

## A REVIEW ON REDUCTION IN FINITE ELEMENT ANALYSIS

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### Abstract

Reduction methods for large structural systems have been reviewed. Main emphasis is put on the dynamic reduction. Recently, the computing resources and technologies have been expanded so fast that the huge matrices involved in the analysis of structural system can be processed without serious difficulties. For most users, however, the computer facilities are limited and the system reductions in some forms are required. The reduction procedure in static problems is simple and straightforward. The major task is the book-keeping in computations. In dynamic problems and structural optimization, however, the problem is much more complicated. The problem is, in general, nonlinear and hence the exact solution is not available. Therefore, approximate solutions are sought in an iterative manner. A proper convergence criterion needs to be employed in order to get an accurate solution efficiently. Several research works have been reported for the structural optimization combined with system reductions.

**key word** : Finite element method, System reduction, System condensation

### Introduction

Nowadays, the finite element method has been widely used for the static and dynamic analysis of structural systems. As the system becomes large, the preparation of the input data can be a big burden to users. The more difficult problem is, however, the huge computation required for the generation of structural matrices and the solution of the problem. In general, more than 70% of the computing time is spent on the solution procedure in the dynamic analysis. It is evident that the development of supercomputers and parallel processing techniques can alleviate many of the difficulties associated with large structural systems.

Even when new computational resources and technologies are available, quite different approaches have been attempted to overcome the limitations of the conventional solution procedure. In the substructure analysis, the whole system is divided into separate groups of elements called substructures. The exact solution is obtained for static problems.

Sometimes, an approximate solution may be acceptable in the engineering sense, particularly in dynamic analysis. Hence, there has been a steady interest in transforming the system into a subspace and reducing the dimension of the structural matrices.

Of course, the reduction procedure should take less computational effort and save the total computing cost. The main difficulty is how to maximize the computational efficiency without losing the solution accuracy. The present study puts emphasis on the reduction schemes for dynamic problems. Various reduction techniques have been considered to transform the original problem into a reduced subspace. Depending on whether the physical degrees of freedom are included in the reduced subspace, the reduction schemes can be divided into two categories; nodal and modal reductions.

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In the nodal reduction, a set of degrees of freedom called secondary are condensed out while the primary ones are retained in the analysis. In the generation of element matrices, the reduction in element level can be performed for the unloaded degrees of freedom. Static and dynamic condensations are applied to dynamic problems. The term "static" indicates that only the stiffness matrix is used in the reduction procedure. In the dynamic condensation, both stiffness and mass matrices are used in an iterative manner.

The modal reduction employs a set of approximate modes which can closely represent the real system. Eigenvectors and Ritz vectors are the most typical ones. Generalized coordinates are introduced to show the contribution of each vector. The original problem is transformed into a subspace of the generalized coordinates. In theory, however, the method is subject to a limitation. The Rayleigh - Ritz method is a common approach. The subspace iteration method is used for the eigenproblem.

Combined methods use both the primary degrees of freedom and the generalized coordinates in a reduced subspace. The generalized dynamic reduction has been implemented in MSC/NASTRAN. Recently, the hybrid dynamic condensation has been presented for dynamic problems. The method is, in fact, a combination of dynamic condensation, modified subspace iteration, and modal reduction.

It is clear that the structural optimization is much more complicated than the analysis. Compared with static systems, the optimization of dynamic system is more difficult and requires more calculations. Therefore, reduction schemes have been applied successfully to dynamic optimization. Much calculation can be saved without loss of accuracy in the optimum design. It is very promising that the structural optimization of large systems becomes feasible through the use of various reduction schemes.

## Reduction for Static Systems

### 1. Nodal Reduction

In finite element analysis of structural systems, the static equilibrium equation is written as

$$[k]\{U\} = \{P\} \quad (1)$$

where  $[k]$  is the stiffness matrix. The nodal displacement and load are denoted as  $\{U\}$  and  $\{P\}$ , respectively. Equation (1) can be expressed in partitioned form,

$$\begin{bmatrix} k_{pp} & k_{ps} \\ k_{sp} & k_{ss} \end{bmatrix} \begin{Bmatrix} U_p \\ U_s \end{Bmatrix} = \begin{Bmatrix} P_p \\ P_s \end{Bmatrix} \quad (2)$$

where  $\{U_p\}$  is the primary degrees of freedom which will be included in the analysis. The secondary part  $\{U_s\}$  will be condensed out. On some degrees of freedom, no external loads are applied. Then, those degrees of freedom belong to  $\{U_s\}$ .

From the second equation of Eq.(2), we get

$$\{U_s\} = -[k_{ss}]^{-1} [k_{sp}]\{U_p\} + [k_{ss}]^{-1}\{P_s\} \quad (3)$$

Substituting Eq.(3) into the first equation in Eq.(2), one gets

$$[K_r]\{U_p\} = \{P_r\} \quad (4)$$

where the reduced stiffness matrix and load are

$$\begin{aligned} [K_r] &= [k_{pp}] - [k_{ps}][k_{ss}]^{-1} [k_{sp}] \\ \{P_r\} &= \{P_p\} - [k_{ps}][k_{ss}]^{-1}\{P_s\} \end{aligned} \quad (5)$$

Note that the transformation of Eq.(3) is exact and hence the reduced equilibrium equation (4) gives the exact solution.

## 2. Modal Reduction

The eigenvector expansion theorem states that any n-dimensional vector can be expressed as a linear combination of n eigenvectors. Hence, the nodal displacement is expressed as

$$\{U\} = [\phi] \{C\} \quad (6)$$

where  $[\phi]$  is the modal matrix composed of the eigenvectors and  $\{C\}$  is a vector of the generalized coordinates. If  $p$  ( $<n$ ) independent vectors are employed in the expression, Eq.(6) is only an approximation. Then, we have

$$\{U\} \cong [\phi_p] \{C_p\} \quad (7)$$

Direct substitution of Eq.(7) into Eq.(1) gives a residual error in equilibrium.

$$\{R\} = [k] [\phi_p] \{C_p\} - \{P\} \quad (8)$$

Assuming that the residual error weighted by  $[\phi_p]^T$  vanishes, one gets

$$[K_r] \{C_p\} = \{P_r\} \quad (9)$$

where the reduced stiffness matrix and load vector are

$$[K_r] = [\phi_p]^T [k] [\phi_p] \quad \{P_r\} = [\phi_p]^T \{P\} \quad (10)$$

## Reduction for Eigenproblems

### 1. Nodal Reduction

▷ Guyan's Static Condensation

For a discrete structural system, the equilibrium equation for undamped free vibration is written as a general eigenproblem.

$$[k] \{\phi\} = \lambda [m] \{\phi\} \quad (11)$$

where  $[k]$  and  $[m]$  are the stiffness and mass matrices. An eigenpair is denoted as  $\lambda$  and  $\{\phi\}$ . Equation (11) can be expressed in partitioned form,

$$\begin{bmatrix} k_{pp} & k_{ps} \\ k_{sp} & k_{ss} \end{bmatrix} \begin{Bmatrix} \phi_p \\ \phi_s \end{Bmatrix} = \lambda \begin{bmatrix} m_{pp} & m_{ps} \\ m_{sp} & m_{ss} \end{bmatrix} \begin{Bmatrix} \phi_p \\ \phi_s \end{Bmatrix} \quad (12)$$

$\{\phi_p\}$  and  $\{\phi_s\}$  are the primary and secondary sets, respectively. The structural matrices are also partitioned into the corresponding submatrices.

In static condensation, the transformation matrix relating the primary and secondary sets is approximated using the stiffness matrix.

$$\{\phi_s\} \cong -[k_{ss}]^{-1} [k_{sp}]\{\phi_p\} \equiv [T_s] \{\phi_p\} \quad (13)$$

and the mode shape is written as

$$\{\phi\} = \begin{Bmatrix} \phi_p \\ \phi_s \end{Bmatrix} \cong \begin{bmatrix} I \\ T_s \end{bmatrix} \{\phi_p\} \equiv [S] \{\phi_p\} \quad (14)$$

Substituting Eq.(14) into Eq.(11), one gets the residual error in equilibrium because the transformation is not exact.

$$\{R\} \equiv [k][S]\{\phi_p\} - \lambda [m][S]\{\phi_p\} \quad (15)$$

If the residual weighted by  $[S]^T$  vanishes, we get an eigenproblem expressed in terms of the primary degrees of freedom.

$$[K_r] \{\phi_p\} = \lambda [M_r] \{\phi_p\} \quad (16)$$

where

$$\begin{aligned} [K_r] &= [k_{pp}] - [k_{ps}][k_{ss}]^{-1} [k_{sp}] \\ [M_r] &= [m_{pp}] - [m_{ps}][k_{ss}]^{-1} [k_{sp}] - [k_{ps}][k_{ss}]^{-1} [m_{sp}] \\ &\quad + [k_{ps}][k_{ss}]^{-1} [m_{ss}][k_{ss}]^{-1} [k_{sp}] \end{aligned} \quad (17)$$

This is Guyan's static condensation for the general eigenproblem. The rate of convergence depends on how many and which degrees of freedom the primary set should have. Engineering judgment and experience are required for a proper choice of primary degrees of freedom. The primary degrees of freedom should be uniformly distributed throughout the whole structure so that all the probable modes may be excited. Improper selections may result in missing of some modes.

The convergence requirements is usually satisfied because the reduction schemes are used to get an approximation for the lowest modes of a large structural system. Hence, the primary set contains the degrees of freedom associated with the lowest modes and those for the higher modes are eliminated. This is the basic assumption to be satisfied in condensation schemes.

Translational degrees of freedom are preferred to rotational ones even though the numerical magnitude of the corresponding terms in mass matrices is larger than the translational counterpart.

#### ▷ Dynamic Condensation

Assume that the exact eigenvalue be written as

$$\lambda = \lambda_0 + \Delta\lambda \quad (18)$$

where  $\lambda_0$  is an approximation. Then, the equilibrium equation (11) can be rewritten as

$$[D]\{\phi\} = \Delta\lambda [m]\{\phi\} \quad (19)$$

where  $[D]=[k]-\lambda_0[m]$  is the dynamic stiffness matrix.

Equation (19) can be written in partitioned form as

$$\begin{bmatrix} D_{pp} & D_{ps} \\ D_{sp} & D_{ss} \end{bmatrix} \begin{Bmatrix} \phi_p \\ \phi_s \end{Bmatrix} = \Delta\lambda \begin{bmatrix} m_{pp} & m_{ps} \\ m_{sp} & m_{ss} \end{bmatrix} \begin{Bmatrix} \phi_p \\ \phi_s \end{Bmatrix} \quad (20)$$

The transformation is performed through the use of dynamic stiffness matrix.

$$\{\phi_s\} \cong -[D_{ss}]^{-1} [D_{sp}]\{\phi_p\} \equiv [T_d] \{\phi_p\} \quad (21)$$

If  $\lambda_0$  is a good approximation to give small  $\Delta\lambda$  or  $\{\phi_s\}$  has negligible mass, the above transformation becomes accurate.

Through the transformation, an eigenproblem in the condensed subspace is obtained.

$$[D_r] \{\phi_p\} = \Delta\lambda [M_r] \{\phi_p\} \quad (22)$$

where

$$\begin{aligned} [D_r] &= [D_{pp}] - [D_{ps}][D_{ss}]^{-1} [D_{sp}] \\ [M_r] &= [m_{pp}] - [m_{ps}][D_{ss}]^{-1} [D_{sp}] - [D_{ps}][D_{ss}]^{-1} [m_{sp}] \\ &\quad + [D_{ps}][D_{ss}]^{-1} [m_{ss}][D_{ss}]^{-1} [D_{sp}] \end{aligned} \quad (23)$$

From the smallest value of  $\Delta\lambda$  in Eq.(22), the approximation is renewed.

$$\lambda_0 + \Delta\lambda \rightarrow \lambda_0 \quad (24)$$

The iteration continues until a converged solution is obtained. Deterioration of the computational efficiency may be attributable to repeated decomposition of  $[D_{ss}]$ .

## 2. Modal Reduction

### ▷ Subspace Iteration Method

The iterative procedures determine the eigenvectors one by one starting from a single initial vector. That is, each eigenvector is calculated with sufficient accuracy before the next one is initiated. If many eigenvectors are desired, the sweeping technique must be applied.

In the subspace iteration method, however, several eigenvectors are calculated simultaneously. Let's consider a general eigenproblem of Eq.(11). The iteration starts from projection of  $[k]$  and  $[m]$  onto a subspace by using  $q$  iteration vectors  $[X_q]^{(k+1)}$ .

$$\begin{aligned} [K_q] &= ([X_q]^{(k+1)})^T [k] [X_q]^{(k+1)} \\ [M_q] &= ([X_q]^{(k+1)})^T [m] [X_q]^{(k+1)} \end{aligned} \quad (25)$$

An eigenproblem in  $q$ -dimensional subspace is solved to get  $q$  eigenpairs.

$$[K_q][\phi_q] = [M_q][\phi_q][\Lambda_q] \quad (26)$$

Since  $[K_q]$  and  $[M_q]$  are symmetric,  $[\phi_q]$  is orthogonal.

A better approximation can be obtained as

$$[Z_q]^{(k+1)} = [X_q]^{(k+1)}[\phi_q] \quad (27)$$

In fact, the subspace eigenproblem can be obtained through the minimization of the Rayleigh's quotient of a linear combination of the vectors in  $[X_q]^{(k+1)}$ .

The eigenproblem in the subspace only provides a set of orthogonal vectors which may be the most probable candidate for the eigenvectors for the original system. Hence, a procedure of the inverse iteration is employed in order to drive the iteration vectors to the lowest exact eigenvectors.

$$[k] [X_q]^{(k+2)} = [m] [Z_q]^{(k+1)} \quad (28)$$

Some acceleration schemes such as overrelaxation of the iteration vectors and shifting are used to get a better convergence. It has been shown that the effectiveness of the basic and the accelerated subspace iteration methods depends on the number of eigenvectors to be sought, eigenvalue distribution, and the bandwidth of the matrix.

### 3. Mixed Reduction

#### ▷ Generalized Dynamic Reduction

The generalized dynamic reduction is used to improve the accuracy of the usual static condensation through the addition of the generalized coordinates. In the static condensation, the transformation matrix is expressed in Eq.(13).

$$\{\phi_s\} = -[k_{ss}]^{-1} [k_{sp}]\{\phi_p\} \equiv [T_s]\{\phi_p\}$$

In the modal reduction, the eigenvector is approximated as

$$\{\phi\} = \begin{Bmatrix} \phi_p \\ \phi_s \end{Bmatrix} = \begin{bmatrix} \phi_{pr} \\ \phi_{sr} \end{bmatrix} \{C_r\}$$

or

$$\{\phi_p\} = [\phi_{pr}]\{C_r\} \quad \{\phi_s\} = [\phi_{sr}]\{C_r\} \quad (29)$$

Of course, a set of "approximate" eigenvectors which are rich in the lowest modes of the system, need to be provided. Usually, the inverse power method (in fact, the subspace iteration method) is applied to get the approximate set. Only several iterations are performed but not until the complete convergence is reached.

Assume that the static condensation can be improved by adding the generalized coordinates as

$$\{\phi_s\} \cong [T_s] \{\phi_p\} + [Q_{sr}] \{C_r\} \quad (30)$$

where

$$[Q_{sr}] = [\phi_{sr}] - [T_s] [\phi_{pr}]$$

Therefore, the transformation is expressed in terms of the primary set and the generalized coordinates.

$$\{\phi\} = \begin{Bmatrix} \phi_p \\ \phi_s \end{Bmatrix} = \begin{bmatrix} I & 0 \\ T_s & Q_{sr} \end{bmatrix} \begin{Bmatrix} \phi_p \\ C_r \end{Bmatrix} \equiv [S] \begin{Bmatrix} \phi_p \\ C_r \end{Bmatrix} \quad (31)$$

It should be noted that the transformation matrix [S] is only an approximation.

Since the structural matrices are symmetric, the Galerkin's method gives the same reduced structural matrices. For eigenproblems, the equation is written as

$$[K_r] \begin{Bmatrix} \phi_p \\ C_r \end{Bmatrix} = \lambda [M_r] \begin{Bmatrix} \phi_p \\ C_r \end{Bmatrix} \quad (32)$$

where

$$\begin{aligned}
 [K_r] &= [S]^T [k] [S] \\
 &= \begin{bmatrix} [k_{pp}] + [k_{ps}][T_s] & 0 \\ 0 & [Q_{sr}]^T [k_{ss}] [Q_{sr}] \end{bmatrix} \\
 [M_r] &= [S]^T [m] [S] \\
 &= \begin{bmatrix} [m_{pp}] + [m_{ps}][T_s] + [T_s]^T [m_{sp}] + [T_s]^T [m_{ss}] [T_s] & \text{SYM.} \\ [Q_{sr}]^T [m_{sp}] + [Q_{sr}]^T [m_{ss}] [T_s] & [Q_{sr}]^T [m_{ss}] [Q_{sr}] \end{bmatrix} \quad (33)
 \end{aligned}$$

It can easily be seen that the reduced matrices are symmetric.

The accuracy and efficiency of the generalized dynamic reduction depends on two factors; the selection of the primary degrees of freedom as in static condensations and the approximate eigenmodes used for the generalized coordinates.

#### 4. Hybrid Dynamic Condensation

In the dynamic condensation, the modal matrix containing  $p$  approximate eigenvectors can be determined.

$$[\phi_p] = \begin{bmatrix} \phi_{pp} \\ \phi_{sp} \end{bmatrix} \quad (34)$$

In theory, the transformation and the eigenproblem are good only for particular modes of small  $\Delta\lambda$ . The vectors in  $[\phi_p]$ , however, satisfy the necessary condition of orthogonality and hence offer a good candidate for the solution.

The approximate eigenvectors can be improved through the conventional inverse iteration.

$$[D][X_p] = [m][\phi_p] \equiv [B_p] \quad (35)$$

which can be written in partitioned form as

$$\begin{bmatrix} D_{pp} & D_{ps} \\ D_{sp} & D_{ss} \end{bmatrix} \begin{bmatrix} X_{pp} \\ X_{sp} \end{bmatrix} = \begin{bmatrix} m_{pp} & m_{ps} \\ m_{sp} & m_{ss} \end{bmatrix} \begin{bmatrix} \phi_{pp} \\ \phi_{sp} \end{bmatrix} \equiv \begin{bmatrix} B_{pp} \\ B_{sp} \end{bmatrix} \quad (36)$$

The above equation can be reformulated to give

$$[D_r][X_{pp}] = [B_r] \quad (37)$$

where

$$[B_r] = [B_{pp}] - [D_{ps}][D_{ss}]^{-1}[B_{sp}] = [B_{pp}] - [T_d]^T [B_{sp}] \quad (38)$$

The reduced equation (37) for inverse iteration is exact while the equilibrium equation (22) in dynamic condensation is an approximation for small  $\Delta\lambda$ .

With  $[X_{pp}]$  determined, we calculate  $[X_{sp}]$  as

$$[X_{sp}] = [D_{ss}]^{-1} (-[D_{sp}][X_{pp}] + [B_{sp}]) = [T_d][X_{pp}] + [D_{ss}]^{-1}[B_{sp}] \quad (39)$$

The rate of convergence is defined for termination criterion as

$$R_c = \frac{\Delta\lambda_{\text{new}} - \Delta\lambda_{\text{old}}}{\Delta\lambda_{\text{old}}} < \epsilon_s \quad (40)$$

where

$$\Delta\lambda_{\text{new}} = \frac{\{X\}_{\text{new}}^T [D] \{X\}_{\text{new}}}{\{X\}_{\text{new}}^T [m] \{X\}_{\text{new}}} \quad (41)$$

Numerical examples show that the proper value of  $\epsilon_s$  may be  $10^{-5} \sim 10^{-6}$ .

When a converged solution is not obtained with maximum number of inverse iterations, the direction of solution search needs to be changed. The approximate eigenvectors can be improved through subspace transformation. Assume that a vector  $\{Z\}$  is a linear combination of the vectors in  $[X_p]$ .

$$\{Z\} = [X_p] \{q\} \quad (42)$$

where  $\{q\}$  is the generalized coordinates.

The Rayleigh quotient of  $\{Z\}$  is written as

$$R(\{Z\}) = \frac{\{Z\}^T [D] \{Z\}}{\{Z\}^T [m] \{Z\}} = \frac{\{q\}^T [X_p]^T [D] [X_p] \{q\}}{\{q\}^T [X_p]^T [m] [X_p] \{q\}} \quad (43)$$

Considering that the Rayleigh quotient has a stationary value in the neighborhood of an eigenvector, one gets

$$[X_p]^T [D] [X_p] \{q\} = \Delta\lambda_R [X_p]^T [m] [X_p] \{q\} \quad (44)$$

where  $\Delta\lambda_R$  is the stationary value.

The eigensolution of Eq.(44) gives  $p$  eigenvectors or  $[Q_p]$ , which in turn yield  $p$  approximate eigenvectors.

$$[Z_p] = [X_p][Q_p] \quad (45)$$

It can easily be seen that  $[Z_p]$  is orthogonal with respect to  $[D]$  and  $[m]$ .

$$\begin{matrix} [Z_p]^T [D] [Z_p] & = & [Z_p]^T [m] [Z_p] [\Delta\lambda_R] \\ \text{diagonal} & & \text{diagonal} \end{matrix} \quad (46)$$

where  $[\Delta\lambda_R]$  is a diagonal matrix containing the approximate eigenvalues.

This is the equation of orthogonality which is a necessary condition for eigenvectors. Hence,  $[Z_p]$  is likely to be a better approximation.

It should be emphasized that the subspace is spanned by the approximate eigenvectors in  $[X_p]$  and the equation of orthogonality in the incomplete subspace is not a sufficient condition. Hence, the improvement obtained through the eigensolution is subject to a limitation. This is one of the most serious problems that subspace transformation methods have in practice.

As the exact solution is approached, the subspace eigensolution can not improve the approximation because the vectors are nearly orthogonal. A threshold value  $\epsilon_I$  can be assigned for the rate of convergence, which dictates that the solution procedure skips the subspace transformation and continues the inverse update. The proper value of  $\epsilon_I$  may be  $10^{-2} \sim 10^{-3}$ .

An expanded subspace, if necessary, can be considered to get a more accurate solution. The formulation is supplemented with the approximate eigenvectors.



$$\{\phi\}_{exact} \cong \{\phi\} + [Z_p]\{C_p\} \tag{47}$$

or

$$\begin{Bmatrix} \phi_p \\ \phi_s \end{Bmatrix}_{exact} \cong \begin{bmatrix} I & Z_{pp} \\ T_d & Z_{sp} \end{bmatrix} \begin{Bmatrix} \phi_p \\ C_p \end{Bmatrix} \equiv [S] \begin{Bmatrix} \phi_p \\ C_p \end{Bmatrix} \tag{48}$$

The original system is transformed into a subspace of dimension 2p.

$$[D_r] \begin{Bmatrix} \phi_p \\ C_p \end{Bmatrix} = \Delta\lambda [M_r] \begin{Bmatrix} \phi_p \\ C_p \end{Bmatrix} \tag{49}$$

where

$$[D_r] = [S]^T [D] [S] \quad [M_r] = [S]^T [m] [S] \tag{50}$$

The eigensolution in the expanded subspace not only gives an improved solution for the current p modes but also provides good starting vectors when next group of adjacent eigenmodes are desired.

## Dynamic Optimization through System Reduction

### 1. Dynamic Condensation

Equation (11) can be rewritten as

$$[D] \{\phi\} = \{0\} \tag{51}$$

where  $[D] = [k] - \lambda[m]$  is the dynamic stiffness matrix.

Equation (51) is expressed in partitioned form as

$$\begin{bmatrix} D_{pp} & D_{ps} \\ D_{sp} & D_{ss} \end{bmatrix} \begin{Bmatrix} \phi_p \\ \phi_s \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{52}$$

For the perturbed system, the equation becomes

$$\begin{bmatrix} D_{pp}' & D_{ps}' \\ D_{sp}' & D_{ss}' \end{bmatrix} \begin{Bmatrix} \phi_p' \\ \phi_s' \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{53}$$

From the second equation of Eq.(53), we get

$$\{\phi_s'\} = \{\phi\} + \{\Delta\phi\} \tag{54}$$

where

$$\{\phi\} \cong -[D_{ss}]^{-1} [D_{sp}] \{\phi_p'\} \tag{55}$$

Substituting Eqs.(54) and (55) into Eq.(53) and eliminating  $\{\Delta\phi\}$  give the nonlinear perturbation equation.

$$\begin{aligned} & [D_{ps}]\{\phi\} + [D_{pp}]\{\phi_p'\} + [\Delta D_{ps}]\{\phi\} + [\Delta D_{pp}]\{\phi_p'\} \\ & - ([D_{ps}] + [\Delta D_{ps}])([D_{ss}] + [\Delta D_{ss}])^{-1}([\Delta D_{sp}]\{\phi\} + [\Delta D_{sp}]\{\phi_p'\}) = \{0\} \end{aligned} \tag{56}$$

The last term of Eq.(56) contains the inverse of an unknown matrix and an iterative procedure is used to solve the nonlinear equation.

## 2. Modal Reduction

The major difficulty in the structural optimization of large systems is the huge computing effort necessary to evaluate the functions and their gradients with respect to the design variables. Hence, various reduction methods have been combined with the structural optimization successfully.

The equation of motion for the baseline system is shown in Eq.(11).

$$[k]\{\phi\} = \lambda [m]\{\phi\}$$

If the system is perturbed, the equation becomes

$$[k']\{\phi'\} = \lambda' [m']\{\phi'\} \quad (57)$$

where the perturbations are expressed as

$$\begin{aligned} [k'] &= [k] + [\Delta k] \\ [m'] &= [m] + [\Delta m] \\ \lambda' &= \lambda + \Delta\lambda \\ \{\phi'\} &= \{\phi\} + \{\Delta\phi\} \end{aligned} \quad (58)$$

The perturbed mode shapes can be approximated as a linear combination of  $p$  eigenvectors.

$$\{\phi'\} \cong [\Phi_p] \{C_p\} \quad (59)$$

In the mathematical programming for the structural optimization, the equality constraints is expressed as the equilibrium equation.

$$[\Phi_p]^T([k] + [\Delta k])[\Phi_p]\{C_p\} = (\lambda + \Delta\lambda) [\Phi_p]^T([m] + [\Delta m])[\Phi_p]\{C_p\} \quad (60)$$

The structural design variables are contained in  $[\Delta k]$  and  $[\Delta m]$ .  $\Delta\lambda$  is a characteristic design variable and the components in  $\{C_p\}$  are the free variables.

## Numerical Examples

### 1. Dynamic Analysis of Helicopter Tail-Boom

The tail-boom structure of Cobra helicopter is solved using the hybrid dynamic condensation. The finite element model consists of 28 nodes, 48 beam elements, and 8 concentrated masses. Figure 1 show the geometrical layout of the tail-boom.

Four nodes at the support are constrained. Hence, the structure has 144 unconstrained degrees of freedom. A finite element program, developed in house for general purpose, was employed for the dynamic analysis.

The geometric and material properties are as follows.

|                                |   |
|--------------------------------|---|
| cross-sectional area of beam   | $A = 4.1494 \times 10^2 \text{ mm}^2$                   |
| area moment of inertia of beam | $I = 5.35 \times 10^5 \text{ mm}^4$                     |
| Young's modulus                | $E = 7.2 \times 10^4 \text{ N/mm}^2$                    |
| mass density                   | $\rho = 2.77 \times 10^{-9} \text{ Nsec}^2/\text{mm}^4$ |
| concentrated mass              | $M = 2.268 \times 10^{-2} \text{ Nsec}^2/\text{mm}$     |

Fig. 1 Helicopter tail-boom

Table. 1 shows the eigenvalues of the system. The formulation of averaged mass is used. Since the tail-boom is nearly symmetric in transverse motions, many sets of close eigenvalues are found. The eight nodes carrying the concentrated masses are included in the analysis and the remaining ones are condensed out. The primary set contains the lateral translations and has 16 degrees of freedom.

Table. 1 Eigenvalues of tail-boom

| Mode | Eigenvalue            | Mode | Eigenvalue            |
|------|-----------------------|------|-----------------------|
| 1    | $1.72244 \times 10^3$ | 9    | $3.70145 \times 10^5$ |
| 2    | $1.79624 \times 10^3$ | 10   | $5.74278 \times 10^5$ |
| 3    | $8.76679 \times 10^3$ | 11   | $6.13599 \times 10^5$ |
| 4    | $1.48218 \times 10^4$ | 12   | $7.07847 \times 10^5$ |
| 5    | $1.51423 \times 10^4$ | 13   | $7.44577 \times 10^5$ |
| 6    | $6.28409 \times 10^4$ | 14   | $1.18694 \times 10^6$ |
| 7    | $1.93407 \times 10^5$ | 15   | $1.27885 \times 10^6$ |
| 8    | $3.67901 \times 10^5$ | 16   | $1.55295 \times 10^6$ |

To begin with, the shifting point is set to zero or  $\lambda_0 = 0$ . Then, the first step is static condensation, in which the relative errors range from  $10^{-3}$  to  $10^{-5}$  for the six lowest modes. Three subspace iterations are performed and the results are illustrated in Table. 2. In each subspace iteration, three inverse updates are used.

**Table. 2 Relative errors in eigenvalue**

| Mode | Errors                   | Mode | Errors                 |
|------|--------------------------|------|------------------------|
| 1    | $5.79 \times 10^{-11}$   | 9    | $1.63 \times 10^{-08}$ |
| 2    | $- 2.60 \times 10^{-11}$ | 10   | $2.81 \times 10^{-05}$ |
| 3    | $2.79 \times 10^{-11}$   | 11   | $1.30 \times 10^{-05}$ |
| 4    | $- 2.52 \times 10^{-08}$ | 12   | $3.98 \times 10^{-10}$ |
| 5    | $2.47 \times 10^{-08}$   | 13   | $5.42 \times 10^{-05}$ |
| 6    | $- 3.49 \times 10^{-11}$ | 14   | $1.12 \times 10^{-05}$ |
| 7    | $- 1.86 \times 10^{-10}$ | 15   | $2.32 \times 10^{-01}$ |
| 8    | $6.56 \times 10^{-11}$   | 16   | $1.47 \times 10^{-01}$ |

## 2. Structural Optimization of Aluminum Casting

As a practical problem of large size, redesign of a small aluminum casting is considered, as shown in Fig. 2. A total of 209 grid points and 312 finite elements are used for the finite element modeling. The elements are divided into 10 design groups reducing the number of structural design variables. The characteristic changes are to modify the fundamental frequency +30%.

**Fig. 2 Aluminum casting and design groups**

Table. 3 shows the rates of thickness change in each design group obtained through mathematical programming and the reanalysis results. The weight of the structure is to be minimized.

**Table. 3 Values of Design Variables**

| Design variables | $h' / h$                                 |
|------------------|--|
| $\alpha_1$       | 1.0804                                   |
| $\alpha_2$       | 1.1242                                   |
| $\alpha_3$       | 1.4779                                   |
| $\alpha_4$       | 1.0915                                   |
| $\alpha_5$       | 1.4568                                   |
| $\alpha_6$       | 1.4556                                   |
| $\alpha_7$       | 1.9265                                   |
| $\alpha_8$       | 0.9179                                   |
| $\alpha_9$       | 0.9493                                   |
| $\alpha_{10}$    | 0.9898                                   |
| <hr/>            |  |
| Goals            | $\omega' \times 10^{-3}$ 3.2136 (30.00%) |
| Reanalysis       | $\omega' \times 10^{-3}$ 3.1221 (26.29%) |
| Baseline system  | $\omega_o = 2.4720 \times 10^3$ rad/sec  |

### 3. Redesign of Curved Shell

A uniform curved shell segment is redesigned using the dynamic condensation. The axial length and outer radius are 76.2 mm. The plate thickness is 3.17 mm. The finite elements are divided into 8 design groups, as shown in Fig. 3.

**Fig. 3 Uniform shell and design group**

The design goal is to modify the fifth mode of chordwise bending. With two top points on

each side as a reference, the translational vibration of the top midpoint will be reduced. The primary set has 3 degrees of freedom and 357 unconstrained degrees of freedom are included in the secondary set. Plate thicknesses of the groups are the structural design variables. The total weight of the structure is to be minimized.

The frequency will be increased by 30% along with the mode shape change. Table. 4 shows the rates of the thickness changes of design groups. The reanalysis results are given in Table. 5.

**Table. 4 Structural changes ( $\Delta h/h$ )**

| Group     | 1, 8   | 2, 7   | 3, 6   | 4, 5   |
|-----------|--------|--------|--------|--------|
| Linear    | -0.300 | -0.200 | +0.174 | -0.300 |
| Nonlinear |        |        |        |        |
| first     | -0.326 | -0.183 | +0.214 | -0.353 |
| second    | -0.333 | -0.167 | +0.222 | +0.358 |
| third     | -0.333 | -0.165 | +0.223 | +0.358 |

**Table. 5 Characteristic Changes**

|                 | Mode shape<br>(top midpoint) | Frequency ( $\omega_5 \times 10^{-4}$ ) |
|-----------------|------------------------------|---|
| Baseline system | -0.46642                     | 3.3451                                  |
| Design goal     | -0.37314 (-20%)              | 4.3486 (+30%)                           |
| Linear          | -0.37173 (-20%)              | 4.0949 (+22%)                           |
| Nonlinear       |                              |   |
| first           | -0.36694 (-21%)              | 4.2858 (+28%)                           |
| second          | -0.37094 (-20%)              | 4.3446 (+30%)                           |
| third           | -0.37180 (-20%)              | 4.3486 (+30%)                           |

## Conclusions

Various reduction schemes for static and dynamic problems have been reviewed. As the dimension of the structural matrices, involved in the analysis and redesign, increases, the computational effort can be a huge burden to users. Hence, some types of system reduction are required for actual problems. The main concern is the computational efficiency and solution accuracy.

Most transformation methods which use a reduced subspace has a limitation in convergence. Therefore, the procedure of inverse iteration is employed in order to accelerate the convergence. At present, it is believed that the hybrid dynamic condensation is very promising for the general eigenproblems.

Reduction methods were applied successfully to the structural optimization. The nonlinear mathematical programming is used for the constrained minimization. More research works are necessary for the development of an efficient and accurate method for both analysis and design.

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### References

- Bathe, K.J., *Finite Element Procedures in Engineering Analysis*, Prentice-Hall, 1982.
- Guyan, R.J., "Reduction of Stiffness and Mass Matrices," *AIAA Journal*, Vol. 3, Feb. 1965, p. 380.
- Haug, E.J. and Arora, J.S., *Applied Optimal Design*, John Wiley & Sons, 1979.
- Kidder, R.L., "Reduction of Structural Frequency Equations," *AIAA Journal*, Vol. 11, June 1973, p. 892.
- Leung, Y.T., "An Accurate Method of Dynamic Condensation in Structural Analysis," *International Journal of Numerical Methods in Engineering*, Vol. 12, 1978, pp. 1705-1715.
- Kim, K. O. and W. J. Anderson, "Generalized Dynamic Reduction in Finite Element Dynamic Optimization," *AIAA Jnl* 22, 1616-1617 (1984).
- Kim, K. O., "Dynamic Condensation for Structural Redesign," *AIAA Jnl* 23, 1830-1831 (1985).
- Kim, K. O., "Hybrid Dynamic Condensation for Eigenproblems," *Computers & Structures*, Vol. 56, No. 1, 1995, pp. 105-112.
- "MSC/NASTRAN Handbook for Dynamic Analysis," The MacNeal-Schwendler Corp., Los Angeles, Calif., 1983.
- Matta, K.W., "Selection of Degrees of Freedom for Dynamic Analysis," *Journal of Pressure Vessel Technology*, Vol. 109, No. 1, 1987, pp. 65-69.
- Ong, J.H., "Improved Automatic Masters for Eigenvalue Economization," *Finite Elements in Analysis and Design*, Vol. 3, 1987, pp. 149-160.
- Paz, M., "Dynamic Condensation," *AIAA Journal*, Vol. 22, May 1984, pp. 724-727.
- Shah, V.N. and Raymund, M., "Analytical Selection of Masters for the Reduced Eigenvalue Problem," *International J. for Numerical Methods in Engineering*, Vol. 18, 1982, pp. 89-98.
- Suarez, L.E. and Singh, M.P., "Dynamic Condensation Method for Structural Eigenvalue Analysis," *AIAA Journal*, Vol. 30, April 1992, pp. 1046-1054.
- Woo, T.W., "Space Frame Optimization Subject to Frequency Constraints," *AIAA Journal*, Vol. 25, No. 10, October 1987, pp. 1396-1404.