J. Korean Math. Soc. **39** (2002), No. 5, pp. 681–692

ON THE CONJUGATE DARBOUX-PROTTER PROBLEMS FOR THE TWO DIMENSIONAL WAVE EQUATIONS IN THE SPECIAL CASE

JONG BAE CHOI AND JONG YEOUL PARK

ABSTRACT. In the article [2], the conjugate Darboux-Protter problem D_n is formulated for the two dimensional wave equation in the class of unbounded functions and the uniqueness of solutions has been established. In this paper, we shall show the existence of solutions for the hyperbolic equations with Bessel operators in another special case.

1. Introduction

In 1954, M. H. Protter [7] formulated the following boundary value problem as an analogue of the plane Darboux problem.

PROBLEM D_0. *Find a solution $u(x,y,\tau)$ to the equation*

$$(1)\qquad \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial \tau^2}\right)u(x,y,\tau) \;=\; 0$$

in the domain $Q : 0<\tau<\rho\equiv\sqrt{x^2+y^2}<1-\tau$ such that $u\in C\left(\overline{Q}\right)\cap C^2(Q)$ and

$$u\Big|_{\overline{K_i}}=\varphi_i,\quad i=0,1,$$

where the $\varphi_i\,(i=0,1)$ are given functions, $K_0:\ 0<\rho=\tau<1/2$ and the circle without the center $K_1:\ \tau=0,\ 0<\rho<1$.

Received September 3, 2001. Revised March 7, 2002.
2000 Mathematics Subject Classification: 35L05, 35L20.
Key words and phrases: the conjugate Darboux-Protter problem, the hyperbolic equation with Bessel operator.
The work was supported by the Korea Research Foundation Grant (KRF-2000-DA0008).

In 1957, K. C Tong [8] noted that the linear space of solutions of the homogeneous Problem D_0 is infinite-dimensional. Examples similar to [8] are presented in [4]: functions

$$\begin{aligned} u_{i,k,n}(x,y,\tau) &= u_{i,k,n}(\rho\cos\theta,\rho\sin\theta,\tau) \\ &= \tau\rho^{n-3-2i}(1-\tau^2/\rho^2)^{n-2i-3/2} \\ &\quad \times F(n-i,-i,3/2;\ \tau^2/\rho^2)Y_{k,n}(\theta), \end{aligned}$$

where $n \geq 3$; $k = 0,1$; $i = 0,1,\dots,(n-3)/2$; $\rho = \sqrt{x^2+y^2} \geq 0$; $0 \leq \theta < 2\pi$; $Y_{0,n}(\theta) = \cos n\theta$, $Y_{1,n}(\theta) = \sin n\theta$; $F(a,b,c;t)$-hypergeometric functions, are nontrivial solutions of the equations (1) in Q and

$$u_{i,k,n}(x,y,\tau) \in C(\overline{Q}) \cup C^2(Q)$$

and they satisfy to the homogeneous boundary conditions:

$$u|_{\overline{K_i}} \equiv 0,\ i = 0,1.$$

Therefore a well-posed formulation of boundary value problems for the equations (1) in Q has attracted the attention of many authors. In [5]–[6], sufficient conditions for the uniqueness of solution of problem D_0 and of the following problem were given.

PROBLEM D_1. *Find a solution* $u(x,y,\tau)$ *to the equation* (1) *in* Q *such that* $u \in C\left(\overline{Q}\right) \cap C^1\left(Q \cup K_1\right) \cup C^2\left(Q\right)$ *and*

$$u\Big|_{\overline{K_0}} = \varphi_0, \quad \frac{\partial u}{\partial \tau}\Big|_{\overline{K_1}} = \varphi_1.$$

The existence of a classical solution satisfying the uniqueness conditions was proved in [6] for the problem D_i $(i = 0,1)$. In [2], the conjugate Darboux-Protter problem was formulated for the two dimensional wave equation in the class of unbounded functions and it was proved the uniqueness of solutions in [1]. In [3], it was shown an existence of the solution of the conjugate Darboux-Protter problem in the case of independence of the polar angle. In this work, we shall show the existence of solutions for the hyperbolic equations with Bessel operators in another special case.

I would like to express my deepest gratitude to Professor Jong Yeoul Park for many hours of inspiring conversations and encouragement. I would also like to thank the referee for helpful suggestions.

2. The setting of Darboux-Protter problems and lemmas

In this paper, we shall consider the following conjugate boundary value problems.

PROBLEM D_0^*. *Find a solution $u(x,y,\tau)$ to the equation (1) in Q such that*

$$u \in C\left(\overline{Q} \setminus O(0,0,0)\right) \cap C^2(Q)$$

and

$$u\Big|_{\overline{K_1}} = \varphi_1, \quad u\Big|_{\overline{K_2}} = \varphi_2, \tag{2}$$

$\varphi_i \ (i=1,2)$ are given functions, K_1 is the circle $\tau = 0,\ 0 < \rho < 1$ with deleted center and $K_2 = \partial Q \setminus (\overline{K}_0 \cap K_1)$ is the surface of the frustum of the cone $\rho + \tau = 1,\ 0 < \tau < 1/2,\ 1/2 < \rho < 1$.

PROBLEM D_1^*. *Find a solution $u(x,y,\tau)$ to the equation (1) in Q such that*

$$u \in C\left(\overline{Q} \setminus O(0,0,0)\right) \cap C^1(Q \cup K_1) \cap C^2(Q)$$

and

$$\frac{\partial u}{\partial \tau}\Big|_{\overline{K_1}} = \varphi_1, \quad u\Big|_{\overline{K_2}} = \varphi_2,$$

where $\varphi_i \ (i=1,2)$ are given functions, K_1 is the circle $\tau = 0,\ 0 < \rho < 1$ with deleted center and K_2 is the conic surface $\rho + \tau = 1,\ 0 < \tau < 1/2,\ 1/2 < \rho < 1$.

We note the conjugate boundary value problem D_0, D_0^* and D_1, D_1^* are overdetermined in the class of functions $C(\overline{Q}) \cap C^2(Q)$, because the homogeneous Problems $D_i \ (i=1,2)$ have the infinite-dimensional linear space of solutions. Let K_0 be the conic surface $0 < \rho = \tau < 1/2$. We denote by U the class of unbounded function

$$u(x,y,\tau) \in C\left(\overline{Q} \setminus O(0,0,0)\right) \cap C^1\left(Q \cup \bigcup_{i=0}^{2} K_i\right) \cup C^2(Q),$$

which is represented in the form

$$\begin{aligned} u(x,y,\tau) &= u(\rho\cos\theta,\ \rho\sin\theta,\ \tau) \\ &= \sum_{n=0}^{\infty} \rho^{-n} \sum_{k=0}^{1} v_{k,n}(\rho,\tau) Y_{k,n}(\theta), \end{aligned} \tag{3}$$

where $Y_{0,n}(\theta) = \cos n\theta,\ Y_{1,n} = \sin n\theta$; the functions

$$v_{k,n}(\rho,\tau) \in C(\overline{G}) \ \cap\ C^1\left(G \cup \bigcup_{i=0}^{2} \Gamma_i\right) \cap\ C^2(G),$$

$k = 0, 1;\ n = 0, 1, 2, \dots$; the domain $G:\ 0 < \tau < \rho < 1 - \tau$; the lines on $\rho - \tau$ plane:

$$\begin{aligned}\Gamma_0\ &:\ 0 < \tau = \rho < 1/2,\\ \Gamma_1\ &:\ \tau = 0,\ 0 < \rho < 1,\\ \Gamma_2\ &:\ 0 < \tau = 1 - \rho < 1/2.\end{aligned}$$

THEOREM 1. (The Uniqueness of Solutions [2]) *Let the functions* $u_i(x, y, \tau),\ i = 1, 2$ *be the solutions of the equation* (1) *in* Q *with the boundary conditions* (2) *and* $u_i(x, y, \tau) \in U,\ i = 1, 2$. *Then* $u_1(x, y, \tau) \equiv u_2(x, y, \tau)$ *in* Q.

THEOREM 2. [3] *Suppose that the functions* φ_i *in the boundary conditions* (2) *are smooth and independent of the polar angle*: $\varphi_1 \equiv \varphi_1(\rho^2) \in C^6(0 \le \rho \le 1),\ \varphi_2 \equiv \varphi_2(\tau) \in C^4(0 \le \tau \le 1/2)$ *and* $\varphi_1(1) = \varphi_2(0)$. *Then there is a unique solution to Problem* D_0^*.

In [3], we have the solution

$$u(x, y, \tau) \equiv u(\rho, \tau), \quad \rho = \sqrt{x^2 + y^2}.$$

Therefore,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial \tau^2}\right) u(x, y) \ =\ \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} - \frac{\partial^2 u}{\partial \tau^2} \ =\ 0 \tag{4}$$

and (see [3])

$$\begin{aligned}u(\rho, \tau) = &\int_0^1 \left[f_1\left((\rho s + \tau)^2\right) + f_1\left((\rho s - \tau)^2\right)\right](1 - s^2)^{-\frac{1}{2}}\, ds\\ &+ \tau \int_0^1\!\!\int_0^1 f_2\left((\rho + \tau s)^2 - 4\rho\tau s t\right)[t(1 - t)]^{-\frac{1}{2}}\, s(1 - s^2)^{-\frac{1}{2}} ds dt\end{aligned} \tag{5}$$

where

(6)
$$f_1(\rho) = \frac{\sqrt{\rho}}{\pi}\frac{d}{d\rho}\int_0^{\rho}(\rho-\xi)^{-\frac{1}{2}}\varphi_1(\xi)\,d\xi, \quad 0 \le \xi \le 1,$$

(7)
$$f_2(\rho) = \frac{2}{\pi^2}\frac{d}{d\rho}\int_{\rho}^{1}(\lambda-\rho)^{-\frac{1}{2}}(1+\sqrt{\lambda})^{\frac{1}{2}}F(\sqrt{\lambda})\,d\lambda, \quad 0 \le \rho \le 1,$$

(8)
$$F(\lambda) = \int_{\lambda}^{1}(\mu-\lambda)^{-\frac{1}{2}}\frac{d}{d\mu}F_1\left(\frac{1-\mu}{2}\right)d\mu, \quad 0 \le \lambda \le 1,$$

(9)
$$F_1(\tau) = \varphi_2(\tau) - \int_0^1\Big\{f_1\left([(1-\tau)s+\tau]^2\right) + f_1\left([(1-\tau)s-\tau]^2\right)\Big\}(1-s^2)^{-1/2}\,ds, \quad 0 \le \tau \le 1/2.$$

REMARK 1. From (4) – (9) it is sufficient that the given functions are from the following class:

$$\varphi_1 \in C^5(0 \le \rho \le 1), \quad \varphi_2 \in C^3(0 \le \rho \le 1/2).$$

Next if the function $u \in U$ and u is a solution of the equation (1) in Q, then we can show that functions $v_{k,n}(\rho,\tau)$ in (3) are solutions of equations (respectively)

(10)
$$L_n v_{k,n} \equiv \left(\frac{\partial^2}{\partial\rho^2} - \frac{2n-1}{\rho}\frac{\partial}{\partial\rho} - \frac{\partial^2}{\partial\tau^2}\right)v_{k,n}(\rho,\tau) \;=\; 0,$$

where $(\rho,\tau) \in G$; $k = 0,1$; $n = 0,1,2,\dots$, by using functions

$$v_{k,n}(\rho,\tau) = \frac{\rho^n}{\tau_1}\int_{-\tau_1}^{\tau_1} u(\rho\cos\theta,\rho\sin\theta,\tau)Y_{k,n}(\theta)\,d\theta.$$

LEMMA 1. [9, 10] *Let* $w_n(\rho,\tau), w_{n+1}(\rho,\tau) \in C^2(G)$, *where*

(11)
$$w_n(\rho,\tau) = \frac{1}{\rho}\frac{\partial w_{n+1}(\rho,\tau)}{\partial\rho}$$

and

(12)
$$L_{n+1}w_{n+1} = 0 \ \text{ in } \ G.$$

Then the function $w_n(\rho,\tau)$ *is a solution of the equation*

(13)
$$L_n w_n(\rho,\tau) \;=\; 0, \quad (\rho,\tau) \;\in\; G.$$

LEMMA 2. (Inverse lemma) *Let the function*

$$w_n(\rho,\tau) \in C^2(G) \cap C^1(G \cup \Gamma_2) \cap C(\overline{G})$$

and $w_n(\rho,\tau)$ *satisfy the equation* (13). *Let* $f_n(\tau) = w_n(1-\tau,\tau)$, $0 \leq \tau \leq 1/2$, *and*

$$f_{n+1}(\tau) \in C^2(0<\tau<1/2) \cap C(0 \leq \tau \leq 1/2)$$

and let $f_{n+1}(\tau)$ *be a solution of the differential equation*

$$\frac{d^2 f_{n+1}(\tau)}{d\tau^2} = -2(1-\tau)\frac{df_n(\tau)}{d\tau} + (1-2n)f_n(\tau), \; 0<\tau<1/2. \tag{14}$$

Then the function

$$w_{n+1}(\rho,\tau) = f_{n+1}(\tau) - \int_\rho^{1-\tau} \xi w_n(\xi,\tau)\, d\xi \tag{15}$$

is a solution of the equation (12) *and* $w_{n+1}(1-\tau,\tau) = f_{n+1}(\tau)$, $0 \leq \tau \leq 1/2$.

Proof. From (15), we have

$$\begin{aligned}
\frac{\partial w_{n+1}}{\partial \rho} &= \rho w_n(\rho,\tau), \\
\frac{\partial^2 w_{n+1}}{\partial \rho^2} &= \rho\frac{\partial w_n(\rho,\tau)}{\partial \rho} + w_n(\rho,\tau), \\
\frac{\partial w_{n+1}}{\partial \tau} &= \frac{df_{n+1}(\tau)}{d\tau} + (1-\tau)f_n(\tau) - \int_\rho^{1-\tau} \xi\frac{\partial w_n(\xi,\tau)}{\partial \tau}\, d\xi, \\
\frac{\partial^2 w_{n+1}}{\partial \tau^2} &= \frac{d^2 f_{n+1}(\tau)}{d\tau^2} - f_n(\tau) + (1-\tau)\frac{df_n(\tau)}{d\tau} \\
&\quad + (1-\tau)\frac{\partial w_n(\rho,\tau)}{\partial \tau}\bigg|_{\rho=1-\tau} - \int_\rho^{1-\tau} \xi\frac{\partial^2 w_n(\xi,\tau)}{\partial \tau^2}\, d\xi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{\partial^2 w_{n+1}}{\partial \rho^2} - \frac{2n+1}{\rho}\frac{\partial w_{n+1}}{\partial \rho} &= \rho\frac{\partial w_n(\rho,\tau)}{\partial \rho} - 2nw_n(\rho,\tau) \\
&= (1-\tau)\frac{\partial w_n(\rho,\tau)}{\partial \rho}\bigg|_{\rho=1-\tau} - 2nf_n(\tau) \\
&\quad - \int_\rho^{1-\tau}\left[\xi\frac{\partial^2 w_n(\xi,\tau)}{\partial \xi^2} - (2n-1)\frac{\partial w_n(\xi,\tau)}{\partial \xi}\right] d\xi
\end{aligned}$$

and

$$L_{n+1}w_{n+1} = -\int_{\rho}^{1-\tau} \xi\Big[\frac{\partial^2 w_n(\xi,\tau)}{\partial\xi^2} - \frac{(2n-1)}{\xi}\frac{\partial w_n(\xi,\tau)}{\partial\xi} - \frac{\partial^2 w_n(\xi,\tau)}{\partial\tau^2}\Big]d\xi$$
$$- \frac{d^2 f_{n+1}(\tau)}{d\tau^2} - (1-\tau)\frac{df_n(\tau)}{d\tau} + (1-2n)f_n(\tau)$$
$$- (1-\tau)\Big[\frac{\partial^2 w_n(\rho,\tau)}{\partial\tau^2} - \frac{\partial w_n(\rho,\tau)}{\partial\rho}\Big]\Big|_{\rho=1-\tau}.$$

Because of (13) if $\rho \le \xi < 1-\tau$, $(\rho,\tau)\in G$, we have

$$\frac{\partial^2 w_n(\xi,\tau)}{\partial\xi^2} + \frac{(1-2n)}{\xi}\frac{\partial w_n(\xi,\tau)}{\partial\xi} - \frac{\partial^2 w_n(\xi,\tau)}{\partial\tau^2} = 0$$

and

$$\Big[\frac{\partial w_n(\rho,\tau)}{\partial\tau} - \frac{\partial w_n(\rho,\tau)}{\partial\rho}\Big]\Big|_{\rho=1-\tau} = \frac{df_n(\tau)}{d\tau}.$$

Therefore, $L_{n+1}w_{n+1} = 0$ in G if and only if the equation (14) holds. Lemma 2 is thus proved. □

LEMMA 3. *Let functions $w_n(\rho,\tau)$ and $w_{n+1}(\rho,\tau)$ satisfy the system of equations* (11) − (13) *and let*

$$f_n(\tau) = w_n(1-\tau,\tau),\ f_{n+1}(\tau) = w_{n+1}(1-\tau,\tau),\ 0\le\tau\le 1/2.$$

Denote the following constants by

$$A_n = f_n(0),\ \ B_n = \frac{df_n(\tau)}{d\tau}\Big|_{\tau=0}, \tag{16}$$

$$A_{n+1} = f_{n+1}(0),\ \ B_{n+1} = \frac{df_{n+1}(\tau)}{d\tau}\Big|_{\tau=0}. \tag{17}$$

Then the following functional formulas hold:

(18)
$$
\begin{aligned}
f_n(\tau) = {} & \Big[A_n + \frac{1}{2}(n+\frac{1}{2})A_{n+1} + \frac{1}{2}B_{n+1}\Big](1-\tau)^{n-1/2} \\
& + \frac{1}{2}(n+\frac{1}{2})(1-\tau)^{-2} f_{n+1}(\tau) - \frac{1}{2}(1-\tau)^{-1}\frac{df_{n+1}(\tau)}{d\tau} \\
& - \frac{1}{2}(n+\frac{1}{2})(n+\frac{3}{2})(1-\tau)^{n-1/2}\int_0^\tau (1-t)^{-n-5/2} f_{n+1}(t)\,dt,
\end{aligned}
$$

(19)
$$
\begin{aligned}
f_{n+1}(\tau) = {} & A_{n+1} + (2A_n + B_{n+1})\tau \\
& - 2\int_0^\tau [1 + (n+1)\tau - (n+2)t] f_n(t)\,dt, \ \ 0 \le \tau \le 1/2.
\end{aligned}
$$

Proof. The proof of Lemma 3 follows from the differential equation (14). □

REMARK 2. Let

$$C_{n+1} = \frac{d^2 f_{n+1}(\tau)}{d\tau^2}\bigg|_{\tau=0}$$

and

$$\frac{1}{\rho}\frac{\partial w_{n+1}(\rho,\tau)}{\partial \rho} = w_n(\rho,\tau) \in C\big(G \cup \Gamma_1 \cup \Gamma_2 \cup \rho(1,0)\big),$$

where the domain $G : 0 < \tau < \rho < 1-\tau$; the lines $\Gamma_1 : \ \tau = 0, \ 0 < \rho < 1$ and $\Gamma_2 : \ 0 < \tau = 1-\rho < 1/2$. Then in Lemma 3, the constants are $A_n = f_n(0) = \frac{\partial}{\partial\rho} w_{n+1}\Big|_{\rho(1,0)}$ and $B_n = \left(\frac{1}{2} - n\right)A_n - \frac{1}{2}C_{n+1}$.

3. The Darboux-Problem for the hyperbolic equation with Bessel operator

Let n be a positive integer. We denote V_n by the class of functions

$$\frac{1}{\rho}\frac{\partial^k v(\rho,\tau)}{\partial \rho^k} \in C(\overline{G}) \ \cap \ C^1\left(G \cup \bigcup_{i=0}^{2}\Gamma_i\right) \cap \ C^2(G), k = 0, 1, 2, \dots, n.$$

PROBLEM D_n. *Find a solution* $v(\rho,\tau)$ *to the equation*

$$L_n v(\rho,\tau) = 0 \quad \text{in} \quad G \tag{20}$$

such that

$$v(\rho,\tau) \in V_n, \tag{21}$$

$$v\Big|_{\overline{\Gamma_1}} = \psi_1(\rho), \quad v\Big|_{\overline{\Gamma_2}} = \psi_2(\tau), \tag{22}$$

where the ψ_i $(i = 1, 2)$ *are given functions and also* $\psi_1(1) = \psi_2(0)$,

$$\psi_1(\rho) \in C^{n+s}(0 \leq \rho \leq 1), \quad \psi_2(\tau) \in C^{n+3}(0 \leq \tau \leq 1/2).$$

THEOREM 3. [1] *There is a unique solution of the Problem* D_n : (20) − (22).

Proof. The uniqueness of solutions follows from the result [2]. Next if $n = 0$, then the existence of solutions follows from the result [3]. If $n \geqq 1$, then for finding a solution of the Problem D_n where n is fixed, we consider the following n auxiliary problems for the system of equations.□

PROBLEM S_k $(k = n-1, n-2, \cdots, 1, 0)$. *Find solutions* $w_k(\rho,\tau)$ *of the system of equations*

$$L_k w_k(\rho,\tau) = 0 \quad \text{in} \quad G \tag{23}$$

such that

$$w_k(\rho,\tau) \in V_k, \tag{24}$$

$$w_k(\rho,\tau) = \frac{1}{\rho}\frac{\partial w_{k+1}(\rho,\tau)}{\partial \rho}, \quad k = n-1, n-2, \ldots, 1, 0, \tag{25}$$

where $w_n(\rho,\tau) = v(\rho,\tau)$ *which is a solution of Darboux boundary Problem* D_n : (20) − (22).

First we find boundary conditions for solutions $w_{k+1}(\rho,\tau)$ of the system of equations (23)–(25). We denote by $\psi_{1,k}(\rho)$ and $\psi_{2,k}(\tau)$ traces of functions $w_k(\rho,\tau)$:

$$w_k(\rho,\tau)\Big|_{\overline{\Gamma_1}} = \psi_{1,k}(\rho), \quad w_k(\rho,\tau)\Big|_{\overline{\Gamma_2}} = \psi_{2,k}(\tau), \tag{26}$$

where $k = n-1, n-2, \dots, 1, 0$. Using Lemma 3 and Remark 2, we have the following functional formulas (see (18) and (25)):

(27)
$$\psi_{1,k}(\rho) = \frac{1}{\rho}\frac{d}{d\rho}\psi_{1,k+1}(\rho), \ (0 \le \rho \le 1),$$
(28)
$$\begin{aligned}\psi_{2,k}(\tau) = &\left[A_k - \frac{1}{2}\left(k+\frac{1}{2}\right)A_{k+1} + \frac{1}{2}B_{k+1}\right](1-\tau)^{k-\frac{1}{2}} \\ &+ \frac{1}{2}\left(k+\frac{1}{2}\right)(1-\tau)^{-2}f_{k+1}(\tau) - \frac{1}{2}(1-\tau)^{-1}\frac{d\psi_{2,k+1}(\tau)}{d\tau} \\ &- \frac{1}{2}\left(k+\frac{1}{2}\right)\left(k+\frac{3}{2}\right)(1-\tau)^{k-\frac{1}{2}}\int_0^\tau (1-t)^{-n-5/2}f_{k+1}(t)\,dt,\end{aligned}$$

where (see Remark 2)

$$A_k = \psi_{2,k}(0) = \psi_{1,k}(1), \ B_k = \left(\frac{1}{2}-k\right)A_k - \frac{1}{2}C_{k+1},$$
$$C_{k+1} = \left.\frac{d^2\psi_{2,k+1}(\tau)}{d\tau^2}\right|_{\tau=0}, \ k = n-1, n-2, \cdots, 1, 0;$$
$$\psi_{1,n}(\rho) = \psi_1(\rho), \ 0 \le \rho \le 1: \ \psi_{2,n}(\tau) = \psi_2(\tau), \ 0 \le \tau \le 1/2.$$

Therefore, there are one-valued boundary functions as solutions $w_k(\rho, t)$, $k = n-1, n-2, \cdots, 1, 0$. From the result [3] (see formulas (5)–(9) of this article), we can find a unique solution $w_0(\rho,\tau)$ for the equation (23) where $k=0$ such that it belongs to (24) where $k=0$ and it satisfies the boundary condition (26) where $k=0$. Now using Lemma 2, we get sequentially unique solutions

$$w_k(\rho,\tau) = \psi_{2,k}(\tau) - \int_\rho^{1-\tau} \xi w_{k-1}(\xi,\tau)\,d\xi, \quad k = 1, 2, \dots, n.$$

Theorem 3 is thus proved. □

4. Conjugate Darboux-Protter problem for the equation (1) in the special case

Let the functions φ_1 and φ_2 in (2) have forms

$$\varphi_1 = \varphi_1(\rho,\tau) = \sum_{n=0}^{N} \rho^{-n} \sum_{k=0}^{1} \varphi_{1,k,n}(\rho) Y_{k,n}(\theta), \tag{29}$$

$$\varphi_2 = \varphi_2(\tau,\theta) = \sum_{n=0}^{N} (1-\tau)^{-n} \sum_{k=0}^{1} \varphi_{2,k,n}(\tau) Y_{k,n}(\theta), \tag{30}$$

where N is any positive integer, the functions

$$\varphi_{1,k,n}(\rho) \in C^{n+5}(0 \le \rho \le 1), \quad \varphi_{2,k,n}(\tau) \in C^{n+3}(0 \le \tau \le 1/2), \tag{31}$$

$$\varphi_{1,k,n}(1) = \varphi_{2,k,n}(0): \ \varphi_{1,1,0}(\rho) \equiv \varphi_{2,1,0}(\tau) \equiv 0;$$

$$(k = 0,1; \ n = 0,1,2,\cdots, N).$$

THEOREM 4. *There is a unique solution* $u(x,y,\tau)$ *of the problem* $D_0^*: (1)-(2)$, *where* $u \in U$ *and the boundary functions in* (2) *have the forms* (29) – (31).

Proof. The uniqueness of solutions follows from the result [2]. An unknown solution is constructed in an explicit form:

$$u(x,y,\tau) = \sum_{n=0}^{N} \rho^{-n} \sum_{k=0}^{1} v_{k,n}(\rho,\tau) Y_{k,n}(\theta),$$

where functions $v_{k,n}(\rho,\tau)$ are the solutions of the Darboux-Protter problems for the hyperbolic equations with Bessel operators:

$$L_n v_{k,n}(\rho,\tau) = 0 \ \text{ in } \ G, \tag{32}$$

$$v_{k,n}(\rho,\tau) \in V_n, \tag{33}$$

$$v_{k,n}\Big|_{\overline{\Gamma_1}} = \varphi_{1,k,n}(\rho); \quad v_{k,n}\Big|_{\overline{\Gamma_2}} = \varphi_{2,k,n}(\tau) \tag{34}$$

$$k = 0,1, \quad n = 0,1,2,\ldots, N.$$

It follows from Theorem 3 that there are unique solutions of the problem (32)–(34) for an arbitrary index. Theorem 4 is thus proved. □

References

[1] J. B. Choi, *On Darboux-Protter problems for the hyperbolic equations with Bessel operators*, Far East J. Appl. Math. **5** (2001), no. 1, 75–85.

[2] J. D. Jeon, K. C. Khe, J. H. Park, Y. H. Jeon, and J. B. Choi, *Protter's conjugate boundary value problems for the two dimensional wave equation*, J. Korean Math. Soc. **33** (1996), no. 4, 857–863.

[3] J. D. Jeon and K. C. Khe, *On the conjugate Darboux-Protter problem in the case of independence of the polar angle*, Math. Zametki YaGU **4** (1997), no. 2, 91–99.

[4] K. C. Khe, *Nonuniqueness of the solution of the Darboux problem for a class of degenerate hyperbolic equations.* (Russian), Sibirsk. Math. Zh. **26** (1985), 159–161.

[5] ______, (Russian) Sibirsk. Math. Zh. **5** (1990), 207; English transl. in Silberian J. Math. Y. **5** (1990).

[6] ______, *Darboux-Protter problems for the two-dimensional wave equation* (Russian), Far East Branch Acad. Sci. USSR, Vladivostok (1990).

[7] M. H. Protter, *New boundary value problems for the wave equation and equations of mixed type*, J. Rational Mech. Anal. **3** (1954), 435–446.

[8] K. C. Tong, *On a boundary value problems for the wave equation*, Science Record, New Series **1** (1957), 277–278.

[9] A. Weinstein, *Sur le probleme de Cauchy pour l'equation de Poisson et l'equation des Ondes*, Comptes Rendas Acad. Sci-Paris **234** (1952), 2584–2585.

[10] ______, *The generalized radiation problem and the Euler-Poisson-Darboux equa -tion*, Samma Brasiliensis Math. **3** (1955), no. 7–9, 125–146.

Jong Bae Choi
Department of Mathematics
Kyung Hee University
Seoul 131-701, Korea
E-mail: byjbchoi@unitel.co.kr

Jong Yeoul Park
Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail: jyepark@hyowon.pusan.ac.kr