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DERIVATIONS OF SEMIPRIME RINGS AND NONCOMMUTATIVE BANACH ALGEBRAS

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ABSTRACT. Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \rightarrow A$ such that $D(x)[D(x), x]D(x) \in \mathrm{rad}(A)$ for all $x \in A$. In this case, we have $D(A) \subseteq \mathrm{rad}(A)$.

1. Introduction

Throughout, R represents an associative ring and A will be a complex Banach algebra. We write $[x, y]$ for the commutator $xy - yx$ for x, y in a ring. Let $\mathrm{rad}(R)$ denote the *(Jacobson) radical* of a ring R. A ring R is said to be *(Jacobson) semisimple* if $\mathrm{rad}(R) = 0$.

A ring R is called *n-torsion free* if $nx = 0$ implies $x = 0$. Recall that R is *prime* if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semiprime* if $aRa = (0)$ implies $a = 0$. On the other hand, let X be an element of a normed algebra. Then for every $a \in X$ the *spectral radius* of a, denoted by $r(a)$, is defined by $r(a) = \inf\{||a^n||^{\frac{1}{n}} : n \in \mathbb{N}\}$. It is well-known that the following theorem holds: if a be an element of a normed algebra, then $r(a) = \lim_{n\rightarrow\infty} ||a^n||^{\frac{1}{n}}$ (see F. F. Bonsall and J. Duncan [1]).

An additive mapping D from R to R is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. And an additive mapping D from R to R is called a *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$.

B. E. Johnson and A. M. Sinclair [5] have proved that any linear derivation on a semisimple Banach algebra is continuous. A result of I.

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M. Singer and J. Wermer [8] states that every continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. From these two results, we can conclude that there are no nonzero linear derivations on a commutative semisimple Banach algebra.
M. P. Thomas [9] has proved that any linear derivation on a commutative Banach algebra maps the algebra into its radical.

J. Vukman [10] has proved the following: let R be a 2-torsion free prime ring. If $D : R \longrightarrow R$ is a derivation such that $[D(x), x]D(x) = 0$ for all $x \in R$, then $D = 0$.

Moreover, using the above result, he has proved that the following holds: let A be a noncommutative semisimple Banach algebra. Suppose that $[D(x), x]D(x) = 0$ holds for all $x \in A$. In this case, $D = 0$.

B. D. Kim [6] has showed that the following results hold: let R be a 3!-torsion free semiprime ring. Suppose there exists a Jordan derivation $D : R \to R$ such that

$$[D(x), x]D(x)[D(x), x] = 0$$

for all $x \in R$. In this case, we have $[D(x), x]^5 = 0$ for all $x \in R$.

And, let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \to A$ such that $[D(x), x]D(x)[D(x), x] \in \mathrm{rad}(A)$ for all $x \in A$. In this case, we have $D(A) \subseteq \mathrm{rad}(A)$.

In this paper, our aim is to prove the following result in the ring theory in order to apply it to the Banach algebra theory: let R be a 3!-torsion free semiprime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that

$$D(x)[D(x), x]D(x) = 0$$

for all $x \in R$. In this case, we obtain $[D(x), x]^9 = 0$ for all $x \in R$.

This generalizes J. Vukman's result [10] as follows: let A be a noncommutative Banach algebra and let $D : A \longrightarrow A$ be a continuous linear Jordan derivation. Suppose that $D(x)[D(x), x]D(x) \in \mathrm{rad}(A)$ holds for all $x \in A$. In this case, $D(A) \subseteq \mathrm{rad}(A)$.

2. Preliminaries and results

The following lemma is due to L. O. Chung and J. Luh [4].

LEMMA 2.1. *Let R be a $n!$-torsion free ring. Suppose there exist elements $y_1, y_2, \cdots, y_{n-1}, y_n$ in R such that $\sum_{k=1}^{n} t^k y_k = 0$ for all $t =$*

$1, 2, \cdots, n$. *Then we have* $y_k = 0$ *for every positive integer* k *with* $1 \leq k \leq n$.

The following theorem is due to M. Brešar [3].

THEOREM 2.2. *Let* R *be a 2-torsion free semiprime ring and let* $D : R \longrightarrow R$ *be a Jordan derivation. In this case,* D *is a derivation.*

We write $Q(A)$ for the set of all quasinilpotent elements in A. M. Brešar [2] has proved the following theorem.

THEOREM 2.3. *Let* D *be a bounded derivation of a Banach algebra* A. *Suppose that* $[D(x), x] \in Q(A)$ *for every* $x \in A$. *Then* D *maps* A *into* $rad(A)$.

We need the lemma to prove the main theorem. After this, by S_m we denote the set $\{k \in \mathbb{N} \mid 1 \leq k \leq m\}$ where m is a positive integer.

LEMMA 2.4. *Let* R *be a ring. Suppose there exists a mapping* $T : R \longrightarrow R$ *such that* $[T(x), x]^p T(x) = 0$ *and* $T(x)[T(x), x]^q = 0$ *holds for all* $x \in R$ *and some positive integers* p, q. *In this case, we have* $[T(x), x]^{p+q+1} = 0$ *for all* $x \in R$.

PROOF. For some positive integers p, q, assume that

$$[T(x), x]^p T(x) = 0, x \in R \tag{1}$$

and

$$T(x)[T(x), x]^q = 0, x \in R. \tag{2}$$

For simplicity, we shall introduce the notations as follows: $F(x) \equiv [T(x), x]$, $G(x) \equiv [F(x), x]$ for all $x \in R$. In order to prove the statement, we consider the three cases.

Case: $p = 1$.

By assumption, the relation (1) gives

$$F(x)T(x) = 0, x \in R. \tag{3}$$

Using (3), we get

$$\begin{aligned} 0 &= [F(x)T(x), x] \\ &= G(x)T(x) + F(x)^2, x \in R. \end{aligned} \tag{4}$$

Right multiplication of (4) by $F(x)^q$ leads to

$$G(x)T(x)F(x)^q + F(x)^{q+2} = 0, x \in R. \tag{5}$$

Combining (2) with (5), we obtain

$$F(x)^{q+2} = 0, x \in R. \tag{6}$$

Case: $p = 2$.
Then

$$F(x)^2T(x) = 0, x \in R. \tag{7}$$

By (7), we get

$$\begin{aligned} 0 &= [F(x)^2T(x), x] \\ &= G(x)F(x)T(x) + F(x)G(x)T(x) + F(x)^3, x \in R. \end{aligned} \tag{8}$$

Right multiplication of (8) by $F(x)^q$ yields

$$G(x)F(x)T(x)F(x)^q + F(x)G(x)T(x)F(x)^q + F(x)^{q+3} = 0, x \in R. \tag{9}$$

According to (2) and (9), we have

$$F(x)^{q+3} = 0, x \in R. \tag{10}$$

Case: $p \geq 3$.
From (1), it follows that

$$\begin{aligned} 0 &= [F(x)^pT(x), x] \\ &= G(x)F(x)^{p-1}T(x) + \sum_{k=1}^{p-2} F(x)^kG(x)F(x)^{p-1-k}T(x) \\ &\quad + F(x)^{p-1}G(x)T(x) + F(x)^{p+1}, x \in R. \end{aligned} \tag{11}$$

Multiplication (12) from the right by $F(x)^q$, one obtains

$$\begin{aligned} &G(x)F(x)^{p-1}T(x)F(x)^q + \sum_{k=1}^{p-2} f(x)^kG(x)F(x)^{p-1-k}T(x)F(x)^q \\ &+ F(x)^{p-1}G(x)T(x)F(x)^q + F(x)^{p+q+1} = 0, x \in R. \end{aligned} \tag{12}$$

Comparing (2) and (12), we arrive at

$$F(x)^{p+q+1} = 0, x \in R. \tag{13}$$

Therefore in any case, we can conclude that

$$F(x)^{p+q+1} = 0, x \in R.$$

□

We need the following result to obtain the main theorem for Banach algebra theory.

THEOREM 2.5. *Let R be a 3!-torsion free noncommutative semiprime ring. Suppose there exists a Jordan derivation $D : R \longrightarrow R$ such that*

$$D(x)[D(x), x]D(x) = 0$$

for all $x \in R$. In this case we have $[D(x), x]^9 = 0$ for all $x \in R$.

PROOF. By Theorem 2.2, we can see that D is a derivation on R. For simplicity, we shall denote the maps $B : R \times R \longrightarrow R$, $f, g : R \longrightarrow R$ by $B(x, y) \equiv [D(x), y] + [D(y), x]$, $f(x) \equiv [D(x), x]$, $g(x) \equiv [f(x), x]$ for all $x, y \in R$ respectively. Then we have the basic properties:

$$\begin{aligned} B(x, y) &= B(y, x), \\ B(x, yz) &= B(x, y)z + yB(x, z) + D(y)[z, x] + [y, x]D(z), \\ B(x, x) &= 2f(x), \\ B(xy, z) &= B(y, z)x + zB(y, x) + D(z)[x, y] + [z, y]D(x), \\ B(x, x^2) &= 2(f(x)x + xf(x)),\ x, y, z \in R. \end{aligned}$$

After this, we use the above relations without specific reference. By assumption,

$$D(x)f(x)D(x) = 0,\ x \in R. \tag{14}$$

Replacing $x + ty$ for x in (14), we have

$$\begin{aligned} & D(x + ty)[D(x + ty), x + ty]D(x + ty) \\ \equiv\ & D(x)f(x)D(x) + t\{D(y)f(x)D(x) + D(x)B(x, y)D(x) \\ & + D(x)f(x)D(y)\} \\ & + t^2 J_1(x, y) + t^3 J_2(x, y) + t^4 D(y)f(y)D(y) \\ =\ & 0,\ x, y \in R \text{ and } t \in S_3, \end{aligned} \tag{15}$$

where $J_i, 1 \leq i \leq 2$, denotes the term satisfying the identity (15). From (14) and (15), we obtain

$$\begin{aligned} & t\{D(y)f(x)D(x) + D(x)B(x, y)D(x) + D(x)f(x)D(y)\} \\ & + t^2 J_1(x, y) + t^3 J_2(x, y) = 0,\ x, y \in R \text{ and } t \in S_3. \end{aligned} \tag{16}$$

Since R is 3!-torsion free by assumption, by Lemma 2.1 the relation (16) yields

$$D(y)f(x)D(x) + D(x)B(x, y)D(x) + D(x)f(x)D(y) = 0, x, y \in R. \tag{17}$$

Let $y = x^2$ in (17). Then

$$\begin{aligned} & 3(D(x)xf(x)D(x) + D(x)f(x)xD(x)) \\ & + xD(x)f(x)D(x) + D(x)f(x)D(x)x = 0,\ x \in R. \end{aligned} \tag{18}$$

From (14) and (18), we arrive at

$$3(D(x)xf(x)D(x) + D(x)f(x)xD(x)) = 0, x \in R. \tag{19}$$

Since we have assumed that R is 3!-torsion free, by Lemma 2.1 the relation (19) gives

$$D(x)xf(x)D(x) + D(x)f(x)xD(x) = 0, x \in R. \tag{20}$$

Using (14), the relation (20) reduces to

$$f(x)^2D(x) - D(x)f(x)^2 = 0, x \in R. \tag{21}$$

On the other hand, we obtain from (14) that

$$\begin{aligned} 0 &= [D(x)f(x)D(x), x] \\ &= f(x)^2D(x) + D(x)g(x)D(x) + D(x)f(x)^2, x \in R. \end{aligned} \tag{22}$$

By (21) and (22), we can conclude that

$$2f(x)^2D(x) + D(x)g(x)D(x) = 0, x \in R. \tag{23}$$

Writing xy for y in (17), we get

$$\begin{aligned} & xD(y)f(x)D(x) + D(x)yf(x)D(x) + D(x)(xB(x,y) + 2f(x)y \\ & + D(x)[y,x])D(x) + D(x)f(x)xD(y) + D(x)f(x)D(x)y \\ = \; & 0, x, y \in R. \end{aligned} \tag{24}$$

Left multiplication of (17) by x leads to

$$\begin{aligned} & xD(y)f(x)D(x) + xD(x)B(x,y)D(x) + xD(x)f(x)D(y) \\ = \; & 0, x, y \in R. \end{aligned} \tag{25}$$

Comparing (25) and (25), we arrive at

$$\begin{aligned} & D(x)yf(x)D(x) + f(x)B(x,y)D(x) + 2D(x)f(x)yD(x) \\ & + D(x)^2[y,x]D(x) + (D(x)g(x) + f(x)^2)D(y) \\ & + D(x)f(x)D(x)y \\ = \; & 0, x, y \in R. \end{aligned} \tag{26}$$

By (14) and (27), it follows that

$$\begin{aligned} & D(x)yf(x)D(x) + f(x)B(x,y)D(x) + 2D(x)f(x)yD(x) \\ & + D(x)^2[y,x]D(x) + (D(x)g(x) + f(x)^2)D(y) \\ = \; & 0, x, y \in R. \end{aligned} \tag{27}$$

Substituting yx for y in (28), we obtain

$$\begin{aligned} & D(x)yxf(x)D(x) + f(x)(B(x,y)x + 2yf(x) \\ & + [y,x]D(x))D(x) + 2D(x)f(x)yxD(x) + D(x)^2[y,x]xD(x) \\ (28) \quad & + (D(x)g(x) + f(x)^2)D(y)x + (D(x)g(x) + f(x)^2)yD(x) \\ = \; & 0, x, y \in R. \end{aligned}$$

Right multiplication of (28) by x gives

$$\begin{aligned} & D(x)yf(x)D(x)x + f(x)B(x,y)D(x)x + 2D(x)f(x)yD(x)x \\ (29) \quad & + D(x)^2[y,x]D(x)x + (D(x)g(x) + f(x)^2)D(y)x = 0, x, y \in R. \end{aligned}$$

According to (29) and (29), we see that

$$\begin{aligned} & - D(x)y(g(x)D(x) + f(x)^2) - f(x)B(x,y)f(x) \\ (30) \quad & + 2f(x)yf(x)D(x) + f(x)[y,x]D(x)^2 - 2D(x)f(x)yf(x) \\ & - D(x)^2[y,x]f(x) + (D(x)g(x) + f(x)^2)yD(x) \\ = \; & 0, x, y \in R. \end{aligned}$$

Right multiplication of (31) by $f(x)D(x)$ leads to

$$\begin{aligned} & - D(x)y(g(x)D(x) + f(x)^2)f(x)D(x) - f(x)B(x,y)f(x)^2D(x) \\ (31) \quad & + 2f(x)yf(x)D(x)f(x)D(x) + f(x)[y,x]D(x)^2f(x)D(x) \\ & - 2D(x)f(x)yf(x)^2D(x) - D(x)^2[y,x]f(x)^2D(x) \\ & + (D(x)g(x) + f(x)^2)yD(x)f(x)D(x) = 0, x, y \in R. \end{aligned}$$

Combining (14) with (32), we get

$$\begin{aligned} & D(x)yf(x)^3D(x) + f(x)B(x,y)f(x)^2D(x) \\ (32) \quad & + 2D(x)f(x)yf(x)^2D(x) + D(x)^2[y,x]f(x)^2D(x) = 0, x, y \in R. \end{aligned}$$

Put $y = x$ in (32). Then

$$\begin{aligned} & D(x)xf(x)^3D(x) + 2f(x)^4D(x) + 2D(x)f(x)xf(x)^2D(x) \\ (33) \quad = \; & 0, x \in R. \end{aligned}$$

Left multiplication of (31) by $D(x)f(x)$ yields

$$\begin{aligned} & - D(x)f(x)D(x)y(g(x)D(x) + f(x)^2) - D(x)f(x)^2B(x,y)f(x) \\ (34) \quad & + 2D(x)f(x)^2yf(x)D(x) + D(x)f(x)^2[y,x]D(x)^2 \\ & - 2D(x)f(x)D(x)f(x)yf(x) - D(x)f(x)D(x)^2[y,x]f(x) \\ & + D(x)f(x)(D(x)g(x) + f(x)^2)yD(x) = 0, x, y \in R. \end{aligned}$$

By (14) and (35), we have

$$(35)\quad \begin{aligned} &- D(x)f(x)^2B(x,y)f(x) + 2D(x)f(x)^2yf(x)D(x) \\ &+ D(x)f(x)^2[y,x]D(x)^2 + D(x)f(x)^3yD(x) = 0, x, y \in R. \end{aligned}$$

Set $y = x$ in (35). Then

$$(36)\quad \begin{aligned} &-2D(x)f(x)^4 + 2D(x)f(x)^2xf(x)D(x) + D(x)f(x)^3xD(x) \\ = &\ 0, x \in R. \end{aligned}$$

On the one hand, right multiplication of (21) by $f(x)D(x)$ leads to

$$(37)\quad f(x)^2D(x)f(x)D(x) - D(x)f(x)^3D(x) = 0, x \in R.$$

From (14) and (37), we get

$$(38)\quad D(x)f(x)^3D(x) = 0, x \in R.$$

Comparing (33) and (38), we arrive at

$$(39)\quad 3f(x)^4D(x) + 2D(x)f(x)xf(x)^2D(x) = 0, x \in R.$$

Since $f(x)^2D(x) = D(x)f(x)^2$ for all $x \in R$ from (21), the relation (39) can be written as

$$(40)\quad 3f(x)^4D(x) + 2D(x)f(x)xD(x)f(x)^2 = 0, x \in R.$$

But also, since $2D(x)f(x)D(x)xf(x)^2 = 0$ for all $x \in R$ from (14), combining this relation with (40) we can see that

$$(41)\quad 3f(x)^4D(x) - 2D(x)f(x)^4 = 0, x \in R.$$

According to (36) and (38), we arrive at

$$(42)\quad -3D(x)f(x)^4 + 2D(x)f(x)^2xf(x)D(x) = 0, x \in R.$$

Since $f(x)^2D(x) = D(x)f(x)^2$ for all $x \in R$ from (21), the expression (42) can be written in the form

$$(43)\quad -3D(x)f(x)^4 + 2f(x)^2D(x)xf(x)D(x) = 0, x \in R.$$

Since $-2f(x)^2xD(x)f(x)D(x) = 0$ for all $x \in R$ from (14), combining this relation with (44) it follows that

$$(44)\quad 2f(x)^4D(x) - 3D(x)f(x)^4 = 0, x \in R.$$

Subtracting (44) from (41), one obtains

$$(45)\quad f(x)^4D(x) + D(x)f(x)^4 = 0, x \in R.$$

On the other hand, left multiplication of (21) by $f(x)^2$ yields

$$(46)\quad f(x)^4D(x) - f(x)^2D(x)f(x)^2 = 0, x \in R.$$

Right multiplication of (21) by $f(x)^2$ gives

$$f(x)^2D(x)f(x)^2 - D(x)f(x)^4 = 0, x \in R. \tag{47}$$

Combining (46) with (47), we get

$$f(x)^4D(x) - D(x)f(x)^4 = 0, x \in R. \tag{48}$$

From (45) and (48), it follows that

$$2f(x)^4D(x) = 0, x \in R. \tag{49}$$

Since we have assumed that R is 2-torsion free, the relation (49) gives

$$f(x)^4D(x) = 0, x \in R. \tag{50}$$

Thus by (45) and (50),

$$D(x)f(x)^4 = 0, x \in R. \tag{51}$$

Therefore since (50) and (51) hold, applying them to Lemma 2.4 we have

$$f(x)^9 = 0, x \in R.$$

□

The proof of the following theorem as our main theorem is the same argument as in the proof of J. Vukman's theorem [11].

THEOREM 2.6. *Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that*

$$D(x)[D(x), x]D(x) \in rad(A)$$

for all $x \in A$. Then we have $D(A) \subseteq rad(A)$.

PROOF. In first place we consider the case that A has no primitive ideals. That is, in case that A is a radical Banach algebra, it is obvious that $A = \mathrm{rad}(A)$ by the definition. Thus it is trivial that $D(A) \subseteq \mathrm{rad}(A)$. Hence suppose that A has at least a primitive ideal. In this case, from the assumption that D is a continuous linear Jordan derivation on a Banach algebra A, and by Sinclair's result we get $D(P) \subseteq P$ for every primitive ideal P of A and the fact that D_P is a continuous derivation for every primitive ideal P of A. Thus we can define a derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$ for all$x \in A$. And so, by the assumption that $D(x)f(x)D(x) \in \mathrm{rad}(A) \subseteq P$, $x \in A$, it follows that $D_P(\hat{x})[D_P(\hat{x}), \hat{x}]D_P(\hat{x}) = 0$, $\hat{x} \in A/P$. The by Theorem 2.5, we obtain $[D_P(x+P), x+P]^9 = 0$ for all $x \in A$. Hence $0 = \mathrm{r}_{A/P}([D_P(x+P), x+P])$ for all $x \in A$ and primitive ideals P of A. Therefore $[D_P(\hat{x}), \hat{x}] \in Q(A/P)$ for all $\hat{x} \in A/P$ and primitive ideals P of A. Here we consider the first case that A/P is a commutative Banach algebra. Then we get $Q(A/P) = \text{rad}(A/P) = \text{rad}(A)/P = (0)$ for all primitive ideals P of A. Thus we obtain $[D_P(\hat{x}), \hat{x}] = 0$ for all $\hat{x} \in A/P$ and primitive ideals P of A. That is, we get $[D(x), x] \in P$ for all $x \in A$ and all primitive ideals P of A. Consequently, it follows that we get $[D(x), x] \in \text{rad}(A)$ for all $x \in A$. Since $\text{rad}(A) \subseteq Q(A)$, we get $[D(x), x] \in Q(A)$ for all $x \in A$. Thus since D is a continuous derivation by Theorem 2.3, we have $D(A) \subseteq \text{rad}(A)$.

On the other hand, we consider the second case that A/P is a noncommutative Banach algebra. Then we have $D_P(\hat{x}) \in \text{rad}(A/P)$ for all $\hat{x} \in A/P$. Since A/P is a noncommutative semisimple Banach algebra, we get $[D(x), x] \in P$ for all $x \in A$ and all primitive ideals P of A. Consequently it follows that we have $\text{rad}(A/P) = 0$ for all primitive ideals P of A. Consequently it follows that $D_P(\hat{x}) = 0$ for all $\hat{x} \in A/P$. we get $[D(x), x] \in P$ for all $x \in A$ and all primitive ideals P of A. Hence we have $D(x) \in P$ for all $x \in A$ and all primitive ideals P of A. we get $D(A) \subseteq \cap P = \text{rad}(A)$. Consequently, in any case it follows that $D(A) \subseteq \text{rad}(A)$. □

The following theorem generalizes Vukman's result [10].

THEOREM 2.7. *Let A be a noncommutative semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$D(x)[D(x), x]D(x) = 0$$

for all $x \in A$. Then we have $D = 0$.

PROOF. According to the result of B. E. Johnson and A. M. Sinclair [5] every linear derivation on a semisimple Banach algebra is continuous. A. M. Sinclair [7] has proved that any continuous linear derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$. From the given assumptions $D(x)f(x)D(x) = 0$, $x \in A$, it follows that $D_P(\hat{x})[D_P(\hat{x}), \hat{x}]D_P(\hat{x}) = 0$, $\hat{x} \in A/P$, since all the assumptions of Theorem 2.5 are fulfilled. The factor algebra A/P is noncommutative, by Theorem 2.5 we have $[D_P(\hat{x}), \hat{x}]^9 = 0$, $\hat{x} \in A/P$. Then $r_P([D_P(\hat{x}), \hat{x}])^9 = r([D_P(\hat{x}), \hat{x}]^9) = 0$ for all $\hat{x} \in A/P$. Hence we obtain $r_P([D_P(\hat{x}), \hat{x}]) = 0$ for all $\hat{x} \in A/P$. Thus $[D_P(\hat{x}), \hat{x}] \in Q(A/P)$ for all $\hat{x} \in A/P$. On the other hand, since D is continuous, we see that D_P is also continuous. Thus by Theorem 2.3, one obtains $D_P(A/P) \subseteq \text{rad}(A/P)$. But since A/P is semisimple, $D_P(A/P) = \{0\}$. Hence we get $D(A) \subseteq P$ for all primitive ideals P of A. Thus $D(A) \subseteq \mathrm{rad}(A$. But since A is semisimple, $D = 0$. On the other hand, in case A/P is a commutative Banach algebra, one can conclude that $D_P = 0$ as well since A/P is semisimple and since we know that there are no nonzero linear derivations on a commutative semisimple Banach algebras. In other words $D(x)$ is the intersection of all primitive ideals is the radical, and since we have assumed that A is semisimple, it follows $D = 0$. □

The proof of Theorem 2.7 is essentially due to J. Vukman [10].

COROLLARY 2.8. *Let A be a noncommutative semisimple Banach algebra. Suppose there exists a linear Jordan derivation $D : A \longrightarrow A$ such that*

$$[D(x), x]D(x) = 0 \quad \text{or} \quad D(x)[D(x), x] = 0$$

for all $x \in A$. Then we have $D = 0$.

As a special case of Theorem 2.7 we get the following result which characterizes commutative semisimple Banach algebras.

COROLLARY 2.9. *Let A be a semisimple Banach algebra. Suppose*

$$[x, y][[x, y], x][x, y] = 0$$

for all $x, y \in A$. In this case, A is commutative.

As a special case of Theorem 2.6 we have the following result.

COROLLARY 2.10. *Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that $[D(x), x]D(x) \in \mathrm{rad}(A)$ and*

$$D(x)[D(x), x] \in \mathrm{rad}(A)$$

for all $x \in A$. Then we have $D(A) \subseteq \mathrm{rad}(A)$.

COROLLARY 2.11. *Let A be a noncommutative Banach algebra. Suppose there exists a continuous linear Jordan derivation $D : A \longrightarrow A$ such that $[[[D(x), x], x], x] \in \mathrm{rad}(A)$ for all $x \in A$. Then we obtain $D(A) \subseteq \mathrm{rad}(A)$.*

PROOF. From the result of B. E. Johnson and A. M. Sinclair [5] any linear derivation on a semisimple Banach algebra is continuous. A. M. Sinclair [7] has proved that every continuous linear Jordan derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a derivation $D_P : A/P \longrightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$. By assumption that $[g(x), x] \in$

rad(A), $x \in A$, we obtain $[[[D_P(\hat{x}), \hat{x}], \hat{x}], \hat{x}] = 0$, $\hat{x} \in A/P$, since all the assumptions of Theorem 1 in [11] are fulfilled. The factor algebra A/P is noncommutative, by Theorem 1 in [11] we have $[D_P(\hat{x}), \hat{x}]D_P(\hat{x}) = 0$ and $D_P(\hat{x})[D_P(\hat{x}), \hat{x}] = 0$ for all $\hat{x} \in A/P$ in the process of the proof. Then for each P, we can conclude that $[D(x), x]D(x) \in \mathrm{rad}(A)$ and $D(x)[D(x), x] \in \mathrm{rad}(A)$ for all $x \in A$. And so, by Corollary 2.10, we get $D(A) \subseteq \mathrm{rad}(A)$. □

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