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# RICCI CURVATURE OF SUBMANIFOLDS IN A QUATERNION PROJECTIVE SPACE*

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ABSTRACT. Recently, Chen establishes sharp relationship between the $k$-Ricci curvature and the squared mean curvature for a submanifold in a Riemannian space form with arbitrary codimension. In this paper, we establish sharp relationships between the Ricci curvature and the squared mean curvature for submanifolds in quaternion projective spaces.

## 1. Preliminary

Let $\bar{M}^m$ be a $4m$-dimensional Riemannian manifold with metric $g$. $\bar{M}^m$ is called a quaternion Kaehlerian manifold if there exists a 3-dimensional vector bundle $V$ of tensors of type (1,1) over $\bar{M}^m$ with local basis of almost Hermitian structures $I$, $J$ and $K$ such that

(a) $IJ = -JI = K$, $JK = -KJ = I$, $KI = -IK = J$, $I^2 = J^2 = K^2 = -1$,

(b) for any local cross-section $\phi$ of $V$, $\bar{\nabla}_X\phi$ is also a cross-section of $V$, where $X$ is an arbitrary vector field on $\bar{M}^m$ and $\bar{\nabla}$ the Riemannian connection on $\bar{M}^m$.

In fact, condition (b) is equivalent to the following condition:

(b') there exist local 1-forms $p$, $q$ and $r$ such that

$$\begin{aligned} \bar{\nabla}_X I &= \phantom{-r(X)I} \quad r(X)J - q(X)K \\ \bar{\nabla}_X J &= -r(X)I \qquad\quad + p(X)K \\ \bar{\nabla}_X K &= q(X)I - p(X)J. \end{aligned} \tag{1}$$

Now let $X$ be a unit vector on $\bar{M}^m$, then $X$, $IX$, $JX$ and $KX$ form an orthonormal frame on $\bar{M}^m$. We denote by $Q(X)$ the 4-plane spanned

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by them. For any two orthonormal vectors $X$, $Y$ on $\bar{M}^m$, if $Q(X)$ and $Q(Y)$ are orthogonal, the plane $\pi(X,Y)$ spanned by $X$, $Y$ is called a totally real plane. Any 2-plane in a $Q(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic plane $\pi$ is called the quaternionic sectional curvature of $\pi$. A quaternion Kaehlerian manifold is a quaternion space form if its quaternionic sectional curvatures are equal to a constant. A quaternion projective space, denoted by $QP^m(4c)$, is a quaternion Kaehlerian manifold of constant quaternionic sectional curvature $4c$.

It is known that a quaternionic Kaehlerian manifold $\bar{M}^m$ is a quaternion space form if and only if its curvature tensor $\bar{R}$ is of the following form [7]:

$$\begin{aligned} \bar{R}(X,Y)Z &= c\{g(Y,Z)X - g(X,Z)Y \\ &+ g(IY,Z)IX - g(IX,Z)IY + 2g(X,IY)IZ \\ &+ g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ \\ &+ g(KY,Z)KX - g(KX,Z)KY + 2g(X,KY)KZ\} \end{aligned} \tag{2}$$

for vectors $X, Y, Z$ tangent to $\bar{M}^m$.

Let $M^n$ be an $n$-dimensional Riemannian manifold isometrically immersed in a quaternion projective space $QP^m(4c)$. We denote by $K(\pi)$ the sectional curvature of $M$ associated with a plane section $\pi \subset T_pM, p \in M$. Also, we denote by $h$ the second fundamental form and $R$ the Riemannian curvature tensor of $M$. We call $M^n$ a totally real submanifold of $QP^m(4c)$ if each 2-plane of $M^n$ is mapped into a totally real plane in $QP^m(4c)$. Consequently, if $M^n$ is a totally real submanifold of $QP^m(4c)$, then $\phi(TM^n) \subset T^{\perp}M^n$ for $\phi = I$, $J$ or $K$, where $T^{\perp}M^n$ is the normal bundle of $M^n$ in $QP^m(4c)$.

We know that if $M^n$ is a totally real submanifold of $QP^m(4c)$, then for any orthonormal vectors $X$, $Y$ in $M^n$, the plane $\pi(X,Y)$ spanned by $X$ and $Y$ is totally real in $QP^m(4c)$, $Q(X)$ and $Q(Y)$ are orthogonal and $g(X,\phi Y) = g(\phi X, Y) = 0$ for $\phi = I, J$ or $K$.

Then the equation of Gauss is given by

$$\begin{aligned} \bar{R}(X,Y,Z,W) &= R(X,Y,Z,W) + g(h(X,W),h(Y,Z)) \\ &\quad - g(h(X,Z),h(Y,W)) \end{aligned} \tag{3}$$

for any vectors $X, Y, Z, W$ tangent to $M$.

We denote by $H$ the mean curvature vector, i.e.

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i), \tag{4}$$

where $\{e_1, ..., e_n\}$ is an orthonormal basis of the tangent space $T_pM, p \in M$.

Also, we denote by

$$h^r_{ij} = g(h(e_i, e_j), e_r) \tag{5}$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{6}$$

For any $p \in M$ and for any $X \in T_pM$, we put $IX = P_1X + F_1X$, $JX = P_2X + F_2X$, $KX = P_3X + F_3X$, where $P_iX \in T_pM, F_iX \in T_p^{\perp}M$.

Let $\{e_1, ..., e_n\}$ be an orthonormal basis of $T_pM$. We denote by

$$\|P_l\|^2 = \sum_{i,j=1}^{n} g^2(P_l e_i, e_j),\ l = 1, 2, 3. \tag{7}$$

Suppose $L$ is a $k$-plane section of $T_pM$ and $X$ a unit vector in $L$. We choose an orthonormal basis $\{e_1, ..., e_k\}$ of $L$ such that $e_1 = X$.

Define the Ricci curvature $Ric_L$ of $L$ at $X$ by

$$Ric_L(X) = K_{12} + K_{13} + ... + K_{1k}, \tag{8}$$

where $K_{ij}$ denotes the sectional curvature of the 2-plane section spanned by $e_i, e_j$. We simply call such a curvature a $k$-Ricci curvature.

The scalar curvature $\tau$ of the $k$-plane section $L$ is given by

$$\tau(L) = \sum_{1 \le i < j \le k} K_{ij}. \tag{9}$$

For each integer $k$, $2 \le k \le n$, the Riemannian invariant $\Theta_k$ on an $n$-dimensional Riemannian manifold $M$ is defined by

$$\Theta_k(p) = \frac{1}{k-1} \inf_{L,X} Ric_L(X), \quad p \in M, \tag{10}$$

where $L$ runs over all $k$-plane sections in $T_pM$ and $X$ runs over all unit vectors in $L$.

Recall that for a submanifold $M$ in a Riemannian manifold, the relative null space of $M$ at a point $p \in M$ is defined by

$$N_p = \{X \in T_pM \,|\, h(X, Y) = 0, \text{ for all } Y \in T_pM\}. \tag{11}$$

We know that when $M^n$ is totally real in $QP^m(4c)$, then $n \le m$. We choose a local field of orthonormal frames in $QP^m(4c)$:

$$\begin{aligned} &e_1, \ldots, e_n, e_{n+1}, \ldots, e_m; e_{I(1)} = Ie_1, \ldots, e_{I(m)} = Ie_m; \\ &e_{J(1)} = Je_1, \ldots, e_{J(m)} = Je_m; e_{K(1)} = Ke_1, \ldots, e_{K(m)} = Ke_m \end{aligned} \tag{12}$$

in such a way that, restricting to $M^n$, $e_1, \ldots, e_n$ are tangent to $M^n$.

Let $A_r = A_{e_r}$ denote the shape operator on $M^n$ in $QP^m(4c)$. Then $A_r$ is related to the second fundamental form $h$ by

$$g(h(X,Y), e_r) = g(A_r X, Y). \tag{13}$$

Let $M^n$ be a totally real submanifold in $QP^m(4c)$, $\{\phi_r, \phi_s, \phi_t\}$ be the set $\{I, J, K\}$ or a set of the circular permutation of the three elements $I$, $J$ and $K$. Then we have

LEMMA 2.1 [6]. *For any $X, Y, Z, W$ in $TM^n$, we have*
(i) $\bar{R}(Z, W, \phi_r X, \phi_r Y) = \bar{R}(Z, W, X, Y)$,
(ii) $g(h(X,Y), \phi_r Z) = g(h(Z,Y), \phi_r X)$, $r = 1, 2, 3$.

In [3], Chen establishes sharp relationship between the $k$-Ricci curvature and the squared mean curvature for a submanifold in a Riemanian space form with arbitrary codimension. In this article, we establish sharp relationships between the Ricci curvature and the squared mean curvature for submanifolds in quaternion projective spaces.

## 2. Ricci curvature and squared mean curvature

Chen [3] established a sharp relationship between $k$-Ricci curvature and the squared mean curvature for submanifolds in real space forms. In this section, we will prove a similar inequality for an $n$-dimensional Riemannian submanifold $M$ of a quaternion projective space $QP^m(4c)$.

THEOREM 2.1. *Let $M$ be an $n$-dimensional submanifold in a quaternion projective space $QP^m(4c)$. Then:*

i) *For each unit vector $X \in T_pM$, we have*

$$\|H\|^2 \geq \frac{4}{n^2}\Big\{Ric(X) - (n-1)c - \frac{3}{2}c\sum_i \|P_i\|^2\Big\}. \tag{14}$$

ii) *If $H(p) = 0$, then a unit tangent vector $X$ at $p$ satisfies the equality case of (14) if and only if $X \in N_p$.*

iii) *The equality case of (14) holds identically for all unit tangent vectors at $p$ if and only if either $p$ is a totally geodesic point or $n = 2$ and $p$ is a totally umbilical point.*

PROOF. Let $X \in T_pM$ be a unit tangent vector $X$ at $p$. We choose an orthonormal basis $e_1, ..., e_n, e_{n+1}, ..., e_{4m}$ such that $e_1, ..., e_n$ are tangent to $M$ at $p$, with $e_1 = X$.

Then, from the equation of Gauss, we have

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 - c[n(n-1) + 3\sum_i \|P_i\|^2]. \tag{15}$$

From (15), we get

$$n^2 \|H\|^2 = 2\tau + \sum_{r=n+1}^{4m} [(h^r_{11})^2 + (h^r_{22} + ... + h^r_{nn})^2 + 2\sum_{i<j}(h^r_{ij})^2]$$

$$-2\sum_{r=n+1}^{4m} \sum_{2\leq i<j\leq n} h^r_{ii}h^r_{jj} - c[n(n-1) + 3\sum_i \|P_i\|^2]$$

$$= 2\tau + \frac{1}{2}\sum_{r=n+1}^{4m} [(h^r_{11} + ... + h^r_{nn})^2 + (h^r_{11} - h^r_{22} - ... - h^r_{nn})^2] \tag{16}$$

$$+2\sum_{r=n+1}^{4m}\sum_{i<j}(h^r_{ij})^2 - 2\sum_{r=n+1}^{4m}\sum_{2\leq i<j\leq n} h^r_{ii}h^r_{jj}$$

$$-c[n(n-1) + 3\sum_i \|P_i\|^2].$$

From the equation of Gauss, we find

$$K_{ij} = \textstyle\sum_{r=n+1}^{4m}[h^r_{ii}h^r_{jj} - (h^r_{ij})^2] + c[1 + 3g^2(e_i, Ie_j) + 3g^2(e_i, Je_j) + 3g^2(e_i, Ke_j)], \tag{17}$$

and consequently

$$\sum_{2\leq i<j\leq n} K_{ij} = \sum_{r=n+1}^{4m}\sum_{2\leq i<j\leq n} [h^r_{ii}h^r_{jj} - (h^r_{ij})^2] + \frac{(n-1)(n-2)}{2}c$$

$$+\frac{3c}{2}\sum_i(\|P_i\|^2 - \|P_ie_1\|^2). \tag{18}$$

Substituting (18) in (16), one gets

$$n^2 \|H\|^2 \geq 2\tau + \frac{1}{2}n^2 \|H\|^2 + 2\sum_{r=n+1}^{4m}\sum_{j=2}^{n}(h^r_{1j})^2 - 2\sum_{2\leq i<j\leq n} K_{ij}$$

$$+[(n-1)(n-2) - n(n-1)]c - 3c\sum_i \|P_ie_1\|^2. \tag{19}$$

Therefore

$$\frac{1}{2}n^2 \|H\|^2 \geq 2Ric(X) - 2(n-1)c - 3c\sum_i \|P_iX\|^2, \tag{20}$$

or equivalently (14).

ii) Assume $H(p) = 0$. Equality holds in (14) if and only if

$$\begin{cases} h^r_{12} = ... = h^r_{1n} = 0, \\ h^r_{11} = h^r_{22} + ... + h^r_{nn}, r \in \{n+1, ..., 4m\}. \end{cases}$$

Then $h^r_{1j} = 0, \forall j \in \{1, ..., n\}, r \in \{n+1, ..., 4m\}$, i.e. $X \in N_p$.

iii) The equality case of (14) holds for all unit tangent vectors at $p$ if and only if

$$\begin{cases} h^r_{ij} = 0, i \neq j, r \in \{n+1, ..., 4m\}, \\ h^r_{11} + ... + h^r_{nn} - 2h^r_{ii} = 0, \quad i \in \{1, ..., n\}, r \in \{n+1, ..., 4m\}. \end{cases}$$

We distinguish two cases:

a) $n \neq 2$, then $p$ is a totally geodesic point;

b) $n = 2$, it follows that $p$ is a totally umbilical point.

The converse is easy to prove. This completes the proof of Theorem 2.1. □

COROLLARY 2.1. *Let $M$ be an $n$-dimensional totally real submanifold in a quaternion projective space $QP^m(4c)$. Then:*

i) *For each unit vector $X \in T_pM$, we have*

$$\|H\|^2 \geq \frac{4}{n^2}\{Ric(X) - (n-1)c\}. \tag{21}$$

ii) *If $H(p) = 0$, then a unit tangent vector $X$ at $p$ satisfies the equality case of (21) if and only if $X \in N_p$.*

iii) *The equality case of (21) holds identically for all unit tangent vectors at $p$ if and only if either $p$ is a totally geodesic point or $n = 2$ and $p$ is a totally umbilical point.*

COROLLARY 2.2. *Let $M$ be an $n$-dimensional totally complex submanifold in a quaternion projective space $QP^m(4c)$. Then:*

i) *For each unit vector $X \in T_pM$, we have*

$$Ric(X) \leq (\frac{5}{2}n - 1)c. \tag{22}$$

ii) *A unit tangent vector $X$ at $p$ satisfies the equality case of (22) if and only if $X \in N_p$.*

iii) *The equality case of (22) holds identically for all unit tangent vectors at $p$ if and only if either $p$ is a totally geodesic point or $n = 2$ and $p$ is a totally umbilical point.*

## 3. $k$-Ricci curvature

In this section, we prove a relationship between the $k$-Ricci curvature and the squared mean curvature for submanifolds in quaternionic projective spaces. First, we state a relationship between the $k$-Ricci curvature and the sectional curvature.

THEOREM 3.1. *Let $M$ be an $n$-dimensional submanifold in a quaternion projective space $QP^m(4c)$. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - c - \frac{3c\sum_i \|P_i\|^2}{n(n-1)}. \tag{23}$$

PROOF. Let $p \in M$ and $\{e_1, \ldots e_n\}$ an orthonormal basis of $T_pM$. From the equation of Gauss for $X = Z = e_i, Y = W = e_j$, by summing, we obtain

$$n^2 \|H\|^2 = 2\tau + \|h\|^2 - c[n(n-1) + 3\sum_i \|P_i\|^2]. \tag{24}$$

We choose an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{4m}\}$ at $p$ such that $e_{n+1}$ is parallel to the mean curvature vector $H(p)$ and $e_1, \ldots, e_n$ diagonalize the shape operator $A_{n+1}$. Then the shape operators take the forms

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ . & . & \ldots & . \\ . & . & \ldots & . \\ 0 & 0 & \ldots & a_n \end{pmatrix},$$

(25)

$$A_r = \left(h_{ij}^r\right),\ i,j = 1, \ldots, n; r = n+2, \ldots, 4m, \quad trace\, A_r = \sum_{i=1}^{n} h_{ii}^r = 0.$$

From (24), we get

$$n^2 \|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{4m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - c[n(n-1) + 3\sum_i \|P_i\|^2]. \tag{26}$$

On the other hand, since

$$0 \leq \sum_{i<j} (a_i - a_j)^2 = (n-1)\sum_i a_i^2 - 2\sum_{i<j} a_i a_j, \tag{27}$$

we obtain

$$n^2 \|H\|^2 = (\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \leq n \sum_{i=1}^{n} a_i^2, \tag{28}$$

which implies

$$\sum_{i=1}^{n} a_i^2 \geq n \|H\|^2 .$$

We have from (26)

$$n^2 \|H\|^2 \geq 2\tau + n \|H\|^2 - c[n(n-1) + 3\sum_i \|P_i\|^2], \tag{29}$$

or, equivalently,

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - c - \frac{3c\sum_i \|P_i\|^2}{n(n-1)}.$$

Using Theorem 3.1, we obtain the following. □

THEOREM 3.2.*Let $M$ be an $n$-dimensional submanifold in in a quaternion projective space $QP^m(4c)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2 (p) \geq \Theta_k(p) - c - \frac{3c\sum \|P_i\|^2}{n(n-1)}. \tag{30}$$

PROOF. Let $\{e_1, ...e_n\}$ be an orthonormal basis of $T_pM$. Denote by $L_{i_1...i_k}$ the $k$-plane section spanned by $e_{i_1}, ..., e_{i_k}$. It follows from (8) and (9) that

$$\tau(L_{i_1...i_k}) = \frac{1}{2} \sum_{i\in\{i_1,...,i_k\}} Ric_{L_{i_1...i_k}}(e_i), \tag{31}$$

$$\tau(p) = \frac{1}{C_{n-2}^{k-2}} \sum_{1\leq i_1<...<i_k\leq n} \tau(L_{i_1...i_k}). \tag{32}$$

Combining (10), (31) and (32), we find

$$\tau(p) \geq \frac{n(n-1)}{2}\Theta_k(p). \tag{33}$$

From (23) and (33), we obtain (30). □

COROLLARY 3.1. *Let $M$ be an $n$-dimensional totally real submanifold in a quaternion projective space $QP^m(4c)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have:*

$$\|H\|^2(p) \geq \Theta_k(p) - c.$$

COROLLARY 3.2. *Let $M$ be an $n$-dimensional totally complex submanifold in a quaternion projective space $QP^m(4c)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have:*

$$\Theta_k(p) \leq \frac{n+2}{n-1}c.$$

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