A NEW UDB-MRL TEST WITH UNKNOWN CHANGE POINT

Myung Hwan Na
Statistical Research Center for Complex Systems, Seoul National University

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Abstract

The problem of trend change in the mean residual life is great interest in the reliability and survival analysis. In this paper, a new test statistic for testing whether or not the mean residual life changes its trend is developed. It is assumed that neither the change point nor the proportion at which the trend change occurs is known. The asymptotic null distribution of test statistic is established and asymptotic critical values of the asymptotic null distribution is obtained. Monte Carlo simulation is used to compare the proposed test with previously known tests.

1. Introduction

Let F be a continuous life distribution(i.e., $F(x)=0$ for $x\leq 0$) with the finite first moment and let X be a nonnegative random variable with distribution F. The mean residual life(MRL) function $e(x)$ is defined as

$$e(x)=E(X-x \mid X>x), \qquad (1.1)$$

The MRL function $e(x)$ in (1.1) can also be written as

$$e(x)=\frac{\int_x^\infty \overline{F}(u)\,du}{\overline{F}(x)},$$

where $\overline{F}(x)=1-F(x)$ is the reliability function.

The MRL function plays a very important role in the area of engineering, medical science, survival studies, social sciences, and many other fields. The MRL is used by engineers in burn-in studies, setting maintenance policies, and in comparison of life distributions of different systems. Social scientists use MRL, also called as inertia, in studies of lengths of wars, duration of strikes, job mobility etc. Medical researchers use MRL in lifetime experiments under various conditions. Actuaries apply MRL to setting rates and benefits for life insurance.

Guess and Proschan (1988) show that various families of life distributions defined in terms of the MRL(e.g. increasing MRL, decreasing MRL) have been used as models for lifetimes for which such prior information is available. One such family of distributions is called as "Upside-down bathtub MRL(UDB-MRL)" distributions if there exists a change point $\tau \geq 0$ such that

$$e(s) \leq e(t) \quad \text{for } 0 \leq s \leq t < \tau, \qquad (1.2)$$
$$e(s) \geq e(t) \quad \text{for } \tau \leq s \leq t.$$

The dual class of "decreasing initially, then increasing MRL (DIMRL)" distributions is obtained by reversing inequalities on the MRL function in (1.2). See Guess and Proschan(1988) and the references therein for examples and applications of the UDB-MRL(DIMRL) class.

It is well known that $e(x)$ is constant for all $x \geq 0$ if and only if F is an exponential distribution (i.e., $F(x) = 1 - \exp(-x/\mu)$ for $x \geq 0$, $\mu > 0$). Due to this "no-aging" property of the exponential distribution, it is of practical interest to know whether a given life distribution F is constant MRL or UDB-MRL. Therefore, we consider the problem of testing H_0: F is constant MRL, against H_1: F is UDB-MRL, (and not constant MRL), based on random samples. When the dual model is proposed, we test H_0 against H_1': F is UDB-MRL, (and not constant MRL).

Guess, Hollander and Proschan (1986) propose two test procedures for constant MRL against the trend change in MRL when the change point is known or when the proportion before the change occurs is known. Na et al. (1998) propose a test for the trend change in MRL when the change point is known. Aly (1990) suggests several tests for monotonicity of MRL. These tests consider the UDB-MRL alternatives when neither the change point nor the proportion is known. Hawkins, Kochar and Loader (HKL, 1992) develop a test for exponentiality against UDB-MRL alternative when neither the change point nor the proportion is known.

In this paper, we develop a new test statistic for testing H_0 against H_1(H_1') alternative. It is assumed that neither the change point nor the proportion at which the trend change occurs is known. The asymptotic null distribution of the proposed test statistic is derived. Monte Carlo simulations are conducted to compare the performance of our test statistics with those of Aly's (1990) and HKL's (1992) tests by the powers of tests. Section 2 is devoted to developing a family of test statistic for testing H_0 against H_1(H_1'). Results of simulations are presented in Section 3.

2. Test Statistic

The test statistic is motivated by a simple observation. If $e(x)$ is differentiable and decreasing(increasing), then

$$\frac{de(x)}{dx} = \frac{f(x)v(x) - \overline{F}^2(x)}{\overline{F}^2(x)} \leq (\geq) 0,$$

where $v(x) = \int_x^{\infty} \overline{F}(u)du$ and $f(x)$ denotes the probability density function corresponding to F. Thus $e(x)$ is decreasing(increasing) if and only if $f(x)v(x) \leq (\geq) \overline{F}^2(x)$. Hence, as a measure of the deviation from the null hypothesis H_0 in favor of H_1, we propose the parameter

$$T(F) = \sup\{\phi(x:F) \ : \ x \geq 0\}$$

where

$$\phi(x:F) = \int_0^x \overline{F}(t)(f(t)v(t) - \overline{F}^2(t))dt + \int_x^{\infty} \overline{F}(t)(\overline{F}^2(t) - f(t)v(t))dt.$$

Note that $\phi(x:F)$ is differentiable in $x > 0$ and $\frac{d}{dx}\phi(x:F) = 2\overline{F}^3(x)e'(x)$ clearly has the same sign as does $e'(x)$. Thus $\phi(x:F)$ is zero for the exponential distribution F and strictly positive for the UDB-MRL F. Using integration by parts, we can rewrite $\phi(x:F)$ as

$$\phi(x:F) = \frac{1}{2}\Big(\int_0^{\infty} \overline{F}(t)dt - 3\int_0^x \overline{F}^3(t)dt + 3\int_x^{\infty} \overline{F}^3(t)dt - 2\overline{F}^2(x)\int_x^{\infty} \overline{F}(t)dt\Big).$$

Let $F_n(x)$ be the empirical distribution formed by a random sample $X_1, \cdots, X_n$ from F and let $\overline{X}$ denote the sample mean. Then our family of test statistics is

$$T_n = \frac{\sqrt{n}\, T(F_n)}{\overline{X}}.$$

For computational purpose, T_n may be written as

$$T_n \equiv \frac{\max_{0 \leq k \leq n} \sqrt{n}(2\eta(k) - \eta(0))}{\overline{X}}$$

where for $k = 0, 1, \cdots, n$

$$\eta(k) = \frac{1}{2}\sum_{i=k}^{n-1}\left\{3\left(\frac{n-i}{n}\right)^3 - \left(\frac{n-k}{n}\right)^2\left(\frac{n-i}{n}\right)\right\} \times (X_{(i+1)} - X_{(i)}),$$

and $0 = X_{(0)} < X_{(1)} < \cdots < X_{(n)}$ denote the order statistics of the sample.

To establish the asymptotic null distribution of T_n, we use the differentiable statistical function approach of von Mises (1947) (cf. Boos and Serfling (1980) and Serfling (1980)) and the classical weak convergence of the empirical process. The asymptotic null distribution of T_n is given in Theorem 2.1.

THEOREM 2.1 Under H_0, i.e. F is exponential distribution with mean μ,

$$T_n \xrightarrow{d} Z^* \equiv \sup\{Z(p) : 0 \leq p \leq 1\},$$

where $Z(p)$ denotes a mean zero Gaussian process with covariance

$$\sigma(p, q) = \frac{1}{5}(1 + 2(1-q)^5 - 2(1-p)^5).$$

for $p \leq q$,

Proof. The proof of Theorem is similar to that of Theorem 2 in HKL (1992) and therefore omitted. □

Using Durbin (1985)'s approximation method, we can see that

$$\Pr\{T_n > c\} = \{2\sqrt{5}c + O(c^{-1})\}\phi(\sqrt{5}c) \quad \text{as } c \to \infty, \qquad (2.2)$$

where ϕ denotes the probability density function of the standard normal distribution. The argument in (2.2) is useful to calculate the approximated quantiles of the distributions of T_n. Table 2.1 shows the values of c with $\lambda \equiv \Pr\{T_n > c\} = 0.10$, 0.05 and 0.01.

<Table 2.1> Approximated quantiles of T_n

λ	0.10	0.05	0.01
quantiles of T_n	1.089	1.230	1.495

3. Simulation Study

In this section, we conduct a simulation study to compare the stability and the power of the proposed test statistic based on T_n with those of Aly's(1990) test based on U_n and HKL's(1992) test based on V_n. For Monte Carlo study we use the subroutine IMSL of the package FORTRAN.

To calculate the empirical size, the random numbers are generated from exponential distribution, $F(x) = 1 - \exp(-x)$, $x \geq 0$. Table 3.1 presents the empirical sizes of U_n, V_n and T_n. The figures in Tables 3.1 are the ratios of the rejection numbers of H_0 out of 1000 replications for the level of significance $\lambda = 0.10$, 0.05, 0.01, and sample size $n = 20, 40, \cdots, 100$. From Table 3.1, we can see that the empirical sizes of T_n approach the nominal level faster than the other test statistics: the sizes are close to the nominal level when $n \geq 30$. Note that U_n overestimates λ and has severe size distortions.

<Table 3.1> Empirical sizes of U_n, V_n and T_n

n	λ	U_n	V_n	T_n
20	.10	.139	.112	.085
	.05	.069	.052	.036
	.01	.024	.008	.011
40	.10	.148	.120	.110
	.05	.095	.061	.052
	.01	.029	.013	.011
60	.10	.145	.117	.105
	.05	.070	.053	.046
	.01	.015	.004	.005
80	.10	.153	.126	.106
	.05	.090	.060	.053
	.01	.024	.014	.011
100	.10	.142	.123	.098
	.05	.072	.054	.047
	.01	.017	.006	.007

In order to evaluate the empirical powers of the proposed test, the random numbers are generated from

$$\overline{F}_{\alpha,\beta,\gamma}(x) = \left\{\frac{\beta}{\beta+\gamma\exp(-\alpha x)(1-\exp(-\alpha x))}\right\} \times \left\{\frac{[1+d]^2-c^2}{[\exp(\alpha x)+d]^2-c^2}\right\}^{1/2\alpha\beta} \times \left\{\frac{\exp(\alpha x)+d-c}{\exp(\alpha x)+d+c}\,\frac{1+d+c}{1+d-c}\right\}^{\gamma/4\alpha\beta^2 c},$$

where $d=\gamma/2\beta$,

$c^2=[4(\beta/\gamma)+1]/[4(\beta/\gamma)^2]$.

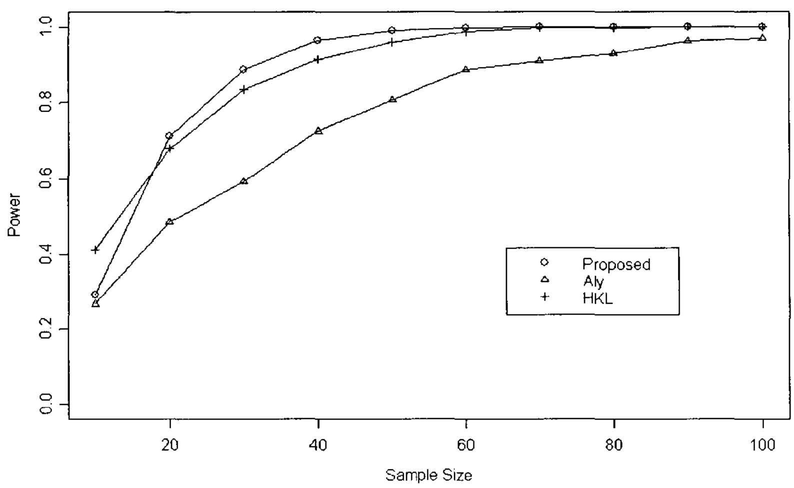
This distribution has MRL function

$$e_{\alpha,\beta,\gamma}(x)=\beta+\gamma\exp(-\alpha x)(1-\exp(-\alpha x)),\quad x\geq 0.$$

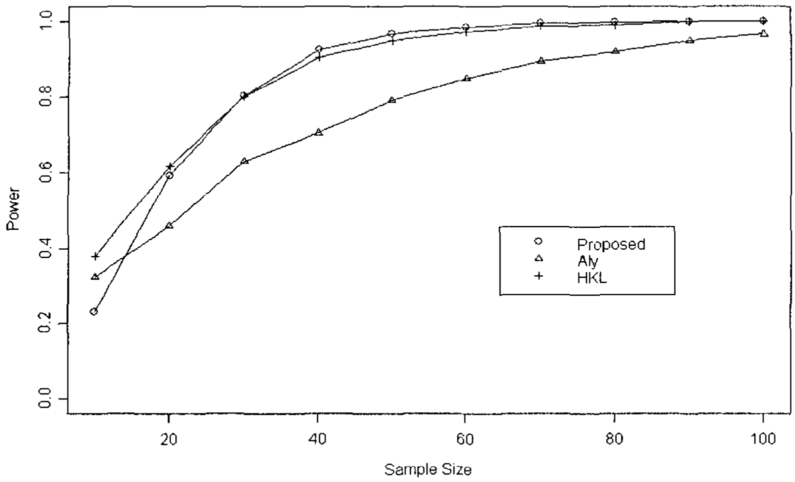
The motivation for choosing $\overline{F}_{\alpha,\beta,\gamma}$ is that $\overline{F}_{\alpha,\beta,\gamma}$ has UDB-MRL structure with the change point $\tau=\ln 2/\alpha$ for any choice of (α,β,γ) and $\overline{F}_{\alpha,\beta,\gamma}$ is exponential distribution if $\gamma=0$.

Figures 3.1~3.4 contain Monte Carlo estimated powers based on 1000 replications of sample size $n=10$, 20, ⋯, 100 from $\overline{F}_{\alpha,\beta,\gamma}$ for $\beta=1$, $\gamma=1$ and a selection of α when the level of significance is 0.05.

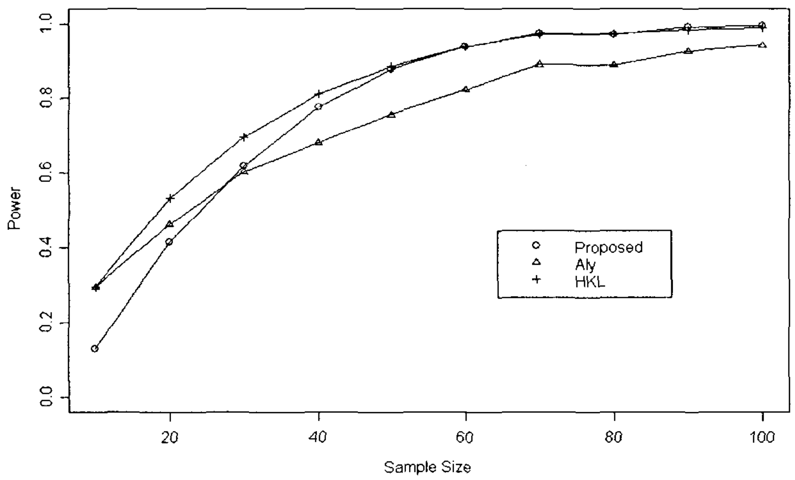
From figures, we notice that the powers of all tests increase as α increases (i.e., the change point τ decreases) when β and γ are fixed. Figures also show that the proposed test generally dominates the other tests except small α. Also the power of the proposed test increase more rapidly than those of the other tests as n increases for any α.



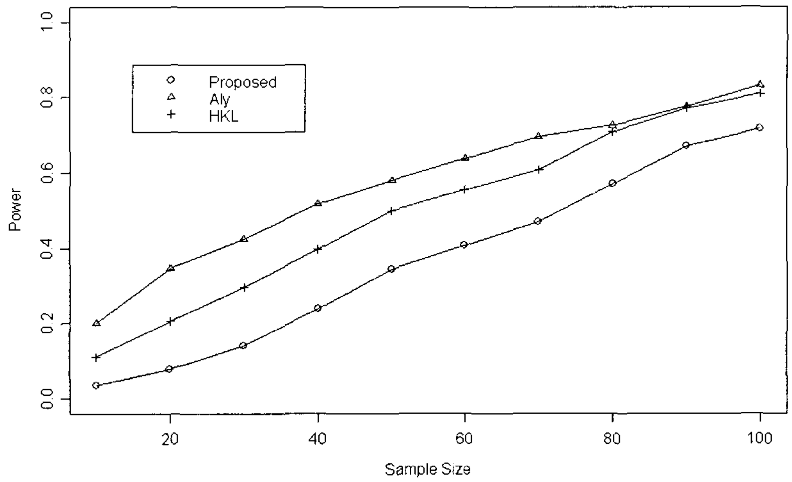
<Figure 3.1> Empirical power of T_n, U_n and V_n tests when testing against alternative $\overline{F}_{\alpha,\beta,\gamma}$ with $\alpha=7,\beta=1$ and $\gamma=1$.



<Figure 3.2> Empirical power of T_n, U_n and V_n tests when testing against alternative $\overline{F}_{\alpha,\beta,\gamma}$ with $\alpha=5, \beta=1$ and $\gamma=1$.



<Figure 3.3> Empirical power of T_n, U_n and V_n tests when testing against alternative $\overline{F}_{\alpha,\beta,\gamma}$ with $\alpha=3, \beta=1$ and $\gamma=1$.



<Figure 3.4> Empirical power of T_n, U_n and V_n tests when testing against alternative $\overline{F}_{\alpha,\beta,\gamma}$ with $\alpha=1, \beta=1$ and $\gamma=1$.

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