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ON A TIME-CONSISTENT SOLUTION OF A COOPERATIVE DIFFERENTIAL TIME-OPTIMAL PURSUIT GAME

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ABSTRACT. In this paper we study a time-optimal model of pursuit in which the players move on a plane with bounded velocities. This game is supposed to be a nonzero-sum group pursuit game. The main point of the work is to construct and compare cooperative and non-cooperative solutions in the game and make a conclusion about cooperation possibility in differential pursuit games. We consider all possible cooperations of the players in the game. For that purpose for every game $\Gamma(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ we construct the corresponding game in characteristic function form $\Gamma_v(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$. We show that in this game there exists the nonempty core for any initial positions of the players. The core can take four various forms depending on initial positions of the players. We study how the core changes when the game is proceeding. For the original agreement (an imputation from the original core) to remain in force at each current instant t it is necessary for the core to be time-consistent. Nonemptiness of the core in any current subgame constructing along a cooperative trajectory and its time-consistency are shown. Finally, we discuss advantages and disadvantages of choosing this or that imputation from the core.

1. Introduction

The process of pursuit represents a typical conflict situation. When only two players are involved in the process of pursuit we deal with a classical zero-sum differential pursuit game. These games grew out of the problem of setting and solving military pursuit games and were

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developed by Isaaks [1]. In the case when more than two players participate in the game and the players' objectives are not strictly opposite it would be rather reasonable to consider such a game as a non zero-sum one. Although even in this case some game theorists used zero-sum models dividing all the players into two groups with opposite interests (see Chikrii [4]).

In contract of this approach to the problem of pursuit we consider a group pursuit game as nonzero-sum (see Petrosjan [6], Tarashnina [10]). Earlier differential games are used to solve military problems. In this paper we expand horizons and consider a nonzero-sum pursuit game. It is obvious that players' goals are not always strictly opposed. We wont to illustrate how differential games can be used for solving economic problems. In this kind of games under "capture" we understand just meeting of players and delivering some goods or information. In other words, players are not aimed to destruct each other. That is so called nonzero-sum pursuit games.

In order to investigate this nonzero-sum game we construct its TU-cooperative version and handle two different tasks: the cooperative one and the noncooperative one. The key moment of this paper is comparing and analyzing of these two solutions.

In the framework of classical cooperative game theory with transferable utilities many optimality principles have been discovered, and among them there is the famous concept of the *core.*

The core was proposed by Gillies [5]. Scarf [8] showed that the core is not empty for a class of convex superadditive games in characteristic function form. Some generalizations of Scarf's result were given by Billera [2] and Shapley [9]. The necessary and sufficient condition for the core to be nonempty was independently formulated by Bondareva [3] and Shapley [8].

In this paper we apply as to the problem of existence of the nonempty core for the considered model of pursuit as to the not less important time-consistency problem.

In dynamic games the property that provides for an optimality principle to be feasible throughout the game is very important. This requirement is called *time-consistency* of a solution of the game. This property and the connected with it imputation distribution procedure (IDP) were introduced by Petrosjan [6].

Moving along cooperative trajectory, the players on some sense travel over the subgames. They differ from each other with initial states and duration. It is obvious that when time is passing either the players'opportunities or the players'interests may change. Therefore, at some instant t, being in the corresponding current subgame, the originally adopted optimal solution may either not exist or not satisfy the players'interests any more. In other words, *time-consistency of a solution of a differential game* means that at each time instant within the game the players do not have any reasons to deviate from the originally adopted "optimal" behaviour.

2. Time-optimal model of pursuit as a normal form game

The game under study is a time-optimal model of pursuit in which three players — two pursuers P_1, P_2 and an evader E — move on a plane with bounded velocities. The players P_1, P_2 and E start their motion at the positions $\mathbf{x}_0$, $\mathbf{y}_0$ and $\mathbf{z}_0$ respectively, and have the possibility of making decisions continuously in time. At each instant they may choose directions of their motion (velocity vectors) and velocities within prescribed limits.

In this case, the motion of the players is described by following system of differential equations

$$\text{(1)} \qquad \begin{array}{lll} \text{for } P_1: & \dot{\mathbf{x}} = \mathbf{u}_{P_1}, & \mathbf{u}_{P_1} \in U_{P_1}; \\ \text{for } P_2: & \dot{\mathbf{y}} = \mathbf{u}_{P_2}, & \mathbf{u}_{P_2} \in U_{P_2}; \\ \text{for } E: & \dot{\mathbf{z}} = \mathbf{u}_{E}, & \mathbf{u}_{E} \in U_{E} \end{array}$$

with initial conditions

$$\text{(2)} \qquad \mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{y}(0) = \mathbf{y}_0, \quad \mathbf{z}(0) = \mathbf{z}_0,$$

where ϱ_1, ϱ_2, σ are constants, $\min\{\varrho_1, \varrho_2\} > \sigma > 0$; $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$, and the sets U_{P_1}, U_{P_2}, U_{P_3} have the forms

$$\begin{aligned} U_{P_i} &= \left\{\mathbf{u}_{P_i} = (u^1_{P_i}, u^2_{P_i}) \colon \left(u^1_{P_i}\right)^2 + \left(u^2_{P_i}\right)^2 \le \varrho_i^2\right\}, \quad i = 1, 2, \\ U_E &= \left\{\mathbf{u}_E = (u^1_E, u^2_E) \colon \left(u^1_E\right)^2 + \left(u^2_E\right)^2 \le \sigma^2\right\}. \end{aligned}$$

Here the vector-parameters $\mathbf{u}_{P_1}$, $\mathbf{u}_{P_2}$ and $\mathbf{u}_E$ are called *controls* of the players in the game.

Now we need to point a method of choosing the controls $\mathbf{u}_{P_i} \in U_{P_i}$ $(i = 1, 2)$, $\mathbf{u}_E \in U_E$ by the players throughout the game according to incoming information. We assume that the pursuers use strategies with discrimination against the evader. This means that at each instant t the players P_1 and P_2 require additional information about the value of the vector-parameter $\mathbf{u}_E$ chosen by the evader E at the same instant t. In such a situation, the evader is said to be discriminated. The evader E uses piecewise open-loop strategies. Denote as $\mathcal{U}_{P_1}$, $\mathcal{U}_{P_2}$ and $\mathcal{U}_E$ the admissible strategy sets of the players P_1, P_2 and E respectively.

The functions $x(\cdot), y(\cdot), z(\cdot)$ which satisfy equations (1) and initial conditions (2) in the situation $\Big(u_{P_1}(\cdot), u_{P_2}(\cdot), u_E(\cdot)\Big)$ are called *trajectories* for the players P_1, P_2 and E.

Now we introduce the surfaces $F_1 = \Big\{(\mathbf{x},\mathbf{z})|\ \|\mathbf{x}-\mathbf{z}\| = 0\Big\}$ and $F_2 = \Big\{(\mathbf{y},\mathbf{z})|\ \|\mathbf{y}-\mathbf{z}\| = 0\Big\}$ which are given in $\mathbb{R}^2 \times \mathbb{R}^2$. We shall say that the surface $F = F_1 \bigcup F_2$ is the terminal surface in the game $\Gamma(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$. The end of the game is determined with the help of the terminal surface F. Denote

$$t_{P_1}\big(\mathbf{x}_0, \mathbf{z}_0; u_{P_1}(\cdot), u_E(\cdot)\big) = \min\Big\{t:\ \big(x(t), z(t)\big) \in F_1\Big\},$$

$$t_{P_2}\big(\mathbf{y}_0, \mathbf{z}_0; u_{P_2}(\cdot), u_E(\cdot)\big) = \min\Big\{t:\ \big(y(t), z(t)\big) \in F_2\Big\}.$$

If there is no t such that $\big(x(t), z(t)\big) \in F_1$, then $t_{P_1}\big(\mathbf{x}_0, \mathbf{z}_0; u_{P_1}(\cdot), u_E(\cdot)\big)$ is supposed to be equal $+\infty$. Similarly, if there is no t such that $\big(y(t), z(t)\big) \in F_2$ then $t_{P_2}\big(\mathbf{y}_0, \mathbf{z}_0; u_{P_2}(\cdot), u_E(\cdot)\big) = +\infty$.
Let $t_E\big(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0; u_{P_1}(\cdot), u_{P_2}(\cdot), u_E(\cdot)\big) = \min\Big\{t_{P_1}, t_{P_2}\Big\}$.

In this differential time-optimal pursuit game we set the payoff for the player E to be

$$\begin{aligned} &K_E\big(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0; u_{P_1}(\cdot), u_{P_2}(\cdot), u_E(\cdot)\big) \\ &\quad = \gamma t_E\big(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0; u_{P_1}(\cdot), u_{P_2}(\cdot), u_E(\cdot)\big). \end{aligned} \tag{3}$$

This means that the payoff of the player E is the first moment when he meets any of the pursuers (his capture time) multiplied by number $\gamma > 0$. Here γ is a price of a time unit.

The payoff function to the player P_i $(i = 1, 2)$ is given as follows

$$\begin{aligned} &K_{P_i}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0; u_{P_1}(\cdot), u_{P_2}(\cdot), u_E(\cdot)) \\ &\quad = -\gamma t_E(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0; u_{P_1}(\cdot), u_{P_2}(\cdot), u_E(\cdot)). \end{aligned} \tag{4}$$

The objective of each player is to maximize his own payoff function. In other words, all this means that each pursuer has a reason to meet the evader before the other does.

So, we define the nonzero-sum pursuit game as a normal form game as follows

$$\Gamma(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) = \Big\langle \mathbf{N}, \{\mathcal{U}_i\}_{i\in\mathbf{N}}, \{K_i\}_{i\in\mathbf{N}}\Big\rangle,$$

where $\mathbf{N} = \{P_1, P_2, E\}$ is the set of players; $\mathcal{U}_i$ is the set of admissible strategies of the player i; K_i is a payoff function of the i-th player.

This game depends on the initial conditions $\mathbf{x}_0$, $\mathbf{y}_0$, $\mathbf{z}_0$, therefore, it is denoted by $\Gamma(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$. Indeed, we define a whole family of games depending on parameters $\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0$.

This game can be interpreted in the following way: imagine that the evader has something what both pursuers need to have. It can be a kind of good or information. Moreover, it is supposed the information to be disappeared once any of the pursuers reaches the evader. Thus each pursuer wants to get it before the other does. It seems quite interesting to consider all possible cooperation between the players in this game, assuming the payoffs to be transferable. It would be rather helpful to know what the best way for the pursuers to share the evader is: whether to cooperate to each other, or to try to win over the evader to his side, or to form the grand coalition of three players. So, the key point of this work is to find the cooperative and noncooperative solutions and compare them. With this aim with every game $\Gamma(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ we associate the corresponding game in characteristic function form $\Gamma_v(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$.

3. A cooperative form of the game $\Gamma(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$

Now we introduce a cooperative form of the game $\Gamma(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$. Assume that utility of any player is transferable.

Let $2^{\mathbf{N}}$ be the set of all subsets of $\mathbf{N}$. The function $v\colon 2^N \to \mathbb{R}^1$ with the following two properties

1. $v(\emptyset)$=0, where $\emptyset$ is an empty set,
2. $v(\mathbf{S} \cup \mathbf{R}) \geq v(\mathbf{S}) + v(\mathbf{R})$ for all $\mathbf{R}, \mathbf{S} \subset \mathbf{N}$ with $\mathbf{S} \bigcap \mathbf{R} = \emptyset$,

is called *the characteristic function* (c.f.). Condition 2 is the superadditivity property.

For any coalition $\mathbf{S} \subset \mathbf{N}$ we define the characteristic function as follows

$$v(\mathbf{S}) = \max_{u_{\mathbf{S}}} \min_{u_{\mathbf{N}\setminus\mathbf{S}}} \sum_{i \in \mathbf{S}} K_i\big(u_{\mathbf{S}}, u_{\mathbf{N}\setminus\mathbf{S}}\big),$$

where $u_{\mathbf{S}}$ and $u_{\mathbf{N}\setminus\mathbf{S}}$ are vectors of admissible strategies of the coalitions $\mathbf{S}$ and $\mathbf{N} \setminus \mathbf{S}$ respectively. Using this approach, we construct the characteristic function for the game $\Gamma(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ in the following form:

$$v(\{P_1\}; \mathbf{x}_0, \mathbf{z}_0) = -\gamma \mathrm{Val}\ \Gamma_{P_1 \setminus E}(\mathbf{x}_0, \mathbf{z}_0) = \frac{\|\mathbf{x}_0 - \mathbf{z}_0\|}{\varrho_1 - \sigma},$$

where $\Gamma_{P_1\backslash E}(\mathbf{x}_0, \mathbf{z}_0)$ is the time-optimal zero-sum game of pursuit played between P_1 and E. Similarly, we define the c.f.'s value for the coalition $\mathbf{S} = \{P_2\}$. Therefore,

$$v(\{P_2\}; \mathbf{y}_0, \mathbf{z}_0) = -\gamma \text{Val } \Gamma_{P_2\backslash E}(\mathbf{y}_0, \mathbf{z}_0) = -\gamma \frac{\|\mathbf{y}_0 - \mathbf{z}_0\|}{\varrho_2 - \sigma}.$$

On account of (4), the c.f.'s value for $\mathbf{S} = \{P_1, P_2\}$ is introduced by the following way

$$v(\{P_1, P_2\}; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) = -\gamma \text{Val } \Gamma_{\mathbf{P}\backslash E}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0),$$

where Val $\Gamma_{\mathbf{P}\backslash E}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ is the value of the pursuit game played between two players $\mathbf{P} = \{P_1, P_2\}$ and E.

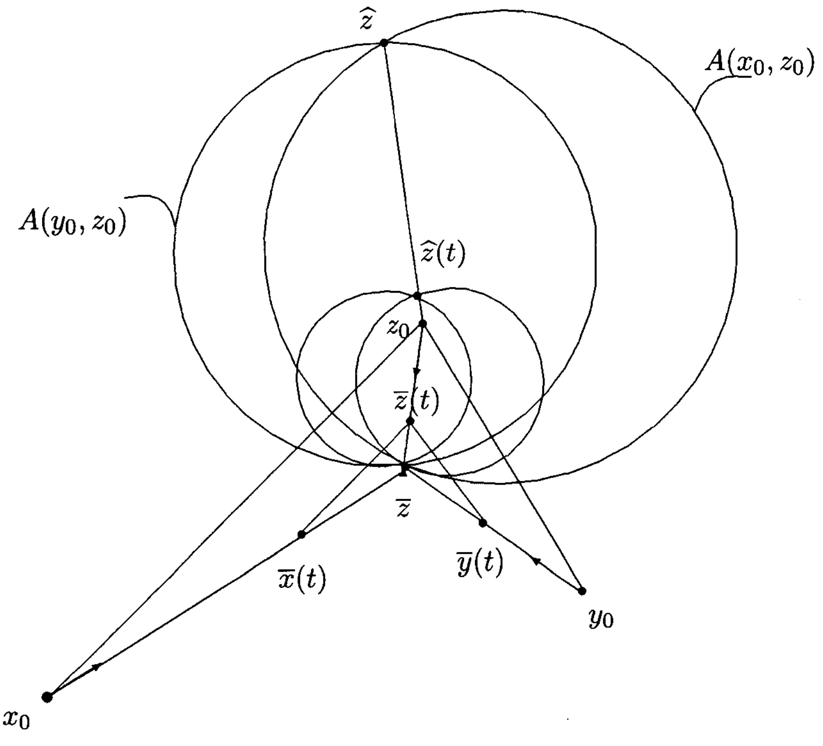

FIGURE 1.

The solution of this game is the following: all the players move to the point $\widehat{\mathbf{z}}$ (see Fig. 1) such that

$$(5) \qquad \|\mathbf{z}_0 - \widehat{\mathbf{z}}\| = \max_{\mathbf{z} \in L(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)} \|\mathbf{z}_0 - \mathbf{z}\|,$$

where $L(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) = A(\mathbf{x}_0, \mathbf{z}_0) \bigcap A(\mathbf{y}_0, \mathbf{z}_0)$, and both $A(\mathbf{x}_0, \mathbf{z}_0)$ and $A(\mathbf{y}_0, \mathbf{z}_0)$ are Apollonius circles in the games $\Gamma_{P_1 \setminus E}(\mathbf{x}_0, \mathbf{z}_0)$ and $\Gamma_{P_2 \setminus E}$ $(\mathbf{y}_0, \mathbf{z}_0)$ respectively. Note that in order to exclude trivial cases we consider only those initial positions $\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0$ for which $L(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) \neq \emptyset$, $A(\mathbf{x}_0, \mathbf{z}_0) \not\subset A(\mathbf{y}_0, \mathbf{z}_0)$, $A(\mathbf{y}_0, \mathbf{z}_0) \not\subset A(\mathbf{x}_0, \mathbf{z}_0)$. Taking into account (5), we get

$$v(\{P_1, P_2\}; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) = -2\gamma \frac{\|\mathbf{z}_0 - \widehat{\mathbf{z}}\|}{\sigma}.$$

As above, in view of (3) we have

$$v(\{E\}; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) = \gamma \mathrm{Val}\ \Gamma_{\mathbf{P} \setminus E}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) = \gamma \frac{\|\mathbf{z}_0 - \widehat{\mathbf{z}}\|}{\sigma}.$$

It is obvious that

$$v(\{P_1, E\}; \mathbf{x}_0, \mathbf{z}_0) = 0, \quad v(\{P_2, E\}; \mathbf{y}_0, \mathbf{z}_0) = 0.$$

And finally, it is not hard to show that

$$\begin{aligned} v(\{P_1, P_2, E\} \; &; \; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) \\ &= \max_{\substack{u_{P_1}(\cdot) \in \mathcal{U}_{P_1} \\ u_{P_2}(\cdot) \in \mathcal{U}_{P_2} \\ u_E(\cdot) \in \mathcal{U}_E}} \big\{ -\gamma t_E\big(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0; u_{P_1}(\cdot), u_{P_2}(\cdot), u_E(\cdot)\big)\big\} \\ &= -\gamma t_E\big(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0; \overline{u}_{P_1}(\cdot), \overline{u}_{P_2}(\cdot), \overline{u}_E(\cdot)\big) \\ &= -\gamma \overline{t}, \end{aligned}$$

where $\overline{t}$ is the minimal total pursuit time.

In this case the objectives of the players consist in choosing of the admissible strategies $\overline{u}_{P_1}(\cdot), \overline{u}_{P_2}(\cdot), \overline{u}_E(\cdot)$ such that the total pursuit time t_E is minimal. According to the strategies $\overline{u}_{P_1}, \overline{u}_{P_2}, \overline{u}_E$, all the players move to the point $\overline{\mathbf{z}}$ (see Fig. 1) such that

$$\|\mathbf{z}_0 - \overline{\mathbf{z}}\| = \min_{\mathbf{z} \in A(\mathbf{x}_0, \mathbf{z}_0) \bigcap A(\mathbf{y}_0, \mathbf{z}_0)} \|\mathbf{z}_0 - \mathbf{z}\|.$$

Hence,

$$v(\{P_1, P_2, E\}; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) = -\gamma \frac{\|\mathbf{z}_0 - \overline{\mathbf{z}}\|}{\sigma}.$$

DEFINITION 3.1. The trajectory $\big(\overline{x}(\cdot), \overline{y}(\cdot), \overline{z}(\cdot)\big)$ of system (1)–(2) such that

$$K_{\mathbf{N}}\big(\overline{x}(\cdot), \overline{y}(\cdot), \overline{z}(\cdot)\big) = \sum_{i \in \mathbf{N}} K_i\big(\overline{x}(\cdot), \overline{y}(\cdot), \overline{z}(\cdot)\big) = v(\mathbf{N}; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$$

is called a *cooperative trajectory* in the game $\Gamma(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$.

All in all, the noncooperative solution is all the players should move to the point $\hat{z}$. It is obvious that this is a Nash equilibrium. The cooperative solution is all the players should move to the point $\overline{z}$.
For simplicity, denote $g_1^0 = \gamma\dfrac{\|\mathbf{x}_0 - \mathbf{z}_0\|}{\varrho_1 - \sigma}$, $g_2^0 = \gamma\dfrac{\|\mathbf{y}_0 - \mathbf{z}_0\|}{\varrho_2 - \sigma}$, $g^0 = \gamma\dfrac{\|\mathbf{z}_0 - \widehat{\mathbf{z}}\|}{\sigma}$, $g^* = \gamma\bar{t} = \gamma\dfrac{\|\mathbf{z}_0 - \overline{\mathbf{z}}\|}{\sigma}$.

So, in this game we completely define the characteristic function v in the form

$$
\begin{aligned}
&v(\{P_1\}; \mathbf{x}_0, \mathbf{z}_0) = -g_1^0, \\
&v(\{P_2\}; \mathbf{y}_0, \mathbf{z}_0) = -g_2^0, \\
&v(\{E\}; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) = g^0, \\
&v(\{P_1, E\}; \mathbf{x}_0, \mathbf{z}_0) = 0, \\
&v(\{P_2, E\}; \mathbf{y}_0, \mathbf{z}_0) = 0, \\
&v(\{P_1, P_2\}; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) = -2g^0, \\
&v(\{P_1, P_2, E\}; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) = -g^*.
\end{aligned}
\tag{6}
$$

Let us show that the constructed function v is superadditive.

For this purpose we remind that

$$g_1^0 = \gamma\frac{\|\mathbf{x}_0 - \mathbf{z}_0|}{\rho_1 - \sigma} = \frac{\gamma}{\sigma}\max_{\mathbf{z}\in A(\mathbf{x}_0,\mathbf{z}_0)}\|\mathbf{z}_0 - \mathbf{z}\| = \frac{\gamma}{\sigma}\|\mathbf{z}_0 - \mathbf{z}_1\|, \tag{7}$$

$$g_2^0 = \gamma\frac{\|\mathbf{y}_0 - \mathbf{z}_0|}{\rho_2 - \sigma} = \frac{\gamma}{\sigma}\max_{\mathbf{z}\in A(\mathbf{y}_0,\mathbf{z}_0)}\|\mathbf{z}_0 - \mathbf{z}\| = \frac{\gamma}{\sigma}\|\mathbf{z}_0 - \mathbf{z}_2\|, \tag{8}$$

$$g^0 = \frac{\gamma}{\sigma}\max_{\mathbf{z}\in A(\mathbf{x}_0,\mathbf{z}_0)\bigcap A(\mathbf{y}_0,\mathbf{z}_0)}\|\mathbf{z}_0 - \mathbf{z}\| = \frac{\gamma}{\sigma}\|\mathbf{z}_0 - \widehat{\mathbf{z}}\|, \tag{9}$$

and

$$g^* = \frac{\gamma}{\sigma}\min_{\mathbf{z}\in A(\mathbf{x}_0,\mathbf{z}_0)\bigcap A(\mathbf{y}_0,\mathbf{z}_0)}\|\mathbf{z}_0 - \mathbf{z}\| = \frac{\gamma}{\sigma}\|\mathbf{z}_0 - \overline{\mathbf{z}}\|. \tag{10}$$

Taking into account expressions (7)–(10), we have

$$g^* \le g^0 \le g_i^0,\ i = 1, 2.$$

Hence, the following inequalities are satisfied

$$
\begin{aligned}
&v(\{P_1\}) + v(\{P_2\}) = -g_1^0 - g_2^0 \le -2g^0 = v(\{P_1, P_2\}), \\
&v(\{P_i\}) + v(\{E\}) = -g_i^0 + g^0 \le 0 = v(\{P_i, E\}),\ i = 1, 2, \\
&v(\{P_i, E\}) + v(\{P_{3-i}\}) = 0 - g_{3-i}^0 \le -g^* = v(\{P_1, P_2, E\}), i = 1, 2, \\
&v(\{P_1, P_2\}) + v(\{E\}) = -2g^0 + g^0 = -g^0 \le -g^* = v(\{P_1, P_2, E\}).
\end{aligned}
$$

And that implies superadditivity of the characteristic function v.

DEFINITION 3.2. The pair $\langle \mathbf{N}, v(\mathbf{S}; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) \rangle$, where $\mathbf{N} = \{P_1, P_2, E\}$ is the set of players, and v the characteristic function defined by (6), is called a *cooperative differential* game in characteristic function form and is denoted by $\Gamma_v(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$.

To avoid negative values of the characteristic function v we consider an equivalent game with nonnegative values of its characteristic function.

DEFINITION 3.3. The cooperative n-person game $\langle \mathbf{N}, v\rangle$ is *equivalent* to the game $\langle \mathbf{N}, v'\rangle$ if there exist a positive number k and n arbitrary real numbers c_i $(i \in \mathbf{N})$ such that for any coalition $\mathbf{S} \subset \mathbf{N}$

$$v'(\mathbf{S}) = kv(\mathbf{S}) + \sum_{i \in \mathbf{S}} c_i.$$

In fact, by setting $k = 1$, $c_{P_i} = g_i(0) = g_i^0$, $c_E = 0$, we construct the game $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$, which is equivalent to the game $\Gamma_v(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$. In such a case the characteristic function v' has the form

$$\begin{aligned}
&v'\big(\{P_1\}; \mathbf{x}_0, \mathbf{z}_0\big) = 0,\\
&v'\big(\{P_2\}; \mathbf{y}_0, \mathbf{z}_0\big) = 0,\\
&v'\big(\{E\}; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0\big) = g^0,\\
&v'\big(\{P_1, E\}; \mathbf{x}_0, \mathbf{z}_0\big) = g_1^0,\\
&v'\big(\{P_2, E\}; \mathbf{y}_0, \mathbf{z}_0\big) = g_2^0,\\
&v'\big(\{P_1, P_2\}; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0\big) = g_1^0 + g_2^0 - 2g^0,\\
&v'\big(\{P_1, P_2, E\}; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0\big) = g_1^0 + g_2^0 - g^*.
\end{aligned}$$

We shall examine this game by using dominance relation. Recall that the imputation ξ dominates the imputation η with respect to the coalition S $(\xi \succ_{\mathbf{S}} \eta)$ if the following conditions hold

$$\xi_i > \eta_i, \quad i \in S,$$

$$\xi(S) = \sum_{i \in S} \xi_i \le v(S).$$

The following theorem is needed for the sequel.

THEOREM 3.4. *Suppose $\langle \mathbf{N}, v\rangle$ and $\langle \mathbf{N}, v'\rangle$ are two equivalent games, then the map $\xi \mapsto \xi'$, where*

$$\xi_i' = k\xi_i + c_i, \quad i \in \mathbf{N},$$

establishes one-to-one mapping between the imputation set of the game $\langle \mathbf{N}, v \rangle$ *and the imputation set of the game* $\langle \mathbf{N}, v' \rangle$ *such that* $\xi \succ_{\mathbf{S}} \eta$ *implies* $\xi' \succ_{\mathbf{S}} \eta'$.

4. Existence of the core

It follows from the superadditivity condition that it is advantageous for the players to form the maximal coalition $\mathbf{N}$ and obtain the maximal total payoff $v'(\mathbf{N}; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ that is possible in the game $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$. Various methods for "equitable" distribution of the total profit between players are considered as optimality principles. The set of distributions that satisfies an optimality principle is called a solution in a cooperative game (in the sense of this optimality principle).

Let us describe the imputation set in the game $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$. Denote by $\xi^{\mathbf{0}} = (\xi^0_{P_1}, \xi^0_{P_2}, \xi^0_E)$ an imputation. The imputation set is defined as follows

$$\mathbf{E}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) = \Big\{ \xi^{\mathbf{0}} \; : \; \xi^0_{P_1} \geq v(P_1),\; \xi^0_{P_2} \geq v(P_2),\; \xi^0_E \geq v(E); \; \sum_{i \in \mathbf{N}} \xi^0_i = v(\mathbf{N}) \Big\},$$

or

$$\mathbf{E}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) = \Big\{ \xi^{\mathbf{0}} \; : \; \xi^0_{P_1} \geq 0,\; \xi^0_{P_2} \geq 0,\; \xi^0_E \geq g^0; \; \sum_{i \in \mathbf{N}} \xi^0_i = g^0_1 + g^0_2 - g^* \Big\}. \tag{11}$$

As an optimality principle in $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ we take the core. An analytical description of the core is provided by the following theorem, which was independently proved by Bondareva (1963) and Shapley (1967).

THEOREM 4.1. *For the imputation* $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ *to belong to the core it is necessary and sufficient that the inequality*

$$\xi(\mathbf{S}) = \sum_{i \in \mathbf{S}} \xi_i \geq v(\mathbf{S})$$

holds for all $\mathbf{S} \subset \mathbf{N}$.

By Theorem 4.1, for the imputation ξ^0 to belong to the core it is necessary and sufficient that the following system of inequalities holds

$$(12)\qquad \begin{cases} \xi^0_{P_1} + \xi^0_{P_2} \geq g^0_1 + g^0_2 - 2g^0, \\ \xi^0_{P_1} + \xi^0_E \geq g^0_1, \\ \xi^0_{P_2} + \xi^0_E \geq g^0_2. \end{cases}$$

So, the core in the game $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ is the set

$$\mathbf{C}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) = \Big\{\xi^0 \quad : \quad \xi^0 \in \mathbf{E}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0), \quad \xi^0 \text{ satisfies system } (12)\Big\}.$$

Theorem 4.1 provides enough reasons to use the core as an important optimality principle in cooperative theory. However, in many cases the core appears to be empty; whereas in the other cases it represents a multiple optimality principle, and the question as to which of the imputation are to be chosen from the core is still open.

LEMMA 1. *In the game $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ there exists the nonempty core for any initial positions of the players.*

Proof. Summing inequalities (12), we obtain

$$(13)\qquad 2v(\mathbf{N}) \geq 2(g^0_1 + g^0_2 - g^0).$$

By substituting $v(\mathbf{N}) = g^0_1 + g^0_2 - g^*$ into (13), we have

$$(14)\qquad g^0 \geq g^*.$$

The last inequality is the necessary condition for existence of nonempty core in the game $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$.
On account of

$$g^0 = \frac{\gamma}{\sigma} \max_{z \in L(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)} \|\mathbf{z}_0 - \mathbf{z}\| \geq \frac{\gamma}{\sigma} \min_{\mathbf{z} \in L(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)} \|\mathbf{z}_0 - \mathbf{z}\| = g^*,$$

we conclude that inequality (14) holds for any $\mathbf{x}_0$, $\mathbf{y}_0$, $\mathbf{z}_0$.

Since $g^0 \geq g^*$, it remains to show that there exists a vector $\eta^0 = (\eta^0_{P_1}, \eta^0_{P_2}, \eta^0_E) \in \mathbf{E}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ satisfying system (12). In particular, let it be $\eta^0 = (g^0_1 - g^0, g^0_2 - g^*, g^0)$. It can be readily seen that η^0 is an imputation from the core, i.e. $\eta^0 \in \mathbf{C}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$. This completes the proof. □

The core is a subset of the imputation set, defined by linear inequalities (12). Geometrically, the imputation set $\mathbf{E}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ in the game $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ is the triangle shown in Fig. 2. The nonempty core is an intersection of the imputation set and a convex polyhedron. In the

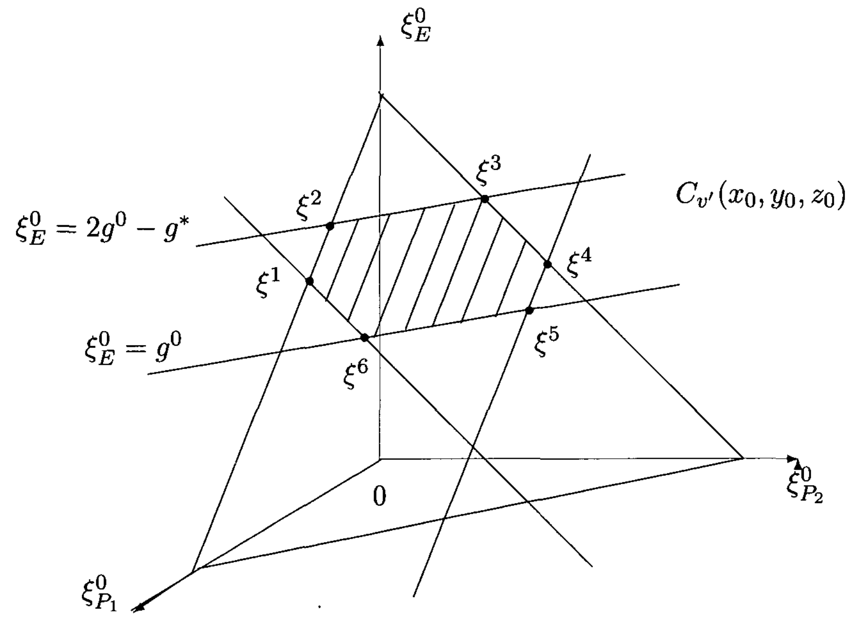

FIGURE 2. The core $\mathbf{C}_{v'}^{22}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$.

game $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ the core can take four various form. The form of the core depends on initial positions $\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0$ of the players and, namely, whether the inequalities $2g^0 - g^* \geq g_i^0$ $(i = 1, 2)$ hold or not. Denote

- by $\mathbf{C}_{v'}^{11}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ the core corresponding to the condition $2g^0 - g^* \geq g_i^0$ for all i, $i = 1, 2$;
- by $\mathbf{C}_{v'}^{22}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ the core corresponding to the condition $2g^0 - g^* < g_i^0$ for all i, $i = 1, 2$;
- by $\mathbf{C}_{v'}^{12}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ and $\mathbf{C}_{v'}^{21}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ the cores corresponding to the conditions $2g^0 - g^* \geq g_i^0$, $2g^0 - g^* < g_j^0$, for all $i, j = 1, 2$, $i \neq j$,

Note that the core $\mathbf{C}_{v'}^{22}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ contains all the others.

5. Time-consistency of the core

We focus our attention on time-consistency of the core $\mathbf{C}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ in the game $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$.

Let an optimality principle be chosen in the game $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$. Let it be the core. The solution of this game constructed at the initial moment $t = 0$ is $\mathbf{C}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$. It follows from Lemma 1 that $\mathbf{C}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0) \neq \emptyset$. Remind that here $(\overline{x}(\cdot), \overline{y}(\cdot), \overline{z}(\cdot))$ is the cooperative trajectory in $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ (see Fig. 1).

We study behavior of the set $\mathbf{C}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ along the cooperative optimal trajectory $(\overline{x}(\cdot), \overline{y}(\cdot), \overline{z}(\cdot))$. With this end in view we enter the notion of a current subgame. At each current state $(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t))$ a current subgame $\Gamma_v(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t))$ is defined like the game $\Gamma_v(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ with the only difference: it starts at the current state lying on the cooperative trajectory and has duration $(\overline{t} - t)$. In the subgame $\Gamma_v(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t))$ we define the characteristic function as it was done for the original game:

$$\begin{aligned}
v(\{P_1\}; \overline{\mathbf{x}}(t), \overline{\mathbf{z}}(t)) &= -\gamma \frac{\|\overline{\mathbf{x}}(t) - \overline{\mathbf{z}}(t)\|}{\varrho_1 - \sigma}, \\
v(\{P_2\}; \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)) &= -\gamma \mathrm{Val}\, \Gamma_{P_2 \setminus E}(\overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)) \\
&= -\gamma \frac{\|\overline{\mathbf{y}}(t) - \overline{\mathbf{z}}(t)\|}{\varrho_2 - \sigma}, \\
v(\{P_1, P_2\}; \overline{x}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)) &= -\gamma \mathrm{Val}\, \Gamma_{\mathbf{P} \setminus E}(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)) \\
&= -\gamma \frac{\|\overline{\mathbf{z}}(t) - \widehat{\mathbf{z}}(t)\|}{\sigma}.
\end{aligned}$$

In this case the players move to the point $\widehat{\mathbf{z}}(t)$ (Fig. 1) such that

$$\|\overline{\mathbf{z}}(t) - \widehat{\mathbf{z}}(t)\| = \max_{\mathbf{z}(t) \in A(\overline{\mathbf{x}}(t), \overline{\mathbf{z}}(t)) \bigcap A(\overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t))} \|\overline{\mathbf{z}}(t) - \mathbf{z}(t)\|,$$

where $A(\overline{\mathbf{x}}(t), \overline{\mathbf{z}}(t))$ and $A(\overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t))$ are Apollonius circles in the games $\Gamma_{P_1 \setminus E}(\overline{\mathbf{x}}(t), \overline{\mathbf{z}}(t))$ and $\Gamma_{P_2 \setminus E}(\overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t))$ respectively. Similarly,

$$\begin{aligned}
v(\{E\}; \overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)) &= \gamma \mathrm{Val}\, \Gamma_{\mathbf{P} \setminus E}(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)) \\
&= \gamma \frac{\|\overline{\mathbf{z}}(t) - \widehat{\mathbf{z}}(t)\|}{\sigma}, \\
v(\{P_1, E\}; \overline{\mathbf{x}}(t), \overline{\mathbf{z}}(t)) &= 0, \\
v(\{P_2, E\}; \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)) &= 0, \\
v(\{P_1, P_2, E\}; \overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)) &= -\gamma(\overline{t} - t).
\end{aligned}$$

Let us consider the functions

$$g_1(t) = \gamma \frac{\|\overline{\mathbf{x}}(t) - \overline{\mathbf{z}}(t)\|}{\varrho_1 - \sigma}, \quad g_2(t) = \gamma \frac{\|\overline{\mathbf{y}}(t) - \overline{\mathbf{z}}(t)\|}{\varrho_2 - \sigma}, \quad g(t) = \gamma \frac{\|\overline{\mathbf{z}}(t) - \widehat{\mathbf{z}}\|}{\sigma}.$$

These functions are continuous monotonically decreasing functions in t on the interval $[0,\bar{t}]$.

REMARK 5.1. From the definition of Apollonius circle it follows that

$$g_1(t)=\gamma\Big(1-\frac{t}{\bar{t}}\Big)\frac{\|\mathbf{x}_0-\mathbf{z}_0\|}{\varrho_1-\sigma},\quad g_2(t)=\gamma\Big(1-\frac{t}{\bar{t}}\Big)\frac{\|\mathbf{y}_0-\mathbf{z}_0\|}{\varrho_2-\sigma}.$$

Now using Remark 5.1 and going to the equivalent game $\Gamma_{v'}\big(\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t)\big)$, we obtain the characteristic function v' in the form

$$\begin{aligned}
&v'\big(\{P_1\};\overline{\mathbf{x}}(t),\overline{\mathbf{z}}(t)\big)=0,\\
&v'\big(\{P_2\};\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t)\big)=0,\\
&v'\big(\{E\};\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t)\big)=g(t),\\
&v'\big(\{P_1,P_2\};\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t)\big)=[g_1^0+g_2^0]\Big(1-\frac{t}{\bar{t}}\Big)-2g(t),\\
&v'\big(\{P_1,E\};\overline{\mathbf{x}}(t),\overline{\mathbf{z}}(t)\big)=g_1^0\Big(1-\frac{t}{\bar{t}}\Big)-2g(t),\\
&v'\big(\{P_2,E\};\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t)\big)=g_2^0\Big(1-\frac{t}{\bar{t}}\Big)-2g(t),\\
&v'\big(\mathbf{N};\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t)\big)=[g_1^0+g_2^0-g^*]\Big(1-\frac{t}{\bar{t}}\Big).
\end{aligned}$$

The imputation set in the game $\Gamma_{v'}\big(\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t)\big)$ is of the form

$$\mathbf{E}_{v'}\big(\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t)\big)=\Big\{\xi^t\ :\ \xi^t_{P_1}\ge 0,\ \xi^t_{P_2}\ge 0,\ \xi^t_E\ge g(t);\ \sum_{i\in\mathbf{N}}\xi^t_i=[g_1^0+g_2^0-g^*]\Big(1-\frac{t}{\bar{t}}\Big)\Big\}.$$

The core of the current game is defined as follows

$$\mathbf{C}_{v'}\big(\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t)\big)=\Big\{\xi^t\ :\ \xi^t\in\mathbf{E}_{v'}\big(\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t)\big),\ \xi^t \text{ satisfies system (15)}\Big\}.$$

$$\text{(15)}\qquad \begin{cases}\xi^t_{P_1}+\xi^t_{P_2}\ge (g_1^0+g_2^0)\left(1-\frac{t}{\bar{t}}\right)-2g(t),\\ \xi^t_{P_1}+\xi^t_E\ge g_1^0\left(1-\frac{t}{\bar{t}}\right),\\ \xi^t_{P_2}+\xi^t_E\ge g_2^0\left(1-\frac{t}{\bar{t}}\right).\end{cases}$$

Suppose $\mathbf{C}_{v'}\big(\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t)\big)\neq\emptyset$ along the cooperative trajectory for any t, $t\in[0,\bar{t}]$. If this condition is not satisfied, then it is impossible for players to adhere to the chosen optimality principle, since at the very first instant t, when $\mathbf{C}_{v'}(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)) = \emptyset$, the players have no possibility to follow this principle. And as we assumed above at the initial state $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ the players agree upon the imputation $\xi^0 \in \mathbf{C}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ such that the share of the i-th player is equal to ξ_i^0. Let the payoff of the player i (his share) on the time interval $[0, t]$ be $\xi_i(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t))$. Then on the remaining time interval $[t, \bar{t}]$, according to the imputation ξ^0, he is to receive the gain $\xi_i^t = \xi_i^0 - \xi_i(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t))$. For the original agreement (the imputation ξ^0) to remain in force at the instant t it is essential that the vector $\xi^t = (\xi_{P_1}^t, \xi_{P_2}^t, \xi_E^t)$ belongs to the set $\mathbf{C}_{v'}(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t))$. If this condition is satisfied at each instant $t \in [0, \bar{t}]$, then the imputation ξ^0 is realized. This is conceptual meaning of time-consistency. In a view of Petrosjan, the definition of time-consistency of an imputation in the game $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ has the form.

Definition 5.2. The imputation $\xi^0 \in \mathbf{C}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ is called time-consistent in the time-optimal game $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ if the following conditions are satisfied:

1. $\mathbf{C}_{v'}(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)) \neq \emptyset$ along the cooperative optimal trajectory $(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t))$ at each instant t, $0 \le t \le \bar{t}$;
2. there exist an integrable function $\beta(t) = (\beta_{P_1}(t), \beta_{P_2}(t), \beta_E(t))$ on $[0, \bar{t}]$ such that $\beta_i(t) \ge 0$ $(i \in \mathbf{N})$ for each $t \in [0, \bar{t}]$ and

$$\xi^0 - \xi(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)) \in \mathbf{C}_{v'}(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)), \tag{16}$$

where

$$\xi(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)) = \Big(\xi_{P_1}(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)),\ \xi_{P_2}(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)),\ \xi_E(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t))\Big)$$

and $\xi_i(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)) = \int_0^t \beta_i(\tau) d\tau$, $i \in \mathbf{N}$.

Remark 5.3. The cooperative differential game $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ has a time consistent solution if each imputation $\xi^0 \in \mathbf{C}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ is time-consistent.

The following theorem is true.

Theorem 5.4. *In the cooperative differential time-optimal pursuit game* $\Gamma_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ *there exists the nonempty core* $\mathbf{C}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ *that is time-consistent.*

Proof. Consider the family of the current subgames

$$\Big\{\Gamma_{v'}\big(\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t)\big),\ 0\le t\le \overline{t}\Big\}.$$

Now our aim is to show that $\mathbf{C}_{v'}\big(\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t)\big)\neq\emptyset$ for each $t\in[0,\overline{t}]$.

Summing inequality (15), we obtain

$$\xi^t_{P_1}+\xi^t_{P_2}+\xi^t_E\ge\big[g^0_1+g^0_2\big]\Big(1-\frac{t}{\overline{t}}\Big)-g(t),$$

or

$$g(t)\ge g^*-\gamma t. \tag{17}$$

Note that

$$g(t)=\frac{\gamma}{\sigma}\big\|\overline{\mathbf{z}}(t)-\widehat{\mathbf{z}}(t)\big\|=\max_{z(t)\in L(\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t))}\big\|\overline{\mathbf{z}}(t)-z(t)\big\|$$

and

$$g^*-t=\frac{\gamma}{\sigma}\big[\|z^0-\widehat{\mathbf{z}}\|-\sigma t\big]=$$

$$\frac{\gamma}{\sigma}\big\|\overline{\mathbf{z}}(t)-\widehat{\mathbf{z}}\big\|=\min_{z(t)\in L(\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t))}\big\|\overline{\mathbf{z}}(t)-z(t)\big\|,$$

where $L(\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t))=A(\overline{\mathbf{x}}(t),\overline{\mathbf{z}}(t))\bigcap A(\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t))$, and $A(\overline{\mathbf{x}}(t),\overline{\mathbf{z}}(t))\subset A(\mathbf{x}_0,\mathbf{z}_0)$ and $A(\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t))\subset A(\mathbf{y}_0,\mathbf{z}_0)$ are Apollonius circles in the games $\Gamma_{P_1\backslash E}(\overline{\mathbf{x}}(t),\overline{\mathbf{z}}(t))$ and $\Gamma_{P_2\backslash E}(\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t))$ respectively. Hence, inequality (17) holds for all $t\in[0,\overline{t}]$.

It is clear that there exists an imputation

$$\eta^t=\left(g^0_1\Big(1-\frac{t}{\overline{t}}\Big)-g(t),\ g^0_2\Big(1-\frac{t}{\overline{t}}\Big)-(g^*-\gamma t),\ g(t)\right)$$

which belongs to $\mathbf{E}_{v'}\big(\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t)\big)$ and satisfies system (15).

So, $\mathbf{C}_{v'}\big(\overline{\mathbf{x}}(t),\overline{\mathbf{y}}(t),\overline{\mathbf{z}}(t)\big)\neq\emptyset$ for all $t\in[0,\overline{t}]$.

Now it remains to check condition 2 in Definition 5.2. According to Remark 5.3, we must prove that condition (16) holds for all imputations from the core.

The core $\mathbf{C}_{v'}(\mathbf{x}_0,\mathbf{y}_0,\mathbf{z}_0)$ represents a convex hull of extreme imputations. In the game $\Gamma_{v'}(\mathbf{x}_0,\mathbf{y}_0,\mathbf{z}_0)$ the core is one of the sets $\mathbf{C}^{11}_{v'}(\mathbf{x}_0,\mathbf{y}_0,\mathbf{z}_0)$, $\mathbf{C}^{12}_{v'}(\mathbf{x}_0,\mathbf{y}_0,\mathbf{z}_0)$, $\mathbf{C}^{21}_{v'}(\mathbf{x}_0,\mathbf{y}_0,\mathbf{z}_0)$, $\mathbf{C}^{22}_{v'}(\mathbf{x}_0,\mathbf{y}_0,\mathbf{z}_0)$. It is obvious that $\mathbf{C}^{11}_{v'}\subset\mathbf{C}^{22}_{v'}$, $\mathbf{C}^{12}_{v'}\subset\mathbf{C}^{22}_{v'}$, $\mathbf{C}^{21}_{v'}\subset\mathbf{C}^{22}_{v'}$. Therefore, if we show that any imputation from $\mathbf{C}^{22}_{v'}(\mathbf{x}_0,\mathbf{y}_0,\mathbf{z}_0)$ is time-consistent, then this implies that any imputation from $\mathbf{C}^{11}_{v'}$, $\mathbf{C}^{12}_{v'}$, or $\mathbf{C}^{21}_{v'}$ is also time-consistent.

The core $\mathbf{C}^{22}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ represents a convex hull of the imputations

$$\begin{aligned}
\xi^1 &= \left(g_1^0 - g^*,\ 0,\ g_2^0\right),\\
\xi^2 &= \left(g_1^0 + g_2^0 - 2g^0,\ 0,\ 2g^0 - g^*\right),\\
\xi^3 &= \left(0,\ g_1^0 + g_2^0 - 2g^0,\ 2g^0 - g^*\right),\\
\xi^4 &= \left(0,\ g_2^0 - g^*,\ g_1^0\right),\\
\xi^5 &= \left(g_1^0 - g^0,\ g_2^0 - g^*,\ g^0\right),\\
\xi^6 &= \left(g_1^0 - g^*,\ g_2^0 - g^0,\ g^0\right).
\end{aligned} \tag{18}$$

Hence, any imputation $\xi^0 \in \mathbf{C}^{22}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ can be represented as

$$\xi^0 = \sum_{j=1}^{6} \lambda_j \xi^j, \quad \sum_{j=1}^{6} \lambda_j = 1, \quad \lambda_j \geq 0 \ \text{ for } \ 1 \leq j \leq 6. \tag{19}$$

By substituting (18) into (19) and denoting

$$\mathbf{s}_1 = \begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_5 + \lambda_6 \\ \lambda_4 \\ \lambda_3 \end{pmatrix}, \quad \mathbf{s}_2 = \begin{pmatrix} \lambda_6 \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \\ \lambda_5 \end{pmatrix},$$
$$\mathbf{s}_3 = \begin{pmatrix} -\lambda_1 - 2\lambda_6 \\ -\lambda_2 - 2\lambda_4 \\ \lambda_1 + \lambda_2 + 2\lambda_4 + 2\lambda_6 \end{pmatrix}, \quad \mathbf{s}_4 = \begin{pmatrix} \lambda_2 + \lambda_5 \\ \lambda_1 + \lambda_3 \\ \lambda_4 + \lambda_6 \end{pmatrix},$$

we have

$$\xi^0 = g_1^0 \mathbf{s}_1 + g_2^0 \mathbf{s}_2 + g^0 \mathbf{s}_3 - g^* \mathbf{s}_4.$$

The main idea is to prove that ξ^0 is time-consistent and, namely, to find an integrable vector function $\beta(t)$ on $[0, \bar{t}]$ such that $\beta_i(t) \geq 0$, $i \in \mathbf{N}$ and condition (16) holds.

Indeed, at the last moment $t = \bar{t}$ the core $\mathbf{C}^{22}_{v'}(\overline{\mathbf{x}}(\bar{t}), \overline{\mathbf{y}}(\bar{t}), \overline{\mathbf{z}}(\bar{t})) = \emptyset$ as a solution of the current game $\Gamma_{v'}\left(\overline{\mathbf{x}}(\bar{t}), \overline{\mathbf{y}}(\bar{t}), \overline{\mathbf{z}}(\bar{t})\right)$ with integral payoffs and zero-duration. Thus, from condition (16) it follows that

$$\xi\left(\overline{\mathbf{x}}(\bar{t}), \overline{\mathbf{y}}(\bar{t}), \overline{\mathbf{z}}(\bar{t})\right) = \int_0^{\bar{t}} \beta(\tau) d\tau = \xi^0. \tag{20}$$

On account of (20), we can put

$$\beta(\tau) = \frac{g_1^0}{\bar{t}} \mathbf{s}_1 + \frac{g_2^0}{\bar{t}} \mathbf{s}_2 + \frac{g^0}{\bar{t}} \mathbf{s}_3 - \gamma \mathbf{s}_4.$$

Finally, according to (20), we have

$$\begin{aligned}\xi^0 &= \int_0^{\bar t} \beta(\tau)d\tau \\ &= \int_0^{\bar t} \Big[\frac{g_1^0}{\bar t}\mathbf{s}_1 + \frac{g_2^0}{\bar t}\mathbf{s}_3 + \frac{g^0}{\bar t}\mathbf{s}_3 - \gamma\mathbf{s}_4\Big]d\tau \\ &= g_1^0\mathbf{s}_1 + g_2^0\mathbf{s}_2 + g^0\mathbf{s}_3 - g^*\mathbf{s}_4.\end{aligned}$$

Now the aim is to show that the imputation $\xi^0 - \xi(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t))$ belongs to the core of the current game $\Gamma_{v'}(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t))$ for any $t \in [0, \bar t]$. Substituting the vector-function $\beta(t)$ into (16), we obtain

$$\begin{aligned}\xi^0 - \xi(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t)) &= \xi^0 - \int_0^t \beta(\tau)d\tau \\ &= \big[g_1^0\mathbf{s}_1 + g_2^0\mathbf{s}_2 + g^0\mathbf{s}_3 - g^*\mathbf{s}_4\big]\Big(1 - \frac{t}{\bar t}\Big) = \xi^t\end{aligned}$$

for all $t \in [0, \bar t]$.

It is not hard to prove that ξ^t belongs to the core $\mathbf{C}_{v'}(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t))$ of the game $\Gamma_{v'}(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t), \overline{\mathbf{z}}(t))$. The proof is left to the reader.

So, condition 2 of definition 5.2 holds for all $t \in [0, \bar t]$ and for all $\xi^0 \in \mathbf{C}_{v'}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$. This completes the prove of the theorem. □

6. Discussion

Now, using Theorem 3.4, we get the solution for the initial game $\Gamma_v(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$. As it was mentioned above we get four families of games depending on initial conditions. They are: $\Gamma_v^{11}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$, $\Gamma_v^{12}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$, $\Gamma_v^{21}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$, $\Gamma_v^{22}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$. Here we would not recommend to follow the Shapley value. If to take the Shapley value as an allocation then we will have the following difficulties: only the game $\Gamma_v^{22}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ is convex, and three others are not. So, for some initial conditions the Shapley value belongs to the core, but for some not. All in all, since the core is never empty in the game $\Gamma_v(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$, then as an allocation in a general case it should be chosen . We offer to choose any allocation which lies strictly inside of the narrowest core $\mathbf{C}_v^{11}(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$, for instance, its mass center. It is obvious that it will be a reasonable solution for all four families of the game $\Gamma_v(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$. Besides that, this allocation would make all the players be better off than they are in the noncooperative case.

Now let us consider the core $\mathbf{C}_v^{22}$ and discuss the advantageous for the players to play cooperatively. The extreme imputations in the core are

$$
\begin{aligned}
\eta^1 &= \big(-g^*,\ -g_2^0,\ g_2^0\big),\\
\eta^2 &= \big(g_2^0 - 2g^0,\ -g_2^0,\ \gamma^0 - g^*\big),\\
\eta^3 &= \big(-g_1^0,\ g_1^0 - 2g^0,\ 2g^0 - g^*\big),\\
\eta^4 &= \big(-g_1^0,\ -g^*,\ g_1^0\big),\\
\eta^5 &= \big(-g^0,\ -g^*,\ g^0\big),\\
\eta^6 &= \big(-g^*,\ -g^0,\ g^0\big).
\end{aligned}
\tag{21}
$$

In order to analyze the solution let us recall that the Nash equilibrium gives the players the payoff profile

$$\big(-g^0,\ -g^0,\ g^0\big). \tag{22}$$

Taking into account that $g^* \leq g^0 < \min\{g_1^0, g_2^0\}$, it can be easily seen that any imputation lying strictly inside of the core gives the players higher payoffs then if they play the Nash equilibrium and get the payoff profile 22. Comparing the cooperative and noncooperative solutions (namely, what the players finely get) we can conclude that if the players have a chance to cooperate they must do it with no doubt. The reason is the core always exists in this game, and moreover, it is time-consistent. So, any above described allocation from the core gives the players strictly higher payoffs than the Nash equilibrium does.

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