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CONVERGENCE THEOREMS AND STABILITY PROBLEMS OF THE MODIFIED ISHIKAWA ITERATIVE SEQUENCES FOR STRICTLY SUCCESSIVELY HEMICONTRACTIVE MAPPINGS

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ABSTRACT. The purpose of this paper is to introduce the concept of a class of strictly successively hemicontractive mappings and construct certain stable and almost stable iteration procedures for the iterative approximation of fixed points for asymptotically nonexpansive and strictly successively hemicontractive mappings in Banach spaces.

1. Introduction

Let X be a Banach space, X^* the dual space of X, $\langle\cdot,\cdot\rangle$ the dual pair between X and X^* and $J : X \to 2^{X^*}$ the normalized duality mapping defined by

$$J(x) = \left\{ f \in X^* : Re\langle x, f\rangle = ||x||||f||, \quad ||x|| = ||f|| \right\}, \quad x \in X.$$

$F(T)$ and N stand for the set of fixed points of an operator T and the set of positive integers, respectively.

DEFINITION 1.1. Let K be a nonempty subset of a Banach space X and $T : K \to K$ be an operator.

(i) T is said to be asymptotically nonexpansive [2] if T^m is continuous for some integer $m \in N$ and if

$$\limsup_{n\to\infty} \sup\left\{ ||T^n x - T^n y|| - ||x - y|| : \quad x, y \in K \right\} \le 0;$$

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(ii) T is said to be strictly hemicontractive [3] if $F(T) \neq \emptyset$ and if there exists $t > 1$ such that for all $x \in K$ and $q \in F(T)$, there exists $j(x-q) \in J(x-q)$ satisfying

$$Re\langle Tx - q, j(x-q)\rangle \leq \frac{1}{t}||x-q||^2 ;$$

(iii) T is said to be strictly successively hemicontractive if $F(T) \neq \emptyset$ and if there exist $t > 1$ and $n_0 \in N$ such that for any $x \in K$ and $q \in F(T)$, there exists $j(x-q) \in J(x-q)$ satisfying

$$Re\langle T^n x - q, j(x-q)\rangle \leq \frac{1}{t}||x-q||^2, \quad n \geq n_0. \tag{1.1}$$

Now we give two examples of selfmappings which are both asymptotically nonexpansive and strictly successively hemicontractive, but not continuous. Moreover, the mapping in Example 1.1 is strictly hemicontractive and the mapping in Example 1.2 is neither strictly hemicontractive nor nonexpansive.

EXAMPLE 1.1. Let $X = (-\infty, \infty)$ with the usual norm and $a_n = 2^{-n}$ for $n \geq 0$. Take $K = [0,1]$ and define $T : K \to K$ by

$$Tx = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{4}, & \text{if } x = 1 \\ a_n - x, & \text{if } x \in [\frac{1}{2}(a_{n+1}+a_n), a_n) \\ x - a_{n+1}, & \text{if } x \in [a_{n+1}, \frac{1}{2}(a_{n+1}+a_n)) \end{cases}$$

for all $n \geq 0$. Then $F(T) = \{0\}$ and T is not continuous at $x = 1$. It is easy to verify that

$$Tx \leq \frac{1}{2}x, \quad x \in K.$$

Therefore T^2 is continuous in K and $T^n K \subseteq [0, 2^{-n}]$ for each $n \in N$. It follows that

$$\limsup_{n\to\infty} \sup\{||T^n x - T^n y|| - ||x-y|| : x, y \in K\} \leq \limsup_{n\to\infty} 2^{-n} = 0.$$

That is, T is asymptotically nonexpansive. Take $t = 2$ and $n_0 = 1$. Then for any $x \in K$, there exists $j(x) \in J(x)$ satisfying

$$Re\langle T^n x, j(x)\rangle = T^n x \cdot x \leq \frac{1}{t}x^2, \quad n \geq n_0.$$

That is, T is strictly successively hemicontractive and strictly hemicontractive.

EXAMPLE 1.2. Let X and $\{a_n\}_{n=0}^{\infty}$ be as in Example 1.1. Take $K = [0,1] \cup \{3\}$ and define $T : K \to K$ by

$$Tx = \begin{cases} 0, & \text{if } x \in \{0,3\} \\ 3, & \text{if } x = 1 \\ a_n - x, & \text{if } x \in [\frac{1}{2}(a_{n+1} + a_n), a_n) \\ x - a_{n+1}, & \text{if } x \in [a_{n+1}, \frac{1}{2}(a_{n+1} + a_n)) \end{cases}$$

for all $n \geq 0$. Clearly, $F(T) = \{0\}$ and T is not continuous at $x = 1$. Since $\|Tx - Ty\| = 3 > 2 = \|x - y\|$ for $x = 1$ and $y = 3$, T is not nonexpansive. Observe that

$$Re\langle Tx - T0, j(x - 0)\rangle = 3 > \frac{1}{t} = \frac{1}{t}||x - 0||^2$$

for $x = 1$ and any $t > 1$. Therefore, T is not strictly hemicontractive. Note that $T^n x \leq \frac{1}{2^n} x$ for $n \geq 2$ and $x \in K$ and that $T^n K \subseteq [0, 2^{-n}]$ for $n \geq 2$. Set $t = 4$ and $n_0 = 2$. As in view of Example 1.1, we can conclude that T is asymptotically nonexpansive and strictly successively hemicontractive.

Let K be a nonempty subset of a Banach space X and $T : K \to K$ be an operator. Assume that $x_0 \in K$ and $x_{n+1} = f(T, x_n)$ defines an iterative scheme which produces a sequence $\{x_n\}_{n=0}^{\infty} \subset K$. Furthermore, suppose that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $q \in F(T) \neq \emptyset$. Let $\{y_n\}_{n=0}^{\infty}$ be any sequence in K and set $\epsilon_n = ||y_{n+1} - f(T, y_n)||$.

DEFINITION 1.2. (i) The iterative scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, x_n)$ is called T-stable on K if $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = q$;

(ii) The iterative scheme $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = f(T, x_n)$ is said to be almost T-stable on K if $\sum_{n=0}^{\infty} \epsilon_n < \infty$ implies that $\lim_{n\to\infty} y_n = q$.

It is easy to see that an iterative scheme $\{x_n\}_{n=0}^{\infty}$ which is T-stable on K is almost T-stable on K. Osilike [11] proved that the converse is not true.

DEFINITION 1.3. Let K be a nonempty convex subset of a Banach space X and let $T : K \to K$ be an operator. Suppose that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are any bounded sequences in K and $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$,

$\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ are arbitrary sequences in $[0,1]$ satisfying certain conditions.

(i) For given $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} y_n = (1-b_n)x_n + b_n T^n x_n, \\ x_{n+1} = (1-a_n)x_n + a_n T^n y_n, \quad n \geq 0, \end{cases}$$

is called the modified Ishikawa iterative sequence [16];

(ii) If $b_n = 0$ for $n \geq 0$ in (i), then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0 \in K, \quad x_{n+1} = (1-a_n)x_n + a_n T^n x_n, \quad n \geq 0$$

is called the modified Mann iterative sequence [14];

(iii) For given $x_0 \in K$ and $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} y_n = a'_n x_n + b'_n T^n x_n + c'_n v_n, \\ x_{n+1} = a_n x_n + b_n T^n y_n + c_n u_n, \quad n \geq 0, \end{cases}$$

is called the modified Ishikawa iterative sequence with errors ;

(iv) If $b'_n = c'_n = 0$ for all $n \geq 0$ in (iii), then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0 \in K, \quad x_{n+1} = a_n x_n + b_n T^n x_n + c_n u_n, \quad n \geq 0,$$

is called the modified Mann iterative sequence with errors.

Goebel-Kirk [4] first introduced the concept of asymptotically nonexpansive mappings. Kirk [9] and Bruck et al. [2] gave also similar concepts. Goebel-Kirk [4] proved that if K is a bounded closed convex subset of a uniformly convex Banach space X, then every asymptotically nonexpansive selfmapping T of K has a fixed point.

The iterative approximation problems of fixed points for asymptotically nonexpansive mappings were studied extensively by Bose [1], Bruck et al. [2], Passty [12], Schu ([14], [15]), Tan-Xu [16] and Xu [17].

Rhoades [13] proved that the Mann and Ishikawa iterative sequences may exhibit different behaviors for different classes of nonlinear mappings. Harder-Hicks [6] revealed the importance of investigating the stability of various iterative procedures for many classes of nonlinear mappings. Harder [5] gave applications of stability results to first order differential equations.

The purpose of this paper is to establish the strong convergence theorem, stability and almost stability of the modified Ishikawa iterative sequences with errors for the iterative approximation of fixed points for asymptotically nonexpansive and strictly successively hemicontractive mappings in Banach spaces.

2. Preliminaries

The following results play important roles in this paper.

LEMMA 2.1. ([10]) *Suppose that* $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$ *and* $\{\omega_n\}_{n=0}^{\infty}$ *are nonnegative sequences such that*

$$\alpha_{n+1} \leq (1-\omega_n)\alpha_n + \beta_n\omega_n + \gamma_n, \quad \forall n \geq 0,$$

with $\{\omega_n\}_{n=0}^{\infty} \subset [0,1]$, $\sum_{n=0}^{\infty}\omega_n = \infty$, $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=0}^{\infty}\gamma_n < \infty$. *Then* $\lim_{n\to\infty}\alpha_n = 0$.

LEMMA 2.2. ([8]) *Let* X *be a Banach space and* $x, y \in X$. *Then* $||x|| \leq ||x+\gamma y||$ *for all* $\gamma > 0$ *if and only if there exists* $j(x) \in J(x)$ *such that* $Re\langle y, j(x)\rangle \geq 0$.

In the sequel, let I denote the identity mapping, $d_n = b_n + c_n$, $d'_n = b'_n + c'_n$ and $k = \frac{t-1}{t}$, where t is the constant appearing in (1.1).

LEMMA 2.3. *Let* K *be a nonempty subset of a Banach space* X *and let* $T : K \to K$ *be an operator. Then the following conditions are equivalent*:

(i) T *is strictly successively hemicontractive*;

(ii) $F(T) \neq \emptyset$ *and there exist* $t > 1$ *and* $n_0 \in N$ *such that*

$$||x-q|| \leq \left|\left|x-q+\gamma\Big[(I-T^n-kI)x-(I-T^n-kI)q\Big]\right|\right| \tag{2.1}$$

for all $x \in K$, $q \in F(T)$, $\gamma > 0$ *and* $n \geq n_0$.

Proof. Observe that (1.1) is equivalent to

$$Re\Big\langle (I-T^n-kI)x-(I-T^n-kI)q, j(x-q)\Big\rangle \geq 0, \quad n \geq n_0. \tag{2.2}$$

Thus, we have the desired result from Lemma 2.2. This completes the proof. □

3. Main results

THEOREM 3.1. *Let* K *be a nonempty convex subset of a Banach space* X. *Assume that* $T : K \to K$ *is asymptotically nonexpansive and strictly successively hemicontractive. Put*

$$s_n = \max\Big\{0, \sup\{||T^n x - T^n y|| - ||x-y|| : x, y \in K\}\Big\}, \quad n \geq 0,$$

so that

$$\lim_{n\to\infty} s_n = 0. \tag{3.1}$$

Let $\{u_n\}_{n=0}^{\infty}$ *and* $\{v_n\}_{n=0}^{\infty}$ *be arbitrary bounded sequences in* K. *Suppose that* $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$, $\{c'_n\}_{n=0}^{\infty}$ *and* $\{r_n\}_{n=0}^{\infty}$ *are arbitrary sequences in* $[0,1]$ *satisfying*

$$a_n + d_n = a'_n + d'_n = 1, \quad n \ge 0; \tag{3.2}$$

$$c_n = r_n d_n, \quad n \ge 0; \tag{3.3}$$

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} d'_n = 0; \tag{3.4}$$

$$\sum_{n=0}^{\infty} d_n = \infty. \tag{3.5}$$

Suppose that $\{x_n\}_{n=0}^{\infty}$ *is the sequence generated from some* $x_0 \in K$ *by*

$$\begin{cases} z_n = a'_n x_n + b'_n T^n x_n + c'_n v_n, \\ x_{n+1} = a_n x_n + b_n T^n z_n + c_n u_n, \quad n \ge 0. \end{cases} \tag{3.6}$$

Let $\{y_n\}_{n=0}^{\infty}$ *be an arbitrary sequence in* K *and define* $\{\epsilon_n\}_{n=0}^{\infty}$ *by*

$$w_n = a'_n y_n + b'_n T^n y_n + c'_n v_n, \quad \epsilon_n = ||y_{n+1} - p_n||, \quad n \ge 0, \tag{3.7}$$

where $p_n = a_n y_n + b_n T^n w_n + c_n u_n$. *Then there exists* $n_1 \in N$ *such that*

(i) *The sequence* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the unique fixed point* q *of* T. *Moreover,*

$$\begin{aligned} ||x_{n+1} - q|| \le (1 - \frac{1}{2}kd_n)||x_n - q|| + (4d_n^2 + 6c_n + 4d_n)s_n \\ + 4c_n||u_n - q|| + (2d_n^2 + 3c_n + 2d_n)c'_n||v_n - q||, \end{aligned}$$

$n \ge n_1$;

(ii) $||y_{n+1} - q|| \le (1 - \frac{1}{2}kd_n)||y_n - q|| + (4d_n^2 + 6c_n + 4d_n)s_n$ $+4c_n||u_n - q|| + (2d_n^2 + 3c_n + 2d_n)c'_n||v_n - q|| + \epsilon_n$, $n \ge n_1$;

(iii) $\sum_{n=0}^{\infty} \epsilon_n < \infty$ *implies that* $\lim_{n\to\infty} y_n = q$, *so that* $\{x_n\}_{n=0}^{\infty}$ *is almost* T*-stable on* K;

(iv) $\lim_{n\to\infty} y_n = q$ *implies that* $\lim_{n\to\infty} \epsilon_n = 0$.

Proof. Note that $F(T) \neq \emptyset$. We claim that $F(T)$ is a singleton. Otherwise, there exist two distinct elements $p, q \in F(T)$. From the strictly successive hemicontractiveness of T, we obtain that for $n \geq n_0$,

$$\begin{aligned}||p-q||^2 &= Re\langle p-q, j(p-q)\rangle \\ &= Re\langle T^n p - q, j(p-q)\rangle \\ &\leq \frac{1}{t}||p-q||^2 \\ &< ||p-q||^2,\end{aligned}$$

which is impossible. Hence $F(T)$ is a singleton. It follows from (3.2) and (3.6) that

$$\begin{aligned}x_n &= x_{n+1} + d_n x_n - d_n T^n z_n - c_n(u_n - T^n z_n) \\ &= (1+d_n)x_{n+1} + d_n(I - T^n - kI)x_{n+1} - (2-k)d_n x_{n+1} \\ &\quad + d_n x_n + d_n(T^n x_{n+1} - T^n z_n) - c_n(u_n - T^n z_n) \\ &= (1+d_n)x_{n+1} + d_n(I - T^n - kI)x_{n+1} + (k-1)d_n x_n \\ &\quad + (2-k)d_n^2(x_n - T^n z_n) + d_n(T^n x_{n+1} - T^n z_n) \\ &\quad - [1 + (2-k)d_n]c_n(u_n - T^n z_n)\end{aligned} \tag{3.8}$$

and

$$q = (1+d_n)q + d_n(I - T^n - kI)q + (k-1)d_n q \tag{3.9}$$

for all $n \geq n_0$. Using (3.8), (3.9) and Lemma 2.3, we know that

$$\begin{aligned}&||x_n - q|| \\ &\geq (1+d_n)||(x_{n+1}-q) + \frac{d_n}{1+d_n}[(I-T^n-kI)x_{n+1} - (I-T^n-kI)q]|| \\ &\quad - (1-k)d_n||x_n - q|| - (2-k)d_n^2||x_n - T^n z_n|| \\ &\quad - d_n||T^n x_{n+1} - T^n z_n|| - [1 + (2-k)d_n]c_n||u_n - T^n z_n|| \\ &\geq (1+d_n)||x_{n+1} - q|| - (1-k)d_n||x_n - q|| \\ &\quad - (2-k)d_n^2||x_n - T^n z_n|| - d_n||T^n x_{n+1} - T^n z_n|| \\ &\quad - 3c_n||u_n - T^n z_n||\end{aligned} \tag{3.10}$$

for all $n \geq n_0$. In view of (3.1), (3.2), (3.6) and (3.10), we conclude that

$$
\begin{aligned}
&||x_{n+1} - q|| \\
&\le \frac{1+(1-k)d_n}{1+d_n}||x_n - q|| + d_n||T^n x_{n+1} - T^n z_n|| \\
&\quad + 2d_n^2||x_n - T^n z_n|| + 3c_n||u_n - T^n z_n|| \\
&\le (1 - kd_n + d_n^2)||x_n - q|| + d_n\Bigg[\Bigg(||T^n x_{n+1} - T^n z_n|| \\
&\quad - ||x_{n+1} - z_n||\Bigg) + ||x_{n+1} - z_n||\Bigg] + 2d_n^2\Bigg(||x_n - q|| \\
&\quad + ||T^n z_n - q||\Bigg) + 3c_n\Bigg(||T^n z_n - q|| + ||u_n - q||\Bigg) \\
&\le (1 - kd_n + 3d_n^2)||x_n - q|| + d_n\Bigg(s_n + b_n||x_n - T^n z_n|| \\
&\quad + c_n||u_n - x_n|| + b'_n||T^n x_n - x_n|| + c'_n||v_n - x_n||\Bigg) \\
&\quad + (2d_n^2 + 3c_n)||T^n z_n - q|| + 3c_n||u_n - q|| \\
&\le (1 - kd_n + 3d_n^2)||x_n - q|| + d_n\Bigg[(d_n + d'_n)||x_n - q|| \\
&\quad + b_n||T^n z_n - q|| + b'_n||T^n x_n - q|| + c_n||u_n - q|| + c'_n \\
&\quad \cdot ||v_n - q||\Bigg] + d_n s_n + (2d_n^2 + 3c_n)||T^n z_n - q|| + 3c_n||u_n - q \\
&\le (1 - kd_n + 4d_n^2 + d_n d')||x_n - q|| + (2d_n^2 + 3c_n + d_n b_n) \\
&\quad \cdot \Bigg(||T^n z_n - q|| - ||z_n - q||\Bigg) + (2d_n^2 + 3c_n + d_n b_n)||z_n - q|| \\
&\quad + d_n b'_n\Bigg(||T^n x_n - q|| - ||x_n - q||\Bigg) + d_n b'_n||x_n - q|| \\
&\quad + (3 + d_n)c_n||u_n - q|| + d_n c'_n||v_n - q|| + d_n s_n \\
&\le [1 - kd_n + 4d_n^2 + d_n(d'_n + b'_n)]||x_n - q|| \\
&\quad + [2d_n^2 + 3c_n + d_n(1 + b_n + b'_n)]s_n + (2d_n^2 + 3c_n + d_n b_n) \\
&\quad \cdot \Bigg[(1 - d'_n)||x_n - q|| + b'_n||T^n x_n - q|| + c'_n||v_n - x_n||\Bigg] \\
&\quad + 4c_n||u_n - q|| + d_n c'_n||v_n - q|| \\
&= (1 - kd_n + 6d_n^2 + 3c_n + d_n d'_n + d_n b'_n + d_n b_n)||x_n - q|| \\
&\quad + (4d_n^2 + 6c_n + 4d_n)s_n + 4c_n||u_n - q||
\end{aligned} \tag{3.11}
$$

$$+ (2d_n^2 + 3c_n + 2d_n)c'_n||v_n - q||$$

for all $n \geq n_0$. It follows from (3.4) that there exists a positive integer $n_1 > n_0$ such that

$$6d_n + 3r_n + d'_n + b'_n + b_n < \frac{k}{2}, \quad n \geq n_1. \tag{3.12}$$

From (3.11) and (3.12), we have

$$\begin{aligned} ||x_{n+1} - q|| \leq (1 - \frac{1}{2}kd_n)||x_n - q|| + (4d_n^2 + 6c_n + 4d_n)s_n \\ + 4c_n||u_n - q|| + (2d_n^2 + 3c_n + 2d_n)c'_n||v_n - q|| \end{aligned} \tag{3.13}$$

for all $n \geq n_1$. Let $\alpha_n = ||x_n - q||$, $\omega_n = \frac{1}{2}kd_n$, $\beta_n = \frac{2}{k}[(4d_n + 6r_n + 4)s_n + 4r_n||u_n - q|| + c'_n(2d_n + 3r_n + 2)||v_n - q||]$ and $\gamma_n = 0$ for $n \geq n_1$. Note that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are bounded. Then Lemma 2.1, (3.1), (3.4), (3.5) and (3.13) ensure that $\lim_{n\to\infty} \alpha_n = 0$. That is, $\lim_{n\to\infty} x_n = q$.

Observe that

$$\begin{aligned} y_n &= p_n + d_n y_n - d_n T^n w_n - c_n(u_n - T^n w_n) \\ &= (1 + d_n)p_n + d_n(I - T^n - kI)p_n + (k-1)d_n y_n \\ &\quad + (2-k)d_n^2(y_n - T^n w_n) + d_n(T^n p_n - T^n w_n) \\ &\quad + [1 + (2-k)d_n]c_n(T^n w_n - u_n) \end{aligned} \tag{3.14}$$

for all $n \geq n_0$. By virtue of (3.14), (3.9) and Lemma 2.3, we obtain that

$$\begin{aligned} &||y_n - q|| \\ &\geq (1 + d_n) \\ &\quad \cdot \left\| (p_n - q) + \frac{d_n}{1 + d_n}\left[(I - T^n - kI)p_n - (I - T^n - kI)q\right] \right\| \\ &\quad - (1-k)d_n||y_n - q|| - (2-k)d_n^2||y_n - T^n w_n|| \\ &\quad - d_n||T^n p_n - T^n w_n|| - [1 + (2-k)d_n]c_n||T^n w_n - u_n|| \\ &\geq (1 + d_n)||p_n - q|| - (1-k)d_n||y_n - q|| \\ &\quad - 2d_n^2||y_n - T^n w_n|| - d_n||T^n p_n - T^n w_n|| - 3c_n||T^n w_n - u_n|| \end{aligned} \tag{3.15}$$

for any $n \geq n_0$. It follows from (3.2), (3.7), (3.12) and (3.15) that

$$
\begin{aligned}
&||p_n - q|| \\
&\leq \frac{1+(1-k)d_n}{1+d_n}||y_n - q|| + 2d_n^2||y_n - T^n w_n|| \\
&\quad + d_n||T^n p_n - T^n w_n|| + 3c_n||T^n w_n - u_n|| \\
&\leq (1 - kd_n + d_n^2)||y_n - q|| + 2d_n^2\Big(||y_n - q|| + ||T^n w_n - q||\Big) \\
&\quad + d_n\Big(s_n + ||p_n - w_n||\Big) + 3c_n\Big(||T^n w_n - q|| + ||u_n - q||\Big) \\
&\leq (1 - kd_n + 3d_n^2)||y_n - q|| + (2d_n^2 + 3c_n)||T^n w_n - q|| \\
&\quad + d_n\Big(b_n||y_n - T^n w_n|| + c_n||y_n - u_n|| + b_n'||y_n - T^n y_n|| \\
&\quad + c_n'||y_n - v_n||\Big) + d_n s_n + 3c_n||u_n - q|| \\
&\leq (1 - kd_n + 4d_n^2 + d_n d_n')||y_n - q|| + (2d_n^2 + 3c_n + d_n b_n) \\
&\quad \cdot ||T^n w_n - q|| + d_n b_n'||T^n y_n - q|| + d_n s_n \\
&\quad + 4c_n||u_n - q|| + d_n c_n'||v_n - q|| \\
&\leq (1 - kd_n + 4d_n^2 + d_n d_n')||y_n - q|| + (2d_n^2 + 3c_n + d_n b_n \\
&\quad + d_n b_n' + d_n)s_n + (2d_n^2 + 3c_n + d_n b_n)||w_n - q|| \\
&\quad + d_n b_n'||y_n - q|| + 4c_n||u_n - q|| + d_n c_n'||v_n - q|| \\
&\leq (1 - kd_n + 4d_n^2 + d_n d_n' + d_n b_n')||y_n - q|| \\
&\quad + (2d_n^2 + 3c_n + 3d_n)s_n + (2d_n^2 + 3c_n + d_n b_n) \\
&\quad \cdot \Big[(1 - d_n')||y_n - q|| + b_n'||T^n y_n - q|| + c_n'||v_n - q||\Big] \\
&\quad + 4c_n||u_n - q|| + d_n c_n'||v_n - q|| \\
&\leq (1 - kd_n + 6d_n^2 + 3c_n + d_n d_n' + d_n b_n' + d_n b_n)||y_n - q|| \\
&\quad + (4d_n^2 + 6c_n + 4d_n)s_n + 4c_n||u_n - q|| \\
&\quad + (2d_n^2 + 3c_n + 2d_n)c_n'||v_n - q|| \\
&\leq (1 - \frac{1}{2}kd_n)||y_n - q|| + (4d_n^2 + 6c_n + 4d_n)s_n \\
&\quad + 4c_n||u_n - q|| + (2d_n^2 + 3c_n + 2d_n)c_n'||v_n - q||
\end{aligned}
\tag{3.16}
$$

for all $n \geq n_1$. Thus (3.8) and (3.16) yield that

$$\begin{aligned} & ||y_{n+1} - q|| \\ & \leq ||p_n - q|| + ||y_{n+1} - p_n|| \\ & \leq (1 - \frac{1}{2}kd_n)||y_n - q|| + (4d_n^2 + 6c_n + 4d_n)s_n \\ & \quad + 4c_n||u_n - q|| + (2d_n^2 + 3c_n + 2d_n)c'_n||v_n - q|| + \epsilon_n \end{aligned} \tag{3.17}$$

for all $n \geq n_1$.

Suppose that $\sum_{n=0}^{\infty} \epsilon_n < \infty$. Put $\alpha_n = ||y_n - q||$, $\omega_n = \frac{1}{2}kd_n$, $\beta_n = \frac{2}{k}\left[(4d_n + 6r_n + 4)s_n + 4r_n||u_n - q|| + (2d_n + 3r_n + 2)c'_n||v_n - q||\right]$ and $\gamma_n = \epsilon_n$ for all $n \geq 0$. Then (3.17) can be rewritten as

$$\alpha_{n+1} \leq (1 - \omega_n)\alpha_n + \omega_n\beta_n + \gamma_n, \quad n \geq n_1.$$

Note that $\lim_{n\to\infty} \beta_n = 0$. It follows from Lemma 2.1, (3.5) and the above inequality that $\lim_{n\to\infty} \alpha_n = 0$. That is, $\lim_{n\to\infty} y_n = q$.

Conversely, suppose that $\lim_{n\to\infty} y_n = q$. Then (3.4), (3.7) and (3.16) imply that

$$\begin{aligned} \epsilon_n & \leq ||y_{n+1} - q|| + ||p_n - q|| \\ & \leq ||y_{n+1} - q|| + (1 - \frac{1}{2}kd_n)||y_n - q|| + (4d_n^2 + 6c_n + 4d_n)s_n \\ & \quad + 4c_n||u_n - q|| + (2d_n^2 + 3c_n + 2d_n)c'_n||v_n - q|| \\ & \to 0 \end{aligned}$$

as $n \to \infty$. Hence $\lim_{n\to\infty} \epsilon_n = 0$. This completes the proof. □

THEOREM 3.2. *Let X, K, T, $\{s_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$, $\{z_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, $\{w_n\}_{n=0}^{\infty}$, $\{p_n\}_{n=0}^{\infty}$ and $\{\epsilon_n\}_{n=0}^{\infty}$ be as in Theorem 3.1. Suppose that $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$, $\{a'_n\}_{n=0}^{\infty}$, $\{b'_n\}_{n=0}^{\infty}$ and $\{c'_n\}_{n=0}^{\infty}$ are arbitrary sequences in $[0,1]$ satisfying (3.2), (3.5) and*

$$\sum_{n=0}^{\infty} c_n < \infty; \tag{3.18}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} d'_n = 0. \tag{3.19}$$

Then the conclusions of Theorem 3.1 hold.

Proof. Since (3.8) implies $\lim_{n\to\infty} r_n = 0$, we can directly get the desired results from Theorem 3.1. □

REMARK 3.1. The following examples reveal that conditions (3.2)–(3.5) are different from conditions (3.2), (3.5), (3.18) and (3.19).

EXAMPLE 3.1. Let $a_n = 1-(n+2)^{-\frac{1}{4}}-(n+2)^{-\frac{1}{2}}$, $b_n = (n+2)^{-\frac{1}{4}}$, $c_n = (n+2)^{-\frac{1}{2}}$, $r_n = (1+(n+2)^{\frac{1}{4}})^{-1}$, $a'_n = nn+2^{-1}$ and $b'_n = c'_n = (n+2)^{-1}$, for $n \geq 0$. Then conditions (3.2)–(3.5) are satisfied. But the condition (3.18) does not hold.

EXAMPLE 3.2. Let a'_n, b'_n and c'_n be as in Example 3.1. Take $c_n = (n+2)^{-2}$, $b_{2n} = 2(2n+2)^{-2}$, $b_{2n+1} = (2n+2)^{-1}$ and $a_n = 1-b_n-c_n$, $n \geq 0$. Then conditions (3.2), (3.5), (3.18) and (3.19) are fulfilled. But conditions (3.3) and (3.4) do not hold since $\lim_{n\to\infty} \frac{c_{2n}}{b_{2n}+c_{2n}} = \frac{1}{3} \neq 0$.

THEOREM 3.3. *Let X, K, T, $\{s_n\}_{n=0}^\infty$, $\{u_n\}_{n=0}^\infty$, $\{v_n\}_{n=0}^\infty$, $\{x_n\}_{n=0}^\infty$, $\{z_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$, $\{\omega_n\}_{n=0}^\infty$, $\{p_n\}_{n=0}^\infty$ and $\{\epsilon_n\}_{n=0}^\infty$ be as in Theorem 3.1. Suppose that $\{a_n\}_{n=0}^\infty$, $\{b_n\}_{n=0}^\infty$, $\{c_n\}_{n=0}^\infty$, $\{a'_n\}_{n=0}^\infty$, $\{b'_n\}_{n=0}^\infty$ and $\{c'_n\}_{n=0}^\infty$ are arbitrary sequences in $[0,1]$ satisfying (3.2) and*

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} c'_n = 0; \tag{3.20}$$

$$kd_n - 6d_n^2 - 3c_n - d_n d'_n - d_n b'_n - d_n b_n \geq \gamma > 0, \quad n \geq n_0, \tag{3.21}$$

where γ is a constant. Then

(i) *The sequence $\{x_n\}_{n=0}^\infty$ converges strongly to the unique fixed point q of T. Moreover,*

$$\begin{aligned} ||x_{n+1}-q|| \leq (1-\gamma)||x_n-q|| &+ (4d_n^2+6c_n+4d_n)s_n \\ &+ 4c_n||u_n-q|| + (2d_n^2+3c_n+2d_n)c'_n||v_n-q||, \quad n \geq n_0; \end{aligned}$$

(ii) $||y_{n+1}-q|| \leq (1-\gamma)||y_n-q||+(4d_n^2+6c_n+4d_n)s_n+4c_n||u_n-q|| +(2d_n^2+3c_n+2d_n)c'_n||v_n-q||+\epsilon_n, \quad n \geq n_0;$

(iii) $\lim_{n\to\infty} y_n = q$ *if and only if* $\lim_{n\to\infty} \epsilon_n = 0$.

Proof. It follows from the proof of Theorem 3.1 that $F(T) = \{q\}$,

$$\begin{aligned} ||x_{n+1}-q|| \leq\ & (1-kd_n+6d_n^2+3c_n+d_n d'_n+d_n b'_n+d_n b_n)||x_n-q|| \\ & + (4d_n^2+6c_n+4d_n)s_n + 4c_n||u_n-q|| \\ & + (2d_n^2+3c_n+2d_n)c'_n||v_n-q|| \end{aligned} \tag{3.22}$$

and

$$
\begin{aligned}
&||p_n - q|| \\
&\leq (1 - kd_n + 6d_n^2 + 3c_n + d_n d_n' + d_n b_n' + d_n b_n)||y_n - q|| \\
&\qquad + (4d_n^2 + 6c_n + 4d_n)s_n + 4c_n||u_n - q|| \\
&\qquad + (2d_n^2 + 3c_n + 2d_n)c_n'||v_n - q||
\end{aligned} \tag{3.23}
$$

for all $n \geq n_0$. Using (3.21) and (3.22), we know that for $n \geq n_0$,

$$
\begin{aligned}
||x_{n+1} - q|| \leq (1-\gamma)||x_n - q|| &+ (4d_n^2 + 6c_n + 4d_n)s_n \\
&+ 4c_n||u_n - q|| + (2d_n^2 + 3c_n + 2d_n)c_n'||v_n - q||.
\end{aligned} \tag{3.24}
$$

Set $\alpha_n = ||x_n - q||$, $\omega_n = r$, $\beta_n = \gamma^{-1}[(4d_n^2 + 6c_n + 4d_n)s_n + 4c_n||u_n - q|| + (2d_n^2 + 3c_n + 2d_n)c_n'||v_n - q||]$ and $\gamma_n = 0$ for $n \geq n_0$. Then Lemma 2.1, (3.1), (3.20), (3.24) and the boundedness of $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ yield that $\lim_{n\to\infty} \alpha_n = 0$. This means that $\lim_{n\to\infty} x_n = q$.

By virtue of (3.21) and (3.23), we get that for all $n \geq n_0$,

$$
\begin{aligned}
||y_{n+1} - q|| &\leq ||p_n - q|| + ||y_{n+1} - p_n|| \\
&\leq (1-\gamma)||y_n - q|| + (4d_n^2 + 6c_n + 4d_n)s_n \\
&\quad + 4c_n||u_n - q|| + (2d_n^2 + 3c_n + 2d_n)c_n'||v_n - q|| + \epsilon_n.
\end{aligned}
$$

Suppose that $\lim_{n\to\infty} y_n = q$. In view of (3.1), (3.20), (3.21) and (3.23), we have

$$
\begin{aligned}
\epsilon_n &\leq ||y_{n+1} - q|| + ||p_n - q|| \\
&\leq ||y_{n+1} - q|| + (1-\gamma)||y_n - q|| + (4d_n^2 + 6c_n + 4d_n)s_n \\
&\quad + 4c_n||u_n - q|| + (2d_n^2 + 3c_n + 2d_n)c_n'||v_n - q|| \\
&\quad \to 0
\end{aligned}
$$

as $n \to \infty$. Hence $\lim_{n\to\infty} \epsilon_n = 0$. Suppose that $\lim_{n\to\infty} \epsilon_n = 0$. Let $\alpha_n = ||y_n - q||$, $\omega_n = \gamma$, $\beta_n = \gamma^{-1}[(4d_n^2 + 6c_n + 4d_n)s_n + 4c_n||u_n - q|| + (2d_n^2 + 3c_n + 2d_n)c_n'||v_n - q|| + \epsilon_n]$ and $\gamma_n = 0$ for $n \geq n_0$. Then Lemma 2.1, (3.1), (3.20), (3.25) and the boundedness of $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ ensure that $\lim_{n\to\infty} y_n = q$. The proof is completed. □

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