha] + X(\eta^\alpha (Y))\, \bar{\xi}_\alpha \\ &\quad + [\overline{X}, \bar{\varphi}\overline{Y}] + \epsilon \overline{X}(\bar{\eta}^\alpha(\overline{Y}))\, \xi_\alpha + \eta^\alpha (Y)\, [\overline{X}, \bar{\xi}_\alpha]\,. \end{aligned} \tag{16}$$

Similarly, the right hand side of (15) is equal to

$$\varphi [X, Y] + \epsilon \bar{\eta}^\alpha ([\overline{X}, \overline{Y}])\, \xi_\alpha + \bar{\varphi}\, [\overline{X}, \overline{Y}] + \eta^\alpha ([X, Y])\, \bar{\xi}_\alpha. \tag{17}$$

Comparing (16) and (17), we get the following four conditions, equivalent to (15):

$$\varphi [X, Y] = [X, \varphi Y]\,, \tag{18}$$

$$\eta^\alpha ([X, Y])\, \bar{\xi}_\alpha = \eta^\alpha (Y)\, [\overline{X}, \bar{\xi}_\alpha] + X(\eta^\alpha (Y))\, \bar{\xi}_\alpha, \tag{19}$$

$$\bar{\varphi}\, [\overline{X}, \overline{Y}] = [\overline{X}, \bar{\varphi}\overline{Y}]\,, \tag{20}$$

and

(21) $$\bar{\eta}^\alpha\left(\left[\overline{X},\overline{Y}\right]\right)\xi_\alpha=\bar{\eta}^\alpha\left(\overline{Y}\right)\left[X,\xi_\alpha\right]+\overline{X}\left(\bar{\eta}^\alpha\left(\overline{Y}\right)\right)\xi_\alpha.$$

From (18) and (20), we get (12) and (13), respectively. Putting $\overline{Y}=\bar{\xi}_\beta$ in (21), we have

(22) $$\bar{\eta}^\alpha\left(\left[\overline{X},\bar{\xi}_\beta\right]\right)\xi_\alpha=\left[X,\xi_\beta\right].$$

Operating by η^γ $(\gamma=1,\ldots,r)$ to above equation, we get (14).

Conversely, let relations (12)–(14) be true. Since (18) and (20) follow from (12) and (13), respectively, putting $Y=\xi_\alpha$ in (18), we have $\varphi\left[X,\xi_\alpha\right]=0$, and hence $\varphi^2\left[X,\xi_\alpha\right]=0$, that is,

(23) $$\left[X,\xi_\alpha\right]=\eta^\beta\left(\left[X,\xi_\alpha\right]\right)\xi_\beta=\bar{\eta}^\beta\left(\left[\overline{X},\bar{\xi}_\alpha\right]\right)\xi_\beta,$$

where (3) and (14) are used. Therefore to prove (21), it is sufficient to prove

(24) $$\bar{\eta}^\alpha\left(\left[\overline{X},\overline{Y}\right]\right)=\overline{X}\left(\bar{\eta}^\alpha\left(\overline{Y}\right)\right)+\bar{\eta}^\alpha\left(\overline{Y}\right)\bar{\eta}^\beta\left(\left[\overline{X},\bar{\xi}_\beta\right]\right).$$

Now, (24) is obvious for $\overline{Y}=\bar{\xi}_\alpha$. Taking $\overline{Y}\in\mathcal{X}\left(\overline{M}\right)$ such that $\bar{\eta}^\alpha\left(\overline{Y}\right)=0$, we get $\bar{\varphi}^2\overline{Y}=\epsilon\overline{Y}$, and then using (20) and (4) we get

$$\bar{\eta}^\alpha\left(\left[\overline{X},\overline{Y}\right]\right)=\bar{\eta}^\alpha\left(\left[\overline{X},\epsilon\bar{\varphi}^2\overline{Y}\right]\right)=\epsilon\bar{\eta}^\alpha\left(\bar{\varphi}\left[\overline{X},\bar{\varphi}\overline{Y}\right]\right)=0.$$

Therefore, (24) holds for $\overline{Y}\in\mathcal{X}\left(\overline{M}\right)$ such that $\bar{\eta}^\alpha\left(\overline{Y}\right)=0$. For an arbitrary vector field $\overline{Y}\in\mathcal{X}\left(\overline{M}\right)$, $\overline{Y}$ can be written as $\overline{Y}=\overline{Y}_1+\bar{\eta}^\alpha\left(\overline{Y}\right)\bar{\xi}_\alpha$, where $\overline{Y}_1=\overline{Y}-\bar{\eta}^\alpha\left(\overline{Y}\right)\bar{\xi}_\alpha$ and $\bar{\eta}^\beta\left(\overline{Y}_1\right)=0$. This shows that (24) is also true for all $\overline{Y}\in\mathcal{X}\left(\overline{M}\right)$. The condition (19) is verified in the similar manner. Thus $X+\overline{X}\in\mathcal{A}\left(M\times\overline{M}\right)$ which proves the theorem. □

5. ϵ-framed Lie algebra

Let G be a connected Lie group. For any element $g\in G$, the left translation L_g of G is defined by

(25) $$L_g(x)=gx,\qquad x\in G.$$

An ϵ-framed structure $\sum=(\varphi,\xi_\alpha,\eta^\alpha)$ on G will be called *left invariant* if $L_g\in A_{\sum}(G)$ for all $g\in G$. In this section we shall show that the problem to find a left invariant normal ϵ-framed structure on a group manifold is reduced to a purely algebraic problem in Lie algebra.

First, we need the following definition.

DEFINITION 5.1. Let $\mathcal{G}$ be a real Lie algebra, φ a linear map of $\mathcal{G}$ into itself, $\xi_\alpha r$ elements of $\mathcal{G}$ and $\eta^\alpha r$ linear functions on $\mathcal{G}$. Then $\mathcal{G}$ is called an *ϵ-framed Lie algebra* if

$$\varphi^2 = \epsilon\left(I - \eta^\alpha \otimes \xi_\alpha\right), \qquad \epsilon = \pm 1, \tag{26}$$

$$\varphi\left(\xi_\alpha\right) = 0, \tag{27}$$

$$\epsilon\left[X, Y\right] + \left[\varphi X, \varphi Y\right] - \varphi\left[X, \varphi Y\right] - \varphi\left[\varphi X, Y\right] = 0. \tag{28}$$

From (26) and (27) it follows that

$$\eta^\alpha \circ \varphi = 0, \tag{29}$$

$$\eta^\alpha\left(\xi_\beta\right) = \delta^\alpha_\beta. \tag{30}$$

Now, we present the main result.

THEOREM 5.2. *An n-dimensional connected Lie group G admits a left invariant normal ϵ-framed structure if and only if the Lie algebra $\mathcal{G}$ of all left invariant vector fields on G is an ϵ-framed Lie algebra.*

PROOF. Let G admit a left invariant normal ϵ-framed structure $\sum = (\varphi, \xi_\alpha, \eta^\alpha)$. Then for any $X \in G$, we have $\varphi X \in G$, because

$$\left(L_g\right)_* \circ \varphi X = \varphi \circ \left(L_g\right)_* X = \varphi X \tag{31}$$

for all $g \in G$. Hence the restriction $\bar{\varphi}$ of φ to $\mathcal{G}$ maps $\mathcal{G}$ into itself. Taking $X \in \mathcal{G}$, since η^α and X are left invariant, we have $\eta^\alpha(X)$ is equal to a constant on $\mathcal{G}$. Hence the restriction $\bar{\eta}^\alpha$ of η^α to $\mathcal{G}$ are linear functions on $\mathcal{G}$. On the other hand, it is clear that $\xi_\alpha \in \mathcal{G}$. Hence by putting $\bar{\xi}_\alpha = \xi_\alpha$, the structure $\overline{\sum} = \left(\bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha\right)$ satisfies (26) and (27). Since $\eta^\alpha(X)$ is constant for $X \in \mathcal{G}$, we see that (7) implies (28), which proves that $\mathcal{G}$ is an ϵ-framed Lie algebra.

Conversely, suppose that $\mathcal{G}$ admits an ϵ-framed structure $\overline{\sum} = (\bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha)$ satisfying (26)–(28). Then for any $X \in \mathcal{X}(G)$ we can find n functions f^i on G such that X can be written uniquely as

$$X = f^i X_i. \tag{32}$$

We now define $\sum = (\varphi, \xi_\alpha, \eta^\alpha)$ as follows:

$$\varphi X = f^i\left(\bar{\varphi} X_i\right), \tag{33}$$

$$\eta^\alpha\left(X\right) = f^i\left(\bar{\eta}^\alpha\left(X_i\right)\right) \tag{34}$$

$$\xi_\alpha = \bar{\xi}_\alpha. \tag{35}$$

Then clearly $\sum$ satisfies (2) and (3), hence $\sum$ is an ϵ-framed structure on G. On the other hand, in view of (28) we have

$$\overset{1}{N}(X_i, X_j) = 0, \qquad i, j \in \{1, ..., r\}.$$

Since $\overset{1}{N}$ is a tensor field on G, $\overset{1}{N}$ vanishes identically, which proves that $\sum$ is normal. Thus Theorem 5.2 is proved. □

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