J. Korean Math. Soc. **39** (2002), No. 5, pp. 775–782

EXISTENCE RESULTS FOR SEMILINEAR DIFFERENTIAL EQUATIONS

R. SAKTHIVEL, Q. H. CHOI, AND T. JUNG

ABSTRACT. In this paper we prove the existence of mild solutions for semilinear differential equations in a Banach space. The results are obtained by using the semigroup theory and the Schaefer fixed point theorem. An example is provided to illustrate the theory.

1. Introduction

The problem of existence of mild solutions for differential and integrodifferential equations in abstract spaces have been studied by several authors [2, 6, 8, 10, 12, 14]. In [4] Frigon and Regan studied the existence of solutions of differential equations by using the Krasnoselski fixed point theorem. Lin and Liu [7] studied the existence of mild solution for autonomous integrodifferential equations in a Banach space by using the resolvent operators and the Banach fixed point theorem. Balachandran and Ilamaran [1] derived the existence of mild and strong solutions for semilinear evolution differential equations in Banach spaces. Dhakne and Pachpatte [3] studied the mild solutions for the functional integrodifferential equations in Banach spaces by using the Schaefer fixed point theorem. In this paper we study the existence of mild solutions for semilinear differential equations in a Banach space by using the semigroup theory and the Schaefer fixed point theorem.

Consider the semilinear differential equation of the form

$$\begin{aligned} (1) \qquad \frac{d}{dt}[x(t) + g(t, x(t))] &= Ax(t) + f(t, x(t)), \ t \in J = [0, b], \\ x(0) &= x_0, \end{aligned}$$

Received December 16, 2001.

2000 Mathematics Subject Classification: 34A05, 34G20, 34A09.

Key words and phrases: semilinear differential equation, mild solution, Schaefer fixed point theorem.

This work was supported by the Korea Research Foundation KRF-2001-015-DP0024.

where the state $x(\cdot)$ takes values in a Banach space X with the norm $\|\cdot\|$, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t), t \geq 0$ in the Banach space X, $f : J \times X \to X$, and $g : J \times X \to X$ are continuous functions. The case $g = 0$ has an extensive literature. The books of Pazy [9], Goldstein [5] and the references contained therein give good account of the most important results.

2. Preliminaries

In order to define the concept of mild solution for (1), by comparison with the abstract Cauchy problem

$$x'(t) = Ax(t) + f(t),$$

whose properties are well known [9], we associate problem (1) to the integral equation

$$\begin{aligned}(2)\quad x(t) = T(t)[x_0 + g(0, x_0)] - g(t, x(t)) - \int_0^t AT(t-s)g(s, x(s))ds \\ + \int_0^t T(t-s)f(s, x(s))ds.\end{aligned}$$

DEFINITION. A function $x \in C([0, b), X)$ is a mild solution of the Cauchy problem (1) if the following holds: $x(0) = x_0$; for each $0 \leq t < b$ and $s \in [0, t)$, the function $AT(t-s)g(s, x(s))$ is integrable and the integral equation (2) is satisfied.

We need the following fixed point theorem due to Schaefer [11]

SCHAEFER'S THEOREM. *Let E be a normed linear space. Let $F : E \to E$ be a completely continuous operator, i.e., it is continuous and the image of any bounded set is contained in a compact set, and let*

$$\zeta(F) = x \in E; x = \lambda Fx \qquad \textit{for some} \qquad 0 < \lambda < 1.$$

Then either $\zeta(F)$ is unbounded or F has a fixed point.

We assume the following hypotheses:

(i) A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in X such that

$$\|T(t)\| \leq M_1, \text{ for some } M_1 \geq 1 \text{ and } \|AT(t)\| \leq M_2,\ M_2 \geq 0.$$

(ii) For each $t \in J$ the function $f(t, .) : X \to X$ is continuous, and for each $x \in X$ the function $f(., x) : J \to X$ is strongly measurable.

(iii) For every positive integer k there exists $\alpha_k \in L^1(0,b)$ such that

$$\sup_{\|x\|\leq k} \|f(t,x)\| \leq \alpha_k(t), \text{ for } t \in J \text{ a.e.}$$

(iv) The function g is completely continuous and uniformly bounded, there exists a constant $M_3 > 0$ such that

$$\|g(t,x)\| \leq M_3, \quad t \in J, \quad x \in X.$$

(v) There exists an integrable function $m : [0,b] \to [0,\infty)$ such that

$$\|f(t,x)\| \leq m(t)\Omega(\|x\|), \ 0 \leq t \leq b, \ x \in X,$$

where $\Omega : [0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(vi)

$$M_1 \int_0^b m(s)ds < \int_c^\infty \frac{ds}{\Omega(s)},$$

where $c = M_1[\|x_0\| + M_3] + M_3 + M_2M_3b$.

3. Existence result

THEOREM. *If the assumptions* (i) *to* (vi) *are satisfied, then the problem* (1) *has a mild solution on* $[0,b]$.

Proof. Consider the space $C = C(J,X)$, the Banach space of all continuous functions from J into X with sup norm.

To prove the existence of a mild solution of (1) we apply Schaefer's theorem. First we obtain a *priori* bounds for the solutions of the following problem, as in [8]

$$(3) \quad \frac{d}{dt}[x(t)+\lambda g(t,x(t))] = Ax(t) + \lambda f(t,x(t)), \ \lambda \in (0,1), \ t \in J=[0,b],$$
$$x(0) = \lambda x_0.$$

Let x be a mild solution of the problem (3). From

$$\begin{aligned} x(t) &= \lambda T(t)[x_0 + g(0,x_0)] - \lambda g(t,x(t)) - \lambda \int_0^t AT(t-s)g(s,x(s))ds \\ &\quad + \lambda \int_0^t T(t-s)f(s,x(s))ds. \end{aligned}$$

From the assumptions, we have

$$\begin{aligned} \|x(t)\| &\leq M_1[\|x_0\| + M_3] + M_3 + M_2M_3b \\ &\quad + M_1 \int_0^t m(s)\Omega(\|x(s)\|)ds. \end{aligned}$$

Denoting by $u(t)$ the right-hand side of the above inequality we have $c = u(0) = M_1[\|x_0\| + M_3] + M_3 + M_2M_3b$, $\|x(t)\| \le u(t)$, $0 \le t \le b$ and

$$\begin{aligned} u'(t) &= M_1 m(t)\Omega(\|x(t)\|) \\ &\le M_1 m(t)\Omega(u(t)). \end{aligned}$$

This implies

$$\int_{u(0)}^{u(t)} \frac{ds}{\Omega(s)} \le M_1 \int_0^b m(s)ds < \int_c^\infty \frac{ds}{\Omega(s)}, \quad 0 \le t \le b.$$

This inequality implies that $u(t) < \infty$. So, there is a constant K such that $u(t) \le K, t \in [0,b]$ and hence $\|x(t)\| \le K, t \in [0,b]$, where K depends only on b and on the functions m and Ω.

In the second step we must prove that the operator $F : C \to C$ defined by

$$\begin{aligned} (Fx)(t) &= T(t)[x_0 + g(0,x_0)] - g(t,x(t)) - \int_0^t AT(t-s)g(s,x(s))ds \\ &\quad + \int_0^t T(t-s)f(s,x(s))ds \end{aligned}$$

is a completely continuous operator.

Let $B_k = \{x \in C : \|x\| \le k\}$ for some $k \ge 1$. We first show that F maps B_k into an equicontinuous family. Let $x \in B_k$ and $t_1, t_2 \in J$. Then if $0 < t_1 < t_2 \le b$,

$$\begin{aligned} &\|(Fx)(t_1) - (Fx)(t_2)\| \\ &\le \|T(t_1) - T(t_2)\|\|x_0 + g(0,x_0)\| + \|g(t_1,x(t_1)) - g(t_2,x(t_2))\| \\ &\quad + \|\int_0^{t_1} A[T(t_1-s) - T(t_2-s)]g(s,x(s))ds\| \\ &\quad + \|\int_{t_1}^{t_2} AT(t_2-s)g(s,x(s))ds\| \\ &\quad + \|\int_0^{t_1} [T(t_1-s) - T(t_2-s)]f(s,x(s))ds\| \\ &\quad + \|\int_{t_1}^{t_2} T(t_2-s)f(s,x(s))ds\| \end{aligned}$$

$$\leq \|T(t_1) - T(t_2)\| \|x_0 + g(0, x_0)\| + \|g(t_1, x(t_1)) - g(t_2, x(t_2))\|$$
$$+ \int_0^{t_1} \|A[T(t_1 - s) - T(t_2 - s)]\| M_3 ds$$
$$+ \int_{t_1}^{t_2} \|AT(t_2 - s)\| M_3 ds$$
$$+ \int_0^{t_1} \|T(t_1 - s) - T(t_2 - s)\| \alpha_k(s) ds$$
$$+ \int_{t_1}^{t_2} \|T(t_2 - s)\| \alpha_k(s) ds.$$

The right hand side is independent of $x \in B_k$ and tends to zero as $t_2 - t_1 \to 0$, since g is completely continuous and the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Thus F maps B_k into an equicontinuous family of functions.

Notice that we considered here only the case $0 < t_1 < t_2$, since the other cases $t_1 < t_2 < 0$ or $t_1 < 0 < t_2$ are very similar.

It is easy to see that the family FB_k is uniformly bounded. Next, we show $\overline{FB_k}$ is compact. Since we have shown FB_k is equicontinuous collection, it suffices by the Arzela-Ascoli theorem to show that F maps B_k into a precompact set in X.

Let $0 < t \leq b$ be fixed and ϵ a real number satisfying $0 < \epsilon < t$. For $x \in B_k$ we define

$$\begin{aligned}
(F_\epsilon x)(t) &= T(t)[x_0 + g(0, x_0)] - g(t, x(t)) \\
&\quad - \int_0^{t-\epsilon} AT(t - s) g(s, x(s)) ds \\
&\quad + \int_0^{t-\epsilon} T(t - s) f(s, x(s)) ds \\
&= T(t)[x_0 + g(0, x_0)] - g(t, x(t)) \\
&\quad - T(\epsilon) \int_0^{t-\epsilon} AT(t - s - \epsilon) g(s, x(s)) ds \\
&\quad + T(\epsilon) \int_0^{t-\epsilon} T(t - s - \epsilon) f(s, x(s)) ds.
\end{aligned}$$

Since $T(t)$ is a compact operator, the set $Y_\epsilon(t) = \{(F_\epsilon x)(t) : x \in B_k\}$ is precompact in X for every ϵ, $0 < \epsilon < t$. Moreover for every $x \in B_k$ we have

$$\begin{aligned}\|(Fx)(t)-(F_\epsilon x)(t)\| &\leq \int_{t-\epsilon}^{t}\|AT(t-s)g(s,x(s))\|ds \\ &\qquad + \int_{t-\epsilon}^{t}\|T(t-s)f(s,x(s))\|ds \\ &\leq \int_{t-\epsilon}^{t}\|AT(t-s)\|M_3 ds \\ &\qquad + \int_{t-\epsilon}^{t}\|T(t-s)\|\alpha_k(s)ds.\end{aligned}$$

Therefore there are precompact sets arbitrarily close to the set $\{(Fx)(t) : x \in B_k\}$. Hence the set $\{(Fx)(t) : x \in B_k\}$ is precompact in X.

It remains to show that $F : C \to C$ is continuous. Let $\{x_n\}_0^\infty \subseteq C$ with $x_n \to x$ in C. Then there is an integer r such that $\|x_n(t)\| \leq r$ for all n and $t \in J$, so $x_n \in B_r$ and $x \in B_r$. By (ii)$f(t,x_n(t)) \to f(t,x(t))$ for each $t \in J$ and since $\|f(t,x_n(t)) - f(t,x(t))\| \leq 2\alpha_r(t)$, and also g is completely continuous, we have by dominated convergence theorem

$$\begin{aligned}\|Fx_n - Fx\| &= \sup_{t\in J}\|[g(t,x_n(t)) - g(t,x(t))] \\ &\quad + \int_0^t AT(t-s)[g(s,x_n(s)) - g(s,x(s))]ds \\ &\quad + \int_0^t T(t-s)[f(s,x_n(s)) - f(s,x(s))]ds\| \\ &\leq \|g(t,x_n(t)) - g(t,x(t))\| \\ &\quad + \int_0^b \|AT(t-s)\|\|g(s,x_n(s)) - g(s,x(s))\|ds \\ &\quad + \int_0^b \|T(t-s)\|\|f(s,x_n(s)) - f(s,x(s))\| \\ &\quad \to 0 \text{ as } n \to \infty.\end{aligned}$$

Thus F is continuous. This completes the proof that F is completely continuous.

Finally the set $\zeta(F) = \{x \in C : x = \lambda Fx, \lambda \in (0,1)\}$ is bounded, as we proved in the first step. Consequently by Schaefer's theorem the operator F has a fixed point in C. This means that the problem (1) has a mild solution on J. □

4. Example

Consider the following partial differential equation of the form

$$
\begin{aligned}
(4)\quad \frac{\partial}{\partial t}[z(t,y)+\mu_1(t,z(t,y))] &= \frac{\partial^2}{\partial y^2}z(t,y)+\mu_2(t,z(t,y)),\\
z(t,0) &= z(t,1)=0,\\
z(0,y) &= z_0(y),\ 0<y<1,\ t\in J=[0,1].
\end{aligned}
$$

Take $X = L^2(J)$ and let $f(t,w)(y) = \mu_2(t,w(y))$ and $g(t,w)(y) = \mu_1(t,w(y))$.
Let $A : X \to X$ be defined by $Aw = w''$ with domain $D(A) = \{w \in X : w, w' \text{ are absolutely continuous},\ w'' \in X,\ w(0) = w(1) = 0\}$.
Then

$$Aw = \sum_{n=1}^{\infty}(-n^2)(w,w_n)w_n,\ w \in D(A),$$

where $w_n(y) = \sqrt{2}\sin ny\,(n = 1,2,3,...)$is the orthogonal set of eigenvectors of A.

It is well known that A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$ in X and is given by [13]

$$T(t)w = \sum_{n=1}^{\infty}\exp(-n^2t)(w,w_n)w_n,\quad w \in X.$$

Since the analytic semigroup $T(t)$ is compact, there exist constants $N \geq 1$ and $N_1 > 0$ such that $\|T(t)\| \leq N$ and $\|AT(t)\| \leq N_1$ for each $t \geq 0$.

Further the function $\mu_1 : J \times X \to X$ is completely continuous and there exists a constant $k_1 > 0$ such that

$$\|\mu_1(t,w)\| \leq k_1,$$

and also there exists an integrable function $l : J \to [0,\infty)$ such that

$$\|\mu_2(t,w)\| \leq l(t)\Omega_1(\|w\|),$$

where $\Omega_1 : [0,\infty) \to (0,\infty)$ is continuous and nondecreasing and

$$N\int_0^1 l(s)ds < \int_c^{\infty}\frac{ds}{\Omega_1(s)},$$

where $c = N[\|z_0\| + k_1] + k_1 + k_1N_1$.

Further, all the conditions of the above theorem are satisfied. Hence, the equation (4) has a mild solution on J.

References

[1] K. Balachandran and S. Ilamaran, *Existence and uniqueness of mild solutions of a semilinear evolution equation with nonlocal conditions,* Indian J. Pure Appl. Math. **25** (1994), 411–418.

[2] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem,* J. Math. Anal. Appl. **162** (1991), 494–505.

[3] M. B. Dhakne and B. G. Pachpatte, *On some abstract functional integrodifferential equations,* Indian J. Pure Appl. Math. **22** (1991), 109–134.

[4] M. Frigon and D. O. Regan, *Existence results for initial value problems in Banach spaces,* Differential Equations Dynam. Systems **2** (1994), 41–48.

[5] A. J. Goldstein, *Semigroups of linear operators and applications,* The Clarendon Press, Oxford University Press, New York, 1985.

[6] E. Hernandez, *Existence results for a class of semilinear evolution equations,* Electron. J. Differential Equations **24** (2001), 1–14.

[7] Y. Lin and J. H. Liu, *Semilinear integrodifferential equations with nonlocal Cauchy problem,* Nonlinear Anal. **26** (1996), 1023–1033.

[8] S. K. Ntouyas and P. Tsamatos, *Global existence for semilinear evolution equations with nonlocal conditions,* J. Math. Anal. Appl. **210** (1997), 679 –687.

[9] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Springer-Verlag, New York, 1983.

[10] S. M. Rankin, *Semilinear evolution equation in Banach spaces with applications to parabolic partial differential equation,* Trans. Amer. Math. Soc. **336** (1993), 523–535.

[11] H. Schaefer, *Uber die methode der a priori schranken,* Math. Ann. **129** (1955), 415–416.

[12] J. S. Shin, *An existence theorem of a functional differential equation,* Funkcial. Ekvac. **30** (1987), 19–29.

[13] C. C. Travis and G. F. webb, *Existence and stability for partial functional integrodifferential equations,* Trans. Amer. Math. Soc. **200** (1974), 395–418.

[14] G. F. Webb, *Growth bounds of solutions of abstract nonlinear differential equations,* Differential Integral Equations **7** (1994), 1145–1152.

R. Sakthivel and Q. H. Choi
Department of Mathematics
Inha University
Inchon 402-751, Korea

T. Jung
Department of Mathematics
Kunsan National University
Kunsan 573-701, Korea
E-mail: krsakthivel@yahoo.com