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Recurrence Relations Between Product Moments of Order Statistics for Truncated Distributions and Their Applications

Jagdish Saran[1] and Narinder Pushkarna[1]

ABSTRACT

In this paper, some general results for obtaining recurrence relations between product moments of order statistics for doubly truncated distributions are established. These results are then applied to some specific doubly truncated distributions, *viz.* doubly truncated Weibull, Exponential, Pareto, power function, Cauchy, Lomax and Rayleigh.

Keywords. Order statistics, moments, recurrence relations, truncated and non-truncated Weibull, exponential, Pareto, power function, Cauchy, Lomax and Rayleigh distributions.
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1. Introduction

Several authors, *viz.* Balakrishnan and Joshi (1981, 1982, 1984), Balakrishnan and Kocherlakota (1986), Balakrishnan and Malik (1987), *etc.* have obtained recurrence relations between product moments of order statistics. In these references, a particular distribution is considered and the relations are obtained. In this paper, we have established some general results for obtaining the product moment of the j^{th} power of the r^{th} order statistic and the k^{th} power of the s^{th} order statistic. Then these results are utilized to determine recurrence relations between product moments of order statistics for some specific doubly truncated distributions, *viz.* doubly truncated Weibull, exponential, Pareto, power function, Cauchy, Lomax and Rayleigh, thus extending the earlier work due to Khan *et al.* (1983). For applications of these distributions, one may refer to Johnson, Kotz and Balakrishnan (1994, 1995).

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[1]Department of Statistics, University of Delhi, Delhi-110007, India (email : dustats@del3.vsnl.net.in)

Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the order statistics obtained from a continuous population having *cdf* $G(x)$ and *pdf* $g(x)$. The joint *pdf* of $X_{r:n}$ and $X_{s:n} (1 \le r < s \le n)$ is given by

$$C_{r,s:n}\{G(x)\}^{r-1}\{G(y) - G(x)\}^{s-r-1}\{1 - G(y)\}^{n-s}g(x)g(y), \tag{1.1}$$

for $-\infty < x < y < \infty$, where

$$C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

Let

$$\begin{aligned}\mu_{r,s:n}^{(j,k)} &= E(X_{r:n}^{j}\, X_{s:n}^{k}) \\ &= C_{r,s:n} \int_{-\infty}^{+\infty} \int_{x}^{+\infty} x^j y^k \{G(x)\}^{r-1}\{G(y) - G(x)\}^{s-r-1}\{1 - G(y)\}^{n-s} \\ &\quad \times\ g(x)g(y)dydx. \end{aligned} \tag{1.2}$$

For simplicity, we shall denote $\mu_{r,s:n}^{(1,1)}$ by $\mu_{r,s:n}$.

The *pdf* in the case of truncation from both the sides is given by

$$f(x) = \frac{g(x)}{P - Q}, \quad Q_1 \le x \le P_1, \tag{1.3}$$

where

$$\int_{-\infty}^{Q_1} g(x)dx = Q \tag{1.4}$$

and

$$\int_{P_1}^{+\infty} g(x)dx = 1 - P, \tag{1.5}$$

i.e., Q and $1 - P$ $(Q < P)$ are, respectively, the proportions of truncation on the left and right of the *pdf* $g(x)$. The quantities Q and P are assumed to be known and Q_1 and P_1 are functions of Q and P.

For $Q = 0$, this distribution reduces to the right truncated distribution, and for $P = 1$, it reduces to the left truncated distribution. Further for $Q = 0$ and $P = 1$, it becomes the original non-truncated distribution with *pdf* $g(x), -\infty < x < \infty$.

Thus, for the doubly truncated distribution having *pdf* $f(x)$ as given in (1.3), and *cdf* $F(x)$, we have, for $Q_1 \le x < y \le P_1$,

$$\begin{aligned}\mu_{r,s:n}^{(j,k)} = C_{r,s:n} \int_{Q_1}^{P_1} \int_{x}^{P_1} & x^j y^k \{F(x)\}^{r-1}\{F(y) - F(x)\}^{s-r-1} \\ \times\ & \{1 - F(y)\}^{n-s} f(x)f(y)dydx. \end{aligned} \tag{1.6}$$

2. Recurrence Relations Between Product Moments of Order Statistics

Relation 2.1. *For* $1 \leq r < s \leq n$, *and* $j, k > 0$,

$$\mu_{r,s:n}^{(j,k)} = \mu_{r+1,s:n}^{(j,k)} - \frac{j\ C_{r,s:n}}{r} \int_{Q_1}^{P_1} \int_{x}^{P_1} x^{j-1} y^k \{F(x)\}^r \{F(y) - F(x)\}^{s-r-1} \times \{1 - F(y)\}^{n-s} f(y) dy dx. \tag{2.1}$$

Proof. Using (1.6), we have

$$\mu_{r+1,s:n}^{(j,k)} - \mu_{r,s:n}^{(j,k)} = \frac{C_{r,s:n}}{r} \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^k \{F(x)\}^{r-1} \{F(y) - F(x)\}^{s-r-2} \{1 - F(y)\}^{n-s} \times [(s-r-1)F(x) - r\{F(y) - F(x)\}] f(x) f(y) dy dx. \tag{2.2}$$

Let

$$H(x,y) = -\{F(x)\}^r \{F(y) - F(x)\}^{s-r-1}, \tag{2.3}$$

then

$$\frac{\partial H(x,y)}{\partial x} = \{F(x)\}^{r-1} \{ F(y) - F(x)\}^{s-r-2} \times [(s-r-1)F(x) - r\{F(y) - F(x)\}] f(x). \tag{2.4}$$

Substituting the value of (2.4) in (2.2), we obtain

$$\mu_{r+1,s:n}^{(j,k)} - \mu_{r,s:n}^{(j,k)} = \frac{C_{r,s:n}}{r} \int_{Q_1}^{P_1} \int_{x}^{P_1} x^j y^k \{1 - F(y)\}^{n-s} f(y) \frac{\partial H(x,y)}{\partial x} dy dx$$
$$= \frac{C_{r,s:n}}{r} \int_{Q_1}^{P_1} y^k \{1 - F(y)\}^{n-s} f(y) \left\{ \int_{Q_1}^{y} x^j \frac{\partial H(x,y)}{\partial x} dx \right\} dy, \tag{2.5}$$

by changing the order of integration. Now, in view of (2.3) and $F(Q_1) = 0$,

$$\int_{Q_1}^{y} x^j \frac{\partial H(x,y)}{\partial x} dx = j \int_{Q_1}^{y} x^{j-1} \{F(x)\}^r \{F(y) - F(x)\}^{s-r-1} dx, \tag{2.6}$$

by integration by parts. Using (2.6) in (2.5), it leads to the required result (2.1).
□

Relation 2.2. *For $1 \le r \le n-1$, and $j,k > 0$,*

$$\mu_{r,r+1:n}^{(j,k)} = \mu_{r+1:n}^{(j+k)} - j(n-r)\binom{n}{r} \int_{Q_1}^{P_1}\int_{x}^{P_1} x^{j-1}y^{k}\{F(x)\}^{r} \times \{1-F(y)\}^{n-r-1} f(y)dydx. \quad (2.7)$$

Proof. This relation follows by setting $s = r+1$ in equation (2.1) and noting that

$$\mu_{r+1,r+1:n}^{(j,k)} = E\left(X_{r+1:n}^{j} X_{r+1:n}^{k}\right) = E\left(X_{r+1:n}^{j+k}\right) = \mu_{r+1:n}^{(j+k)}.$$

□

Relation 2.3. *For $n \ge 2$, and $j,k > 0$,*

$$\mu_{n-1,n:n}^{(j,k)} = \mu_{n:n}^{(j+k)} - jn \int_{Q_1}^{P_1}\int_{x}^{P_1} x^{j-1}y^{k}\{F(x)\}^{n-1} f(y)dydx. \quad (2.8)$$

Proof. Putting $r = n-1$ in equation (2.1), it leads to the desired result. □

Relation 2.4. *For $n \ge 2$, and $j,k > 0$,*

$$\mu_{1,2:n}^{(j,k)} = \mu_{2:n}^{(j+k)} - jn(n-1) \int_{Q_1}^{P_1}\int_{x}^{P_1} x^{j-1}y^{k}\{F(x)\}\{1-F(y)\}^{n-2} f(y)dydx. \quad (2.9)$$

Proof. The result follows on putting $s = 2$ in equation (2.1). □

Relation 2.5. *For $1 \le r < s \le n$, and $k > 0$,*

$$\mu_{r,s:n}^{(0,k)} = \mu_{r+1,s:n}^{(0,k)} = \mu_{r+2,s:n}^{(0,k)} = \cdots = \mu_{s-1,s:n}^{(0,k)} = \mu_{s:n}^{(k)}, \quad (2.10)$$

where $\mu_{s:n}^{(k)} = E(X_{s:n}^{k})$.

Proof. Setting $j = 0$ in (2.5) and applying induction on r (or applying the definition of the product moment directly), we obtain equation (2.10). □

Further, on setting $k = 0$ and $s = r+1$ in (2.1) and noting that $\mu_{r,s:n}^{(j,0)} = E\left(X_{r:n}^{j} X_{s:n}^{0}\right) = E\left(X_{r:n}^{j}\right) = \mu_{r:n}^{(j)}$, it reduces to

$$\mu_{r+1:n}^{(j)} - \mu_{r:n}^{(j)} = j\binom{n}{r} \int_{Q_1}^{P_1} x^{j-1}\{F(x)\}^{r}\{1-F(x)\}^{n-r} dx, \quad (2.11)$$

which for $j = 1$ and $Q = 0, P = 1$ (*i.e.*, the non-truncated case) reduces to a result of Pearson (1902) and David and Groeneveld (1982):

$$\mu_{r+1:n}^{(1)} - \mu_{r:n}^{(1)} = \binom{n}{r} \int_{-\infty}^{\infty} \{F(x)\}^r \{1 - F(x)\}^{n-r} dx. \tag{2.12}$$

Thus (2.11) can be regarded as a generalization of (2.12) from the non-truncated case to the doubly truncated case.

Remark. It may be noted that the corresponding results for the non-truncated case are obtained by setting $P = 1$ and $Q = 0$ in equations (2.1), (2.7), (2.8) and (2.9). In a like manner, the results for the left and the right truncated cases may be easily written down.

3. Applications

In the following we shall use the results of Section 2 to obtain recurrence relations between product moments of order statistics for some specific distributions considered below. For this the usual technique will be to express $F(x)$ as a function of x and $f(x)$. It can be seen from the examples considered in Sub-sections 3.1 to 3.6 that the recurrence relations for specific distributions are obtained rather easily by just substitution.

3.1. Doubly truncated Weibull and exponential distributions

Let the random variable X have a doubly truncated Weibull distribution with *cdf*

$$F(x) = \begin{cases} 0, & x^p < -\log(1-Q), \\ \dfrac{1 - Q - e^{-x^p}}{P - Q}, & -\log(1-Q) \le x^p \le -\log(1-P), \\ 1, & x^p > -\log(1-P), \end{cases} \tag{3.1}$$

for $p > 0$, where Q and $1 - P$ $(Q < P)$ are, respectively, the proportions of truncation on the left and right of the standard Weibull distribution. Denoting $-\log(1-Q)$ by Q_1^p, $-\log(1-P)$ by P_1^p, $(1-Q)/(P-Q)$ by Q_2 and $(1-P)/(P-Q)$ by P_2, it is easy to see that

$$F(x) = Q_2 - \frac{1}{p} x^{1-p} f(x). \tag{3.2}$$

Substituting the value of $F(x)$ from (3.2) in (2.1), we get

$$\mu_{r+1,s:n}^{(j,k)} - \mu_{r,s:n}^{(j,k)} = \frac{j}{r} C_{r,s:n} \int_{Q_1}^{P_1} \int_{x}^{P_1} x^{j-1} y^k \{F(x)\}^{r-1} \{F(y) - F(x)\}^{s-r-1} \times \{1 - F(y)\}^{n-s} \left\{ Q_2 - \frac{1}{p} x^{1-p} f(x) \right\} f(y)\, dy\, dx. \quad (3.3)$$

Now on using (2.1) in the first term and the definition of $\mu_{r,s:n}^{(j,k)}$ in the second term on the right-hand side of (3.3), we get for $1 \le r < s \le n, s - r \ge 2$,

$$\mu_{r+1,s:n}^{(j,k)} - \mu_{r,s:n}^{(j,k)} = \frac{n}{r} Q_2 \left\{ \mu_{r,s-1:n-1}^{(j,k)} - \mu_{r-1,s-1:n-1}^{(j,k)} \right\} - \frac{j}{rp} \mu_{r,s:n}^{(j-p,k)}. \quad (3.4)$$

In a similar manner, one can easily obtain the following special cases:
From Relation 2.2, for $s = r + 1$, (3.4) reduces to

$$\mu_{r,r+1:n}^{(j,k)} = \mu_{r+1:n}^{(j+k)} - \frac{n}{r} Q_2 \left\{ \mu_{r:n-1}^{(j+k)} - \mu_{r-1,r:n-1}^{(j,k)} \right\} + \frac{j}{rp} \mu_{r,r+1:n}^{(j-p,k)}, \quad (3.5)$$

for $1 < r \le n - 1,\ \ n \ge 3$.
From Relation 2.3, for $r = n - 1$, (3.4) reduces to

$$\mu_{n-1,n:n}^{(j,k)} = \mu_{n:n}^{(j+k)} - \frac{n}{n-1} Q_2 \left\{ \mu_{n-1:n-1}^{(j+k)} - \mu_{n-2,n-1:n-1}^{(j,k)} \right\} + \frac{j}{p(n-1)} \mu_{n-1,n:n}^{(j-p,k)}, \quad (3.6)$$

for $n \ge 3$.
From Relation 2.4, for $r = 1$ and $s = 2$, (3.4) reduces to

$$\mu_{1,2:n}^{(j,k)} = \mu_{2:n}^{(j+k)} - nQ_2 \left\{ \mu_{1:n-1}^{(j+k)} - Q_1^j\, \mu_{1:n-1}^{(k)} \right\} + \frac{j}{p} \mu_{1,2:n}^{(j-p,k)}, \ \ n \ge 2. \quad (3.7)$$

It may be noted that the expression in (3.7) could also be obtained from (3.5) by putting therein $r = 1$. In doing so, we get an undefined term, *viz.* $\mu_{0,1:n-1}^{(j,k)}$ which can be interpreted as $E\left(Q_1^j X_{1:n-1}^k\right) = Q_1^j \mu_{1:n-1}^{(k)}$, where Q_1 is the lower limit of the Weibull variate. This interpretation can be seen as follows:

Using (3.2) in (2.9), we get

$$\mu_{1,2:n}^{(j,k)} = \mu_{2:n}^{(j+k)} - jn(n-1) \int_{Q_1}^{P_1} \int_{x}^{P_1} x^{j-1} y^k \{1 - F(y)\}^{n-2} \times \left\{ Q_2 - \frac{x^{1-p}}{p} f(x) \right\} f(y)\, dy\, dx$$

$$= \mu_{2:n}^{(j+k)} - n(n-1)Q_2 \int_{Q_1}^{P_1} y^k\{1-F(y)\}^{n-2} f(y) \left\{ \int_{Q_1}^{y} x^{j-1} dx \right\} dy + \frac{j}{p}\mu_{1,2:n}^{(j-p,k)}$$

$$= \mu_{2:n}^{(j+k)} - n(n-1)Q_2 \int_{Q_1}^{P_1} y^{j+k}\{1-F(y)\}^{n-2} f(y)\, dy$$
$$+ n(n-1)Q_2\, Q_1^j \int_{Q_1}^{P_1} y^k\{1-F(y)\}^{n-2} f(y)\, dy + \frac{j}{p}\mu_{1,2:n}^{(j-p,k)}$$
$$= \mu_{2:n}^{(j+k)} - nQ_2\mu_{1:n-1}^{(j+k)} + nQ_2\, Q_1^j \mu_{1:n-1}^{(k)} + \frac{j}{p}\mu_{1,2:n}^{(j-p,k)}, \quad n \geq 2.$$

Now the required interpretation follows by comparing the above expression with (3.5) for $r = 1$.

If we put $p = 1$ in the above expressions (3.4) to (3.7), we get corresponding results for the exponential distribution.

Remark. For the non-truncated case, the corresponding results could be obtained from equations (3.3) to (3.7) by putting therein $P = 1$ and $Q = 0$. Similarly for the right and the left truncated cases, one has to put $Q = 0$ and $P = 1$, respectively.

3.2. Doubly truncated power function distribution

Let the random variable X have a doubly truncated power function distribution with *cdf*

$$F(x) = \begin{cases} 0, & x < aQ^{1/v}, \\ \dfrac{a^{-v}x^v - Q}{P-Q}, & aQ^{1/v} \leq x \leq aP^{1/v}, \\ 1, & x > aP^{1/v}, \end{cases} \tag{3.8}$$

for $a > 0,\ v > 0$, where Q and $1 - P$ $(Q < P)$ are, respectively, the proportions of truncation on the left and right of the standard power function distribution. Denoting $aQ^{1/v}$ by Q_1, $aP^{1/v}$ by P_1, $Q/(P-Q)$ by Q_2 and $P/(P-Q)$ by P_2, it is easy to see that

$$F(x) = \frac{x}{v} f(x) - Q_2. \tag{3.9}$$

Putting the value of $F(x)$ from equation (3.9) in equation (2.1), we get, for $1 \leq r < s \leq n,\ s - r \geq 2$,

$$\mu_{r+1,s:n}^{(j,k)} = \mu_{r,s:n}^{(j,k)} \left\{ 1 + \frac{j}{vr} \right\} - \frac{n}{r} Q_2 \left\{ \mu_{r,s-1:n-1}^{(j,k)} - \mu_{r-1,s-1:n-1}^{(j,k)} \right\}. \tag{3.10}$$

The marginal results in this case corresponding to (2.7), (2.8) and (2.9) can easily be seen to be equal to

$$\mu_{r,r+1:n}^{(j,k)} = \frac{vr}{vr+j}\left[\mu_{r+1:n}^{(j+k)} + \frac{n}{r}Q_2\left\{\mu_{r:n-1}^{(j+k)} - \mu_{r-1,r:n-1}^{(j,k)}\right\}\right], \tag{3.11}$$

for $1 < r \leq n-1,\ n \geq 3$,

$$\mu_{n-1,n:n}^{(j,k)} = \frac{v(n-1)}{v(n-1)+j}\left[\mu_{n:n}^{(j+k)} + \frac{n}{n-1}Q_2\left\{\mu_{n-1:n-1}^{(j+k)} - \mu_{n-2,n-1:n-1}^{(j,k)}\right\}\right], \tag{3.12}$$

for $n \geq 3$, and

$$\mu_{1,2:n}^{(j,k)} = \frac{v}{v+j}\left[\mu_{2:n}^{(j+k)} + nQ_2\left\{\mu_{1:n-1}^{(j+k)} - Q_1^j\,\mu_{1:n-1}^{(k)}\right\}\right],\ n \geq 2. \tag{3.13}$$

The corresponding results for the non-truncated, right truncated and left truncated cases follow immediately.

3.3. Doubly truncated Pareto distribution

Let the random variable X have a doubly truncated Pareto distribution with *cdf*

$$F(x) = \begin{cases} 0, & x < a(1-Q)^{-1/v}, \\ \dfrac{1-Q-a^v x^{-v}}{P-Q}, & a(1-Q)^{-1/v} \leq x \leq a(1-P)^{-1/v}, \\ 1, & x > a(1-P)^{-1/v}, \end{cases} \tag{3.14}$$

for $a > 0$, $v > 0$, where Q and $1-P$ $(Q < P)$ are, respectively, the proportions of truncation on the left and right of the standard Pareto distribution. Denoting $a(1-Q)^{-1/v}$ by Q_1, $a(1-P)^{-1/v}$ by P_1, $(Q-1)/(P-Q)$ by Q_2 and $(P-1)/(P-Q)$ by P_2, it is easy to see that

$$F(x) = -\left\{Q_2 + \frac{x}{v}f(x)\right\}. \tag{3.15}$$

In view of (3.15) and (2.1),

$$\mu_{r+1,s:n}^{(j,k)} = \mu_{r,s:n}^{(j,k)}\left\{\frac{vr-j}{vr}\right\} - \frac{n}{r}Q_2\left\{\mu_{r,s-1:n-1}^{(j,k)} - \mu_{r-1,s-1:n-1}^{(j,k)}\right\}, \tag{3.16}$$

for $1 \leq r < s \leq n,\ s-r \geq 2$ and $vr \neq j$. However, if $vr = j$, then (3.16) implies that

$$\mu_{r+1,s:n}^{(j,k)} = \frac{n}{r}Q_2\left\{\mu_{r-1,s-1:n-1}^{(j,k)} - \mu_{r,s-1:n-1}^{(j,k)}\right\}, \tag{3.17}$$

for $1 \leq r < s \leq n, \quad s - r \geq 2$.

Likewise, it can be shown that the marginal results are as follows:

$$\mu_{r,r+1:n}^{(j,k)} = \frac{vr}{vr-j}\left[\mu_{r+1:n}^{(j+k)} + \frac{n}{r}Q_2\left\{\mu_{r:n-1}^{(j+k)} - \mu_{r-1,r:n-1}^{(j,k)}\right\}\right], \tag{3.18}$$

for $vr \neq j, \ 1 < r \leq n-1, \ n \geq 3$,

$$\mu_{n-1,n:n}^{(j,k)} = \frac{v(n-1)}{v(n-1)-j}\left[\mu_{n:n}^{(j+k)} + \frac{n}{n-1}Q_2\left\{\mu_{n-1:n-1}^{(j+k)} - \mu_{n-2,n-1:n-1}^{(j,k)}\right\}\right], \tag{3.19}$$

for $v(n-1) \neq j, \quad n \geq 3$, and

$$\mu_{1,2:n}^{(j,k)} = \frac{v}{v-j}\left[\mu_{2:n}^{(j+k)} + nQ_2\left\{\mu_{1:n-1}^{(j+k)} - Q_1^j \ \mu_{1:n-1}^{(k)}\right\}\right], \tag{3.20}$$

for $v \neq j, \quad n \geq 2$.

For the non-truncated and singly truncated cases, the corresponding results can easily be written down.

It may be mentioned that the recurrence relations for single and product moments of order statistics from a doubly truncated Pareto distribution have been obtained by Khurana and Jha (1991, 1995).

3.4. Doubly truncated Cauchy distribution

The *pdf* of the doubly truncated Cauchy distribution is given by

$$f(x) = \frac{1}{(P-Q)\pi} \cdot \frac{1}{(1+x^2)}, \quad Q_1 \leq x \leq P_1,$$

where Q_1 and P_1 are obtained as given in Section 1. Therefore,

$$1 = (P-Q)\pi f(x) + (P-Q)\pi x^2 f(x). \tag{3.21}$$

In view of (3.21) and (2.1), we have for $1 \leq r < s \leq n$ and $s - r \geq 2$,

$$\mu_{r+1,s:n}^{(j,k)} = \mu_{r,s:n}^{(j,k)} + \frac{j(P-Q)\pi}{n+1}\left\{\mu_{r+1,s+1:n+1}^{(j-1,k)} + \mu_{r+1,s+1:n+1}^{(j+1,k)}\right\}. \tag{3.22}$$

The marginal results are as follows:

$$\mu_{r,r+1:n}^{(j,k)} = \mu_{r+1:n}^{(j+k)} - \frac{j(P-Q)\pi}{n+1}\left\{\mu_{r+1,r+2:n+1}^{(j-1,k)} + \mu_{r+1,r+2:n+1}^{(j+1,k)}\right\} \tag{3.23}$$

for $1 \leq r \leq n-1, \quad n \geq 4$,

$$\mu_{n-1,n:n}^{(j,k)} = \mu_{n:n}^{(j+k)} - \frac{j(P-Q)\pi}{n+1}\left\{\mu_{n,n+1:n+1}^{(j-1,k)} + \mu_{n,n+1:n+1}^{(j+1,k)}\right\}, \quad n \geq 2 \tag{3.24}$$

and

$$\mu_{1,2:n}^{(j,k)} = \mu_{2:n}^{(j+k)} - \frac{j(P-Q)\pi}{n+1}\left\{\mu_{2,3:n+1}^{(j-1,k)} + \mu_{2,3:n+1}^{(j+1,k)}\right\}, \quad n \geq 2. \tag{3.25}$$

For the non-truncated, right truncated and left truncated cases, one has to put $(Q = 0, P = 1), Q = 0$ and $P = 1$, respectively.

3.5. Doubly truncated Lomax distribution

The *cdf* of the doubly truncated Lomax distribution is given by

$$F(x) = \begin{cases} 0, & x < (1-Q)^{-1/\alpha} - 1, \\ \dfrac{1 - Q - (1+x)^{-\alpha}}{P-Q}, & (1-Q)^{-1/\alpha} - 1 \leq x \leq (1-P)^{-1/\alpha} - 1, \\ 1, & x > (1-P)^{-1/\alpha} - 1, \end{cases} \tag{3.26}$$

for $\alpha > 0$, where Q and $1 - P$ $(Q < P)$ are, respectively, the proportions of truncation on the left and right of the standard Lomax distribution having *pdf* $\alpha(1+x)^{-(\alpha+1)}, \quad x \geq 0, \ \alpha > 0$. Lomax (1954) used this distribution in the analysis of business failure data; this distribution is also known as the Pareto Type-II distribution (see Arnold, 1983). Denoting $(1-Q)/(P-Q)$ by Q_2, it is easy to see that

$$(1+x)f(x) = \alpha[Q_2 - F(x)], \tag{3.27}$$

or, equivalently,

$$F(x) = Q_2 - \frac{(1+x)}{\alpha}f(x). \tag{3.28}$$

In view of (3.28) and (2.1) we have for $1 \leq r < s \leq n$ and $s - r \geq 2$,

$$r\alpha\mu_{r+1,s:n}^{(j,k)} = (r\alpha - j)\mu_{r,s:n}^{(j,k)} + n\alpha Q_2\left\{\mu_{r,s-1:n-1}^{(j,k)} - \mu_{r-1,s-1:n-1}^{(j,k)}\right\} - j\mu_{r,s:n}^{(j-1,k)}. \tag{3.29}$$

The marginal results in this case corresponding to (2.7), (2.8) and (2.9) can easily be seen to be equal to

$$(r\alpha - j)\mu_{r,r+1:n}^{(j,k)} = j\mu_{r,r+1:n}^{(j-1,k)} + r\alpha\mu_{r+1:n}^{(j+k)} - n\alpha Q_2\left\{\mu_{r:n-1}^{(j+k)} - \mu_{r-1,r:n-1}^{(j,k)}\right\}, \tag{3.30}$$

for $2 \leq r \leq n-1, \quad n \geq 2$,

$$\begin{aligned}[(n-1)\alpha - j]\mu_{n-1,n:n}^{(j,k)} &= j\mu_{n-1,n:n}^{(j-1,k)} + (n-1)\alpha\mu_{n:n}^{(j+k)} \\ &\quad - n\alpha Q_2\left\{\mu_{n-1:n-1}^{(j+k)} - \mu_{n-2,n-1:n-1}^{(j,k)}\right\},\end{aligned} \tag{3.31}$$

for $n \geq 3$, and

$$(\alpha - j)\mu_{1,2:n}^{(j,k)} = j\mu_{1,2:n}^{(j-1,k)} + \alpha\mu_{2:n}^{(j+k)} \\ -n\alpha Q_2 \left\{ \mu_{1:n-1}^{(j+k)} - Q_1^j \mu_{1:n-1}^{(k)} \right\}, \qquad n \geq 2, \tag{3.32}$$

which generalize the corresponding results quoted in Balakrishnan, Malik and Ahmed (1988, p. 2684) for the case of untruncated Lomax distribution.

The corresponding results for the non-truncated, right truncated and left truncated cases follow immediately.

3.6. Doubly truncated Rayleigh distribution

Let the random variable X have a doubly truncated Rayleigh distribution with *cdf*

$$F(x) = \begin{cases} 0, & x < Q_1, \\ \dfrac{1}{P-Q}\left[e^{-\frac{1}{2}\nu Q_1^2} - e^{-\frac{1}{2}\nu x^2}\right], & Q_1 \leq x \leq P_1, \\ 1, & x > P_1, \end{cases} \tag{3.33}$$

for $\nu > 0$, where Q and $1 - P$ $(0 < Q < P < 1)$ are, respectively, the proportions of truncation on the left and right of the standard Rayleigh distribution, and

$$Q_1 = \left[\log \frac{1}{(1-Q)^{2/\nu}}\right]^{\frac{1}{2}} \quad \text{and} \quad P_1 = \left[\log \frac{1}{(1-P)^{2/\nu}}\right]^{\frac{1}{2}}, \tag{3.34}$$

are, respectively, the points of truncation on the left and the right. Denoting $(1 - Q)/(P - Q)$ by Q_2, it is easy to see that the characterizing differential equation for the doubly truncated Rayleigh distribution is

$$f(x) = \nu\, x\, (Q_2 - F(x)), \tag{3.35}$$

or, equivalently,

$$xF(x) = xQ_2 - \frac{f(x)}{\nu}. \tag{3.36}$$

In view of (3.36) and (2.1) we have for $1 \leq r < s \leq n$ and $s - r \geq 2$,

$$\mu_{r+1,s:n}^{(j,k)} = \mu_{r,s:n}^{(j,k)} + \frac{n}{r} Q_2 \left\{ \mu_{r,s-1:n-1}^{(j,k)} - \mu_{r-1,s-1:n-1}^{(j,k)} \right\} - \frac{j}{r\nu}\mu_{r,s:n}^{(j-2,k)}. \tag{3.37}$$

Likewise, the marginal results are as follows:

$$\mu_{r,r+1:n}^{(j,k)} = \mu_{r+1:n}^{(j+k)} - \frac{n}{r} Q_2 \left\{ \mu_{r:n-1}^{(j+k)} - \mu_{r-1,r:n-1}^{(j,k)} \right\} + \frac{j}{r\nu}\mu_{r,r+1:n}^{(j-2,k)}, \tag{3.38}$$

for $2 \leq r \leq n-1$,

$$\mu_{n-1,n:n}^{(j,k)} = \mu_{n:n}^{(j+k)} - \frac{n}{n-1} Q_2 \left\{ \mu_{n-1:n-1}^{(j+k)} - \mu_{n-2,n-1:n-1}^{(j,k)} \right\} + \frac{j}{(n-1)\nu} \mu_{n-1,n:n}^{(j-2,k)}, \tag{3.39}$$

for $n \geq 2$, and

$$\mu_{1,2:n}^{(j,k)} = \mu_{2:n}^{(j+k)} - nQ_2 \left\{ \mu_{1:n-1}^{(j+k)} - Q_1^j \, \mu_{1:n-1}^{(k)} \right\} + \frac{j}{\nu} \mu_{1,2:n}^{(j-2,k)}, \tag{3.40}$$

for $n \geq 2$, which generalize the corresponding results obtained by Balakrishnan and Malik (1986, p. 186) for the case of untruncated Rayleigh distribution.

For the non-truncated , right truncated and left truncated cases, one has to put $(Q = 0, P = 1), Q = 0$ and $P = 1$, respectively.

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