STRONG LAWS OF LARGE NUMBERS FOR RANDOM UPPER-SEMICONTINUOUS FUZZY SETS

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ABSTRACT. In this paper, we concern with SLLN for sums of independent random upper-semicontinuous fuzzy sets. We first give a generalization of SLLN for sums of independent and level-wise identically distributed random fuzzy sets, and establish a SLLN for sums of random fuzzy sets which is independent and compactly uniformly integrable in the strong sense. As a result, a SLLN for sums of independent and strongly tight random fuzzy sets is obtained.

1. Introduction

Since Puri and Ralescu [16] introduced the concept of fuzzy random variables, there has been increasing interest in limit theorems for fuzzy random variables because of its usefulness in several applied fields. Among others, strong laws of large numbers for sums of independent fuzzy random variables have been studied by several people. Klement, Puri and Ralescu [13] proved some limit theorems which includes a SLLN. Inoue [8] obtained a SLLN for sums of independent tight fuzzy random sets. Hong and Kim [7] studied Marcinkiewicz-type law of large numbers for fuzzy random variables under additional assumption. Molchanov [15] gave a short proof of SLLN for i.i.d. fuzzy random variables. Joo and Kim [11] obtained a SLLN for independent and levelwise identically distributed fuzzy random variables.

In this paper, we first give a generalization of the result in Joo and Kim [11] in a more general setting. Also, we establish a SLLN for sums

of random fuzzy sets which is independent and compactly uniformly integrable in the strong sense and satisfy Chung's condition.

2. Preliminaries

Let $\mathcal{K}(R^p)$ denote the family of non-empty compact subsets of the Euclidean space R^p. For $A, B \in \mathcal{K}(R^p)$, let us denote

$$\delta(A, B) = \sup_{a \in A} \inf_{b \in B} |a - b|, \qquad .$$

where $|.|$ denotes the Euclidean norm. Then the space $\mathcal{K}(R^p)$ is metrizable by the Hausdorff metric h defined by

$$h(A, B) = \max\{\delta(A, B), \delta(B, A)\}.$$

A norm of $A \in \mathcal{K}(R^p)$ is defined by

$$\|A\| = h(A, \{0\}) = \sup_{a \in A} |a|.$$

It is well-known that $\mathcal{K}(R^p)$ is complete and separable with respect to the Hausdorff metric h (See Debreu [6]). The addition and scalar multiplication on $\mathcal{K}(R^p)$ are defined as usual:

$$A \oplus B = \{a + b : a \in A, b \in B\}$$
$$\lambda A = \{\lambda a : a \in A\}$$

for $A, B \in \mathcal{K}(R^p)$ and $\lambda \in R$.

In what follows, clA denotes the closure of a set $A \subset R^p$. Let $\mathcal{F}(R^p)$ denote the family of all fuzzy sets $\tilde{u} : R^p \to [0, 1]$ with the following properties;

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R^p$ such that $\tilde{u}(x) = 1$;
(2) $\tilde{u}$ is upper semicontinuous;
(3) supp $\tilde{u} = cl\{x \in R^p : \tilde{u}(x) > 0\}$ is compact.

For a fuzzy set $\tilde{u}$ in R^p, the α-level set of $\tilde{u}$ is defined by

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1 \\ \text{supp } \tilde{u}, & \text{if } \alpha = 0. \end{cases}$$

Then, it follows immediately that for each $\alpha \in [0, 1]$,

$$\tilde{u} \in \mathcal{F}(R^p) \quad \text{if and only if} \quad L_\alpha \tilde{u} \in \mathcal{K}(R^p).$$

The linear structure on $\mathcal{F}(R^p)$ is defined as usual;

$$(\tilde{u} \oplus \tilde{v})(z) = \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y)),$$

$$(\lambda\tilde{u})(z) = \begin{cases} \tilde{u}(z/\lambda), & \text{if } \lambda \neq 0 \\ \tilde{0}, & \text{if } \lambda = 0 \end{cases}$$

for $\tilde{u}, \tilde{v} \in \mathcal{F}(R^p)$ and $\lambda \in R$, where $\tilde{0} = I_{\{0\}}$ denotes the indicator function of $\{0\}$. Then it is known that for each $\alpha \in [0, 1]$,

$$L_\alpha(\tilde{u} \oplus \tilde{v}) = L_\alpha \tilde{u} \oplus L_\alpha \tilde{v}$$

and

$$L_\alpha(\lambda\tilde{u}) = \lambda L_\alpha \tilde{u}.$$

LEMMA 2.1. *For* $\tilde{u} \in \mathcal{F}(R^p)$, *we define*

$$f_{\tilde{u}} : [0, 1] \longrightarrow (\mathcal{K}(R^p), h), f_{\tilde{u}}(\alpha) = L_\alpha \tilde{u}.$$

Then the followings hold;

(1) $f_{\tilde{u}}$ *is non-increasing, i.e.,* $\alpha \leq \beta$ *implies* $f_{\tilde{u}}(\alpha) \supset f_{\tilde{u}}(\beta)$,

(2) $f_{\tilde{u}}$ *is left continuous on* $(0, 1]$,

(3) $f_{\tilde{u}}$ *has right-limits on* $[0, 1)$ *and* $f_{\tilde{u}}$ *is right-continuous at* 0.

Conversely, if $g : [0, 1] \to \mathcal{K}(R^p)$ *is a function satisfying the above conditions* (1) $-$ (3), *then there exists a unique* $\tilde{v} \in \mathcal{F}(R^p)$ *such that* $g(\alpha) = L_\alpha \tilde{v}$ *for all* $\alpha \in [0, 1]$.

Proof. See Lemma 2.2 of Joo and Kim [10]. □

We denote $cl\{x \in R^p : \tilde{u}(x) > \alpha\}$ by $L_{\alpha+}\tilde{u}$. Then the right limit of $f_{\tilde{u}}$ at α is $L_{\alpha+}\tilde{u}$. Now we define for $J \subset [0, 1]$,

$$(2.1) \qquad w_{\tilde{u}}(J) = \sup_{\alpha_1, \alpha_2 \in J} h(L_{\alpha_1}\tilde{u}, L_{\alpha_2}\tilde{u})$$

then it follows that for $0 \leq \alpha < \beta \leq 1$,

$$w_{\tilde{u}}(\alpha, \beta) = w_{\tilde{u}}(\alpha, \beta] = h(L_{\alpha+}\tilde{u}, L_\beta \tilde{u}),$$

and

$$w_{\tilde{u}}[\alpha, \beta) = w_{\tilde{u}}[\alpha, \beta] = h(L_\alpha \tilde{u}, L_\beta \tilde{u}).$$

LEMMA 2.2. *For each* $\tilde{u} \in \mathcal{F}(R^p)$ *and* $\epsilon > 0$, *there exists a partition* $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ *of* $[0,1]$ *such that*

$$(2.2) \qquad w_{\tilde{u}}(\alpha_{i-1}, \alpha_i] < \epsilon, \ i = 1, 2, \ldots, r.$$

Proof. See of Lemma 2.3 of Joo and Kim [10].   □

Now, in order to generalize the Hausdorff metric on $\mathcal{K}(R^p)$ to $\mathcal{F}(R^p)$, we define the two metrics d_1, d_∞ on $\mathcal{F}(R^p)$ by

$$(2.3) \qquad d_1(\tilde{u}, \tilde{v}) = \int_0^1 h(L_\alpha \tilde{u}, L_\alpha \tilde{v}) \, d\alpha,$$

$$(2.4) \qquad d_\infty(\tilde{u}, \tilde{v}) = \sup_{0 \le \alpha \le 1} h(L_\alpha \tilde{u}, L_\alpha \tilde{v}).$$

Also, the norm of $\tilde{u}$ is defined as

$$\|\tilde{u}\| = d_\infty(\tilde{u}, \tilde{0}) = \sup_{x \in L_0 \tilde{u}} |x|.$$

Then it is well-known that $\mathcal{F}(R^p)$ is complete with respect to two metrics d_1 and d_∞, and that $\mathcal{F}(R^p)$ is separable with respect to d_1 but is not with respect to d_∞ (see Klement et al. [13]). Joo and Kim [10] introduced a new metric d_s on $\mathcal{F}(R^p)$ which makes it a separable metric space as follows:

DEFINITION 2.3. Let T denote the class of strictly increasing, continuous mapping of $[0,1]$ onto itself. For $\tilde{u}, \tilde{v} \in \mathcal{F}(R^p)$, we define

$$d_s(\tilde{u}, \tilde{v}) = \inf\{\epsilon > 0 : \text{there exists a } t \in T \text{ such that}$$
$$\sup_{0 \le \alpha \le 1} |t(\alpha) - \alpha| \le \epsilon \text{ and } d_\infty(\tilde{u}, t(\tilde{v})) \le \epsilon\},$$

where $t(\tilde{v})$ denotes the composition of $\tilde{v}$ and t.

3. Main results

Throughout this paper, let $(\Omega, \mathcal{A}, P)$ be a probability space. A set-valued function $X : \Omega \to \mathcal{K}(R^p)$ is called measurable if for each closed subset B of R^p,

$$X^{-1}(B) = \{\omega : X(\omega) \cap B \neq \emptyset\} \in \mathcal{A}.$$

It is well-known that the measurability of X is equivalent to the measurability of X considered as a map from Ω to the metric space $\mathcal{K}(R^p)$ endowed with the Hausdorff metric h. A set-valued function $X : \Omega \to \mathcal{K}(R^p)$ is called a random set if it is measurable.

A random set X is called integrably bounded if

$$E\|X\| < \infty.$$

The expectation of integrably bounded random set X is defined by

$$E(X) = \{E(f) : f \in L(\Omega, R^p) \text{ and } f(\omega) \in X(\omega) \text{ a.s.}\},$$

where $L(\Omega, R^p)$ denotes the class of all R^p-valued random variables f such that $E|f| < \infty$.

It is well-known that if X and Y are integrably bounded random sets, then

(1) $E(X) \in \mathcal{K}(R^p)$,
(2) $E(X \oplus Y) = E(X) \oplus E(Y)$,
(3) $E(\lambda X) = \lambda E(X)$.

The following SLLN for random sets was proved by Artstein and Vitale [2] and generalized by Artstein and Hansen [1].

THEOREM 3.1. *Let* $\{X_n\}$ *be a sequence of independent and identically distributed random sets. If* $E\|X_1\| < \infty$, *then*

$$\lim_{n\to\infty} h(\frac{1}{n} \bigoplus_{i=1}^{n} X_i, co(EX_1)) = 0 \text{ a.s.}$$

where $co(EX_1)$ *denotes the convex hull of* EX_1.

The above SLLN for random sets was generalized to the case of independent and compactly uniformly integrable random sets by Taylor and Inoue [17], Uemura [18]. Note that $\{X_n\}$ is called compactly uniformly integrable if for each $\epsilon > 0$ there exists a compact subset A of the metric space $\mathcal{K}(R^p)$ endowed with the Hausdorff metric h such that

$$\int_{\{X_n \notin A\}} \|X_n\| \, dP < \epsilon \text{ for all } n.$$

THEOREM 3.2. *Let $\{X_n\}$ be a sequence of independent random sets. If $\{X_n\}$ is compactly uniformly integrable and*

$$\sum_{n=1}^{\infty} \frac{1}{n^r} E\|X_n\|^r < \infty \quad \text{for some } 1 \leq r \leq 2,$$

then

$$\lim_{n\to\infty} h\left(\frac{1}{n} \bigoplus_{i=1}^{n} X_i, \frac{1}{n} \bigoplus_{i=1}^{n} co(EX_i)\right) = 0 \text{ a.s.}$$

Now we want to generalize the above SLLN for random sets to the case of independent random fuzzy sets with respect to the metric d_∞ defined as in (2.4). In earlier works which include Hong and Kim [7], Inoue [8] and Klement et al. [13], the metric d_1 defined as in (2.3) have been used. SLLN with respect to the metric d_∞ can be found in Molchanov [15], Joo and Kim [11] and SLLN with respect to the metric d_s in Joo [9].

A fuzzy set valued function $\tilde{X} : \Omega \to \mathcal{F}(R^p)$ is called measurable if for each closed subset B of R^p,

$$\tilde{X}^{-1}(B)(\omega) = \sup_{x \in B} \tilde{X}(\omega)(x)$$

is measurable when considered as a function from Ω to $[0, 1]$. This definition of measurability for a fuzzy set valued function was introduced by Butnariu [4]. It turned out that $\tilde{X}$ is measurable if and only if for each $\alpha \in [0, 1]$, $L_\alpha \tilde{X}$ is measurable as a set-valued function. A fuzzy set valued function $\tilde{X} : \Omega \to \mathcal{F}(R^p)$ is called a random fuzzy set (or fuzzy random variable) if it is measurable. Recent work of Kim [12] shows that a random fuzzy set can be identified with a random element of the metric space $\mathcal{F}(R^p)$ endowed with the metric d_s.

A random fuzzy set $\tilde{X}$ is called integrably bounded if $E\|\tilde{X}\| < \infty$. The expectation of integrably bounded random fuzzy set $\tilde{X}$ is a fuzzy subset $E(\tilde{X})$ of R^p defined by

$$E(\tilde{X})(x) = \sup\{\alpha \in [0, 1] : x \in E(L_\alpha \tilde{X})\}.$$

It is well-known that if $\tilde{X}$ and $\tilde{Y}$ are integrably bounded random fuzzy sets, then

(1) $E(\tilde{X}) \in \mathcal{F}(R^p)$, and $L_\alpha E(\tilde{X}) = E(L_\alpha \tilde{X})$ for all $\alpha \in [0, 1]$.
(2) $E(\tilde{X} \oplus \tilde{Y}) = E(\tilde{X}) \oplus E(\tilde{Y})$.
(3) $E(\lambda \tilde{X}) = \lambda E(\tilde{X})$.

We will also need the concepts of the convex hull of a fuzzy set in R^p. A fuzzy set $\tilde{u}$ in R^p is said to be convex if

$$\tilde{u}(\lambda x + (1 - \lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$$

for $x, y \in R^p$ and $\lambda \in [0, 1]$. The convex hull of $\tilde{u}$ is defined by

$$co(\tilde{u}) = \inf\{\tilde{v} | \tilde{v} \text{ is convex and } \tilde{v} \geq \tilde{u}\}.$$

It follows from Lowen [14] that for all $\alpha \in [0, 1]$,

$$L_\alpha(co(\tilde{u})) = co(L_\alpha \tilde{u}).$$

We denote by $F(R^p)$ the family of all $\tilde{u} \in \mathcal{F}(R^p)$ such that $\tilde{u}$ is convex. Thus if $\tilde{u} \in \mathcal{F}(R^p)$, then $co(\tilde{u}) \in F(R^p)$.

DEFINITION 3.3. Let $\tilde{X}$ and $\tilde{Y}$ be two random fuzzy sets. $\tilde{X}$ and $\tilde{Y}$ are called level-wise identically distributed if for each $\alpha \in [0, 1]$, $L_\alpha \tilde{X}$ and $L_\alpha \tilde{Y}$ are identically distributed random sets.

The following theorem is a generalization of Joo and Kim [11].

THEOREM 3.4. Let $\{\tilde{X}_n\}$ be a sequence of independent and level-wise identically distributed random fuzzy sets. If $E\|\tilde{X}_1\| < \infty$, then

$$\lim_{n \to \infty} d_\infty(\frac{1}{n} \bigoplus_{i=1}^n \tilde{X}_i, co(E\tilde{X}_1)) = 0 \text{ a.s.}$$

Proof. Let $\tilde{S}_n = \bigoplus_{i=1}^n \tilde{X}_i$ and let $\epsilon > 0$ be given. Then applying Lemma 2.2 to $\tilde{u} = co(E\tilde{X}_1)$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r$ of $[0, 1]$ such that

$$(3.1) \qquad h(L_{\alpha_{i-1}^+} co(E\tilde{X}_1), L_{\alpha_i} co(E\tilde{X}_1)) < \epsilon, \, i = 1, 2, \ldots, r.$$

If $0 < \alpha \leq 1$, then $\alpha_{k-1} < \alpha \leq \alpha_k$ for some k. Since $L_\alpha \tilde{S}_n \subset L_{\alpha_{k-1}^+} \tilde{S}_n$ and $L_\alpha co(E\tilde{X}_1) \supset L_{\alpha_k} co(E\tilde{X}_1)$, we have, by (3.1),

$$\delta(\frac{1}{n} L_\alpha \tilde{S}_n, L_\alpha co(E\tilde{X}_1)) \leq \delta(\frac{1}{n} L_{\alpha_{k-1}^+} \tilde{S}_n, L_{\alpha_k} co(E\tilde{X}_1))$$

$$\leq h(\frac{1}{n} L_{\alpha_{k-1}^+} \tilde{S}_n, L_{\alpha_k} co(E\tilde{X}_1))$$

$$\leq h(\frac{1}{n} L_{\alpha_{k-1}^+} \tilde{S}_n, L_{\alpha_{k-1}^+} co(E\tilde{X}_1)) + \epsilon.$$

Similarly, since $L_\alpha co(E\tilde{X}_1) \subset L_{\alpha_{k-1}^+} co(E\tilde{X}_1)$ and $L_\alpha \tilde{S}_n \supset L_{\alpha_k} \tilde{S}_n$, we obtain

$$\delta(L_\alpha co(E\tilde{X}_1), \frac{1}{n} L_\alpha \tilde{S}_n) \leq \delta(L_{\alpha_{k-1}^+} co(E\tilde{X}_1), \frac{1}{n} L_{\alpha_k} \tilde{S}_n)$$

$$\leq h(L_{\alpha_{k-1}^+} co(E\tilde{X}_1), \frac{1}{n} L_{\alpha_k} \tilde{S}_n)$$

$$\leq h(L_{\alpha_k} co(E\tilde{X}_1), \frac{1}{n} L_{\alpha_k} \tilde{S}_n) + \epsilon.$$

Since $L_\alpha co(E\tilde{X}_1) = co(E(L_\alpha \tilde{X}_1))$ for each $\alpha \in [0,1]$, we conclude that

$$d_\infty(\frac{1}{n} \tilde{S}_n, co\ E(\tilde{X}_1)) \leq \max_{1 \leq k \leq r} h(\frac{1}{n} L_{\alpha_{k-1}^+} \tilde{S}_n, co(E(L_{\alpha_{k-1}^+} \tilde{X}_1)))$$

$$+ \max_{1 \leq k \leq r} h(\frac{1}{n} L_{\alpha_k} \tilde{S}_n, co(E(L_{\alpha_k} \tilde{X}_1))) + \epsilon.$$

Since $L_\alpha \tilde{S}_n = \bigoplus_{i=1}^{n} L_\alpha \tilde{X}_i$ for each $\alpha \in [0,1]$, we obtain by Theorem 3.1,

$$\varlimsup_{n \to \infty} d_\infty(\frac{1}{n} \tilde{S}_n, co(E\tilde{X}_1)) \leq \epsilon \text{ a.s.}$$

This completes the proof. $\square$

EXAMPLE. Let $\tilde{u} \in \mathcal{F}(R^p)$ be fixed and let $\{Y_n\}$ be i.i.d. R^p-valued random variables with $E|Y_1| < \infty$ in the usual sense. We identify Y_n with the indicator function I_{Y_n} of Y_n. If we define $\tilde{X}_n = \tilde{u} \oplus Y_n$, then

$$\tilde{X}_n(\omega)(x) = \tilde{u}(x - Y_n(\omega))$$

i.e., $\tilde{X}_n(\omega)$ is the translation of $\tilde{u}$ by $Y_n(\omega)$. Now for each $\alpha \in [0,1]$,

$$L_\alpha X_n(\omega) = L_\alpha \tilde{u} \oplus \{Y_n(\omega)\}.$$

Hence, $E(L_\alpha X_1) = L_\alpha \tilde{u} \oplus \{EY_1\}$ and so $E\tilde{X}_1 = \tilde{u} \oplus EY_1$, i.e., $(E\tilde{X}_1)(x) = \tilde{u}(x - EY_1)$. By the above theorem, we have

$$\frac{1}{n} \bigoplus_{i=1}^{n} \tilde{X}_i \xrightarrow{d_\infty} co(E\tilde{X}_1) \text{ a.s.}$$

We note that $co(E\tilde{X}_1) = co(\tilde{u}) \oplus EY_1$. Furthermore, if $\tilde{u} \in F(R^p)$, then

$$\frac{1}{n} \bigoplus_{i=1}^{n} \tilde{X}_i = \tilde{u} \oplus (\frac{1}{n} \sum_{i=1}^{n} Y_i),$$

that is,

$$\frac{1}{n} \bigoplus_{i=1}^{n} \tilde{X}_i(\omega)(x) = \tilde{u}(x - \frac{1}{n} \sum_{i=1}^{n} Y_i(\omega))$$

and $co(E\tilde{X}_1) = E\tilde{X}_1 = \tilde{u} \oplus EY_1$. Thus, in this case,

$$\tilde{u} \oplus (\frac{1}{n} \sum_{i=1}^{n} Y_i) \xrightarrow{d_\infty} \tilde{u} \oplus EY_1 \text{ a.s.}$$

DEFINITION 3.5. Let $\{\tilde{X}_n\}$ be a sequence of random fuzzy sets.

(1) $\{\tilde{X}_n\}$ is said to be tight if for each $\epsilon > 0$ there exists a compact subset A of $\mathcal{F}(R^p)$ relative to d_s-topology such that

$$P(\tilde{X}_n \notin A) < \epsilon \text{ for all } n.$$

If A is compact relative to d_∞-topology, then $\{\tilde{X}_n\}$ is said to be strongly tight.

(2) $\{\tilde{X}_n\}$ is said to be compactly uniformly integrable if for each $\epsilon > 0$ there exists a compact subset A of $\mathcal{F}(R^p)$ relative to d_s-topology such that

$$\int_{\{\tilde{X}_n \notin A\}} \|\tilde{X}_n\| \, dP < \epsilon \text{ for all } n.$$

If A is compact relative to d_∞-topology, then $\{\tilde{X}_n\}$ is said to be compactly uniformly integrable in the strong sense.

Now we wish to obtain SLLN for independent and strongly compactly uniformly integrable random fuzzy sets. To this end, we need the following lemmas.

LEMMA 3.6. Let $\mathcal{F}_0(R^p)$ be a separable subspace of $\mathcal{F}(R^p)$ with respect to the metric d_∞. If $\tilde{X}$ is a $\mathcal{F}_0(R^p)$-valued random fuzzy set, then $\tilde{X}$ is a random element of the metric space $\mathcal{F}_0(R^p)$ endowed the metric d_∞.

Proof. Let $\mathcal{B}_\infty$ and $\mathcal{B}_s$ be the Borel σ-fields of $\mathcal{F}_0(R^p)$ with respect to the metric d_∞ and d_s, respectively. Since d_s-open sets are d_∞-open sets, it is clear that $\mathcal{B}_s \subset \mathcal{B}_\infty$. On the other hand, by Remark 1 of Kim [12], every d_∞-open ball is $\mathcal{B}_s$-measurable. Since $\mathcal{F}_0(R^p)$ is separable with respect to the metric d_∞, every d_∞-open subsets of $\mathcal{F}_0(R^p)$ can be represented by a countable union of d_∞-open balls of $\mathcal{F}_0(R^p)$. Thus, every d_∞-open subsets of $\mathcal{F}_0(R^p)$ is $\mathcal{B}_s$-measurable, and so $\mathcal{B}_s \supset \mathcal{B}_\infty$. This completes the proof. $\qquad\square$

LEMMA 3.7. *Let* $\tilde{u}_1, \ldots, \tilde{u}_n \in \mathcal{F}(R^p)$. *If* $\|\tilde{u}_i\| \leq M$ *for all* $i = 1, \ldots, n$, *then*

$$d_\infty\Big(\bigoplus_{i=1}^{n} \tilde{u}_i, \bigoplus_{i=1}^{n} co(\tilde{u}_i)\Big) \leq \sqrt{p}M.$$

Proof. Since $\|L_\alpha \tilde{u}_i\| \leq \|\tilde{u}_i\| \leq M$ for all $\alpha \in [0,1]$, we have by Shapley-Folkman inequality,

$$h\Big(L_\alpha \bigoplus_{i=1}^{n} \tilde{u}_i, L_\alpha \bigoplus_{i=1}^{n} co(\tilde{u}_i)\Big)$$
$$= h\Big(\bigoplus_{i=1}^{n} L_\alpha \tilde{u}_i, \bigoplus_{i=1}^{n} co(L_\alpha \tilde{u}_i)\Big)$$
$$\leq \sqrt{p}M \quad \text{for all } \alpha \in [0,1].$$

This gives the desired result. $\qquad\square$

LEMMA 3.8. *Let* $\tilde{u} \in \mathcal{F}(R^p)$ *and* $\{\lambda_n\}$ *be a sequence of 0 and 1's. Then*

(1) $d_\infty(\frac{1}{n} \overbrace{\tilde{u} \oplus \cdots \oplus \tilde{u}}^{n \text{ times}}, co(\tilde{u})) \to 0$ *as* $n \to \infty$.

(2) $\frac{1}{n} d_\infty(\bigoplus_{i=1}^{n} \lambda_i \tilde{u}, \bigoplus_{i=1}^{n} \lambda_i co(\tilde{u})) \to 0$ *as* $n \to \infty$.

Proof. (1) By Lemma 3.7, we have

$$d_\infty\Big(\frac{1}{n} \overbrace{\tilde{u} \oplus \cdots \oplus \tilde{u}}^{n \text{ times}}, co(\tilde{u})\Big)$$
$$= \frac{1}{n} d_\infty\Big(\overbrace{\tilde{u} \oplus \cdots \oplus \tilde{u}}^{n \text{ times}}, \overbrace{co(\tilde{u}) \oplus \cdots \oplus co(\tilde{u})}^{n \text{ times}}\Big)$$
$$\leq \frac{1}{n} \sqrt{p}\|\tilde{u}\| \to 0.$$

(2) Let k_n be the number of $\{i|\lambda_i = 1, 1 \le i \le n\}$. If $\frac{k_n}{n} \to 0$, then

$$\frac{1}{n} d_\infty \big(\bigoplus_{i=1}^{n} \lambda_i \tilde{u}, \bigoplus_{i=1}^{n} \lambda_i co(\tilde{u}) \big)$$

$$= \frac{1}{n} d_\infty \big(\overbrace{\tilde{u} \oplus \cdots \oplus \tilde{u}}^{k_n \text{ times}}, \overbrace{co(\tilde{u}) \oplus \cdots \oplus co(\tilde{u})}^{k_n \text{ times}} \big)$$

$$\le \frac{k_n}{n} d_\infty (\tilde{u}, co(\tilde{u})) \to 0.$$

If $k_n \to \infty$, then

$$\frac{1}{n} d_\infty \big(\bigoplus_{i=1}^{n} \lambda_i \tilde{u}, \bigoplus_{i=1}^{n} \lambda_i co(\tilde{u}) \big)$$

$$= \frac{k_n}{n} d_\infty \big(\frac{1}{k_n} \overbrace{\tilde{u} \oplus \cdots \oplus \tilde{u}}^{k_n \text{ times}}, \frac{1}{k_n} \overbrace{co(\tilde{u}) \oplus \cdots \oplus co(\tilde{u})}^{k_n \text{ times}} \big)$$

$$\le d_\infty \big(\frac{1}{k_n} \overbrace{\tilde{u} \oplus \cdots \oplus \tilde{u}}^{k_n \text{ times}}, co(\tilde{u}) \big) \to 0 \text{ by } (1).$$

This completes the proof. $\square$

Now we state one of our main theorems which is a generalization of Theorem 3.2.

THEOREM 3.9. *Let $\{\tilde{X}_n\}$ be a sequence of independent random fuzzy sets. If*

(3.2) $\{\tilde{X}_n\}$ *is compactly uniformly integrable in the strong sense,*

and

(3.3) $$\sum_{n=1}^{\infty} \frac{1}{n^r} E\|\tilde{X}_n\|^r < \infty \text{ for some } 1 \le r \le 2,$$

then

$$\lim_{n\to\infty} \frac{1}{n} d_\infty \big(\bigoplus_{i=1}^{n} \tilde{X}_i, \bigoplus_{i=1}^{n} co(E\tilde{X}_i) \big) = 0 \text{ a.s.}$$

Proof. The proof will be proceeded by similar arguments in Taylor and Inoue [17], Uemura [18]. Let $\epsilon > 0$ be given. By compactly uniform integrability in the strong sense, there exists a compact subset K of $\mathcal{F}(R^p)$ relative to d_∞-topology such that

$$(3.4) \qquad E\|I_{\{\tilde{X}_n \notin K\}}\tilde{X}_n\| < \epsilon \text{ for all } n.$$

Since K is compact, there exist $\tilde{u}_1, \ldots, \tilde{u}_m \in K$ such that

$$K \subset \bigcup_{k=1}^m N(\tilde{u}_k, \epsilon),$$

where $N(\tilde{u}_k, \epsilon) = \{\tilde{v} \in \mathcal{F}(R^p) : d_\infty(\tilde{v}, \tilde{u}_k) < \epsilon\}$. Note that $N(\tilde{u}_k, \epsilon) \in \mathcal{B}_s$ by Remark 1 of Kim [12]. Now let us denote

$$B_1 = N(\tilde{u}_1, \epsilon),$$
$$B_k = N(\tilde{u}_k, \epsilon)\backslash(\cup_{j=1}^{k-1} N(\tilde{u}_j, \epsilon)), \ k = 2, \ldots, m,$$

and define

$$\tilde{Y}_n = \bigoplus_{k=1}^m I_{\{\tilde{X}_n \in B_k\}}\tilde{u}_k.$$

Then for each n,

$$\frac{1}{n}d_\infty\left(\bigoplus_{i=1}^n \tilde{X}_i, \bigoplus_{i=1}^n co(E\tilde{X}_i)\right)$$

(I) $\qquad \le \dfrac{1}{n}d_\infty\left(\bigoplus_{i=1}^n \tilde{X}_i, \bigoplus_{i=1}^n I_{\{\tilde{X}_i \in K\}}\tilde{X}_i\right)$

(II) $\qquad + \dfrac{1}{n}d_\infty\left(\bigoplus_{i=1}^n I_{\{\tilde{X}_i \in K\}}\tilde{X}_i, \bigoplus_{i=1}^n I_{\{\tilde{X}_i \in K\}}\tilde{Y}_i\right)$

(III) $\qquad + \dfrac{1}{n}d_\infty\left(\bigoplus_{i=1}^n I_{\{\tilde{X}_i \in K\}}\tilde{Y}_i, \bigoplus_{i=1}^n I_{\{\tilde{X}_i \in K\}} co(\tilde{Y}_i)\right)$

(IV) $\qquad + \dfrac{1}{n}d_\infty\left(\bigoplus_{i=1}^n (I_{\{\tilde{X}_i \in K\}} co(\tilde{Y}_i), \bigoplus_{i=1}^n EI_{\{\tilde{X}_i \in K\}}co(\tilde{Y}_i)\right)$

(V) $\qquad + \dfrac{1}{n}d_\infty\left(\bigoplus_{i=1}^n EI_{\{\tilde{X}_i \in K\}} co(\tilde{Y}_i), \bigoplus_{i=1}^n EI_{\{co(\tilde{X}_i) \in K\}}co(\tilde{X}_i)\right)$

(VI) $\qquad + \dfrac{1}{n}d_\infty\left(\bigoplus_{i=1}^n EI_{\{co(\tilde{X}_i) \in K\}}co(\tilde{X}_i), \bigoplus_{i=1}^n Eco(\tilde{X}_i)\right).$

For (I), first we note that

$$(I) \leq \frac{1}{n}\sum_{i=1}^{n} d_{\infty}(\tilde{X}_i, I_{\{\tilde{X}_i \in K\}}\tilde{X}_i)$$

$$\leq \frac{1}{n}\sum_{i=1}^{n} \|I_{\{\tilde{X}_i \notin K\}}\tilde{X}_i\|$$

$$\leq \frac{1}{n}\sum_{i=1}^{n} (\|I_{\{\tilde{X}_i \notin K\}}\tilde{X}_i\| - E\|I_{\{\tilde{X}_i \notin K\}}\tilde{X}_i\|)$$

$$+ \frac{1}{n}\sum_{i=1}^{n} E\|I_{\{\tilde{X}_i \notin K\}}\tilde{X}_i\|.$$

Since $\{\|I_{\{\tilde{X}_n \notin K\}}\tilde{X}_n\| : n \geq 1\}$ is a sequence of independent real-valued random variables and

$$\sum_{n=1}^{\infty} \frac{1}{n^r} E\|I_{\{\tilde{X}_n \notin K\}}\tilde{X}_n\|^r < \infty$$

from (3.3), Chung's strong law of large numbers and (3.4) imply that

$$\limsup_{n\to\infty}(I)$$

$$\leq \limsup_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} E\|I_{\{\tilde{X}_i \notin K\}}\tilde{X}_i\| \leq \epsilon \text{ a.s.}$$

For (II), note that for each $\tilde{X}_i(\omega) \in K$, there is a k such that $\tilde{X}_i(\omega) \in B_k$ and so

$$d_{\infty}(\tilde{X}_i(\omega), \tilde{u}_k) < \epsilon.$$

Thus, by construction of $\tilde{Y}_n$,

$$(3.5) \qquad d_{\infty}(I_{\{\tilde{X}_i \in K\}}\tilde{X}_i, I_{\{\tilde{X}_i \in K\}}\tilde{Y}_i) < \epsilon \text{ for each } i$$

so that for all n,

$$(II) \leq \frac{1}{n}\sum_{i=1}^{n} d_{\infty}(I_{\{\tilde{X}_i \in K\}}\tilde{X}_i, I_{\{\tilde{X}_i \in K\}}\tilde{Y}_i) < \epsilon.$$

For (III), it follows from Lemma 3.8(2) that

$$\text{(III)} = \frac{1}{n} d_\infty \Big(\overset{n}{\underset{i=1}{\oplus}} \overset{m}{\underset{k=1}{\oplus}} I_{\{\tilde{Y}_i = \tilde{u}_k\}} \ \tilde{u}_k, \ \overset{n}{\underset{i=1}{\oplus}} \overset{m}{\underset{k=1}{\oplus}} I_{\{\tilde{Y}_i = \tilde{u}_k\}} \ co(\tilde{u}_k) \Big)$$

$$\leq \frac{1}{n} \sum_{k=1}^{m} d_\infty \Big(\overset{n}{\underset{i=1}{\oplus}} I_{\{\tilde{Y}_i = \tilde{u}_k\}} \ \tilde{u}_k, \ \overset{n}{\underset{i=1}{\oplus}} I_{\{\tilde{Y}_i = \tilde{u}_k\}} \ co(\tilde{u}_k) \Big) \to 0 \text{ for all } \omega.$$

Now for (IV), first we note that the space $F(R^p)$ endowed the metric d_∞ can be embedded into a Banach space. Since $\{I_{\{\tilde{X}_i \in K\}} \ co(\tilde{Y}_i)\}$ is $F(R^p)$-valued and the range of $I_{\{\tilde{X}_i \in K\}} \ co(\tilde{Y}_i)$ is finite, by Lemma 3.7, $\{I_{\{\tilde{X}_i \in K\}} \ co(\tilde{Y}_i)\}$ can be identified with random elements in a separable Banach space. Since

$$\|I_{\{\tilde{X}_i \in K\}} \ co(\tilde{Y}_i)\| \leq \max_{1 \leq k \leq m} \|co(\tilde{u}_k)\| < \infty,$$

we have by Theorem 2.3 of Daffer and Taylor [5],

$$\text{(IV)} \ \to \ 0 \text{ a.s.}$$

For (V), it follows from (3.5) that

$$\text{(V)} \leq \frac{1}{n} \sum_{i=1}^{n} E[d_\infty(I_{\{\tilde{X}_i \in K\}} \ co(\tilde{Y}_i), I_{\{co(\tilde{X}_i) \in K\}} \ co(\tilde{X}_i))]$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} E[d_\infty(I_{\{\tilde{X}_i \in K\}} \tilde{Y}_i, I_{\{\tilde{X}_i \in K\}} \tilde{X}_i)]$$

$$< \epsilon \text{ for all } n.$$

Finally, for (VI), we have from (3,4),

$$\text{(VI)} \leq \frac{1}{n} \sum_{i=1}^{n} E\|I_{\{co(\tilde{X}_i) \notin K\}} co(\tilde{X}_i)\|$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} E\|I_{\{\tilde{X}_i \notin K\}} \tilde{X}_i\| < \epsilon.$$

Therefore, we conclude that

$$\limsup_{n \to \infty} d_\infty \Big(\frac{1}{n} \overset{n}{\underset{i=1}{\oplus}} \tilde{X}_i, \frac{1}{n} \overset{n}{\underset{i=1}{\oplus}} Eco(\tilde{X}_i) \Big) \leq 4\epsilon \text{ a.s.}$$

This completes the proof.   □

COROLLARY 3.10. *Let* $\{\tilde{X}_n\}$ *be a sequence of independent random fuzzy sets. If*

(1) $\{\tilde{X}_n\}$ *is strongly tight,*

(2) $\sup_n E\|\tilde{X}_n\|^r = M < \infty$ *for some* $r > 1$,

then

$$\lim_{n\to\infty} \frac{1}{n} d_\infty \left(\bigoplus_{i=1}^{n} \tilde{X}_i, \ \bigoplus_{i=1}^{n} co(E\tilde{X}_i) \right) = 0 \text{ a.s.}$$

Proof. For each subset A of $\mathcal{F}(R^p)$, we have

$$\int_{\{\tilde{X}_n \notin A\}} \|\tilde{X}_n\| \ dP \leq (E\|\tilde{X}_n\|^r)^{1/r} P[\tilde{X}_n \notin A]^{(r-1)/r}$$

$$\leq M^{1/r} P[\tilde{X}_n \notin A]^{(r-1)/r}.$$

For $\epsilon > 0$, by the strong tightness of $\{\tilde{X}_n\}$, if we choose a compact subset A of $\mathcal{F}(R^p)$ relative to d_∞ such that

$$P[\tilde{X}_n \notin A] \leq \epsilon^{r/(r-1)} M^{-1/(r-1)}$$

for all n, then

$$\int_{\{\tilde{X}_n \notin A\}} \|\tilde{X}_n\| \ dP < \epsilon \text{ for all } n.$$

Therefore, condition(3.2) is satisfied. Condition (3.3) of Theorem 3.9 is easily satisfied by (2).   □

REMARK. It remains open problem whether SLLN with respect to the metric d_s hold if strong tightness and compactly uniform integrability in the strong sense are replaced by tightness and compactly uniform integrability, respectively. The difficulties arise from the fact that the d_s is not translation invariant and that the inequality $d_s(E\tilde{X}, E\tilde{Y}) \leq E(d_s(\tilde{X}, \tilde{Y}))$ does not hold.

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