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Sequential Estimation with β-Protection of the Difference of Two Normal Means When an Imprecision Function Is Variable

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ABSTRACT

For two normal distribution with unknown means and unknown variances, a sequential procedure for estimating the difference of two normal means which satisfies both the coverage probability condition and the β-protection is proposed under some smoothness of variable imprecision function, and the asymptotic normality of the proposed stopping time after some centering and scaling is given.

Keywords. Sequential procedure, coverage probability condition, β-protection, confidence interval, uniformly in, asymptotic normality.
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1. Introduction

Consider two sequences of mutually independent random variables X, X_1, $X_2, \cdots$ and Y, Y_1, Y_2, $\cdots$ where the X's are *iid* $N(\mu_x,\ \sigma_x^2)$ and the Y's are *iid* $N(\mu_y,\ \sigma_y^2)$. Let the unknown parameter space be $\Theta = \{\theta = (\mu_x,\ \mu_y,\ \sigma_x,\ \sigma_y) : -\infty < \mu_x,\ \mu_y < \infty,\ 0 < \sigma_x,\ \sigma_y < \infty\}$. Any confidence interval (CI) of the difference of the two normal means $\mu = \mu_x - \mu_y$ to be considered will be subject to two requirements. The first requirement concerns the probability of covering the true value of μ, *i.e.*, for any given α $(0 < \alpha < 1)$

$$P_\theta\{\mu \in \text{CI}\} \geq 1 - \alpha \qquad \text{for every } \theta \in \Theta. \tag{1.1}$$

The second requirement concerns the precision of the confidence interval, which can be specified in various ways. For example, fixed-width confidence intervals, or minimun risk function. In this study we adopt the "β-protection

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at $\mu - \delta(\mu)$" as a measure of precision of the confidence interval proposed by Wijsman (1981), *i.e.*, for a given imprecision function $\delta(x) > 0$ and a probability $0 < \beta < 1$,

$$P_\theta \{\mu - \delta(\mu) \in \text{CI}\} \leq \beta \qquad \text{for every } \theta \in \Theta. \tag{1.2}$$

A β-protection of the form (1.2) leads to a confidence interval of the form CI$= \big[L(X_1,\ X_2,\ \cdots,\ X_{N_x},\ Y_1,\ Y_2,\ \cdots,\ Y_{N_y}),\ \infty\big)$ where L is some measurable function of the stopped sequences of random variables when employing a stopping time $N = (N_x,\ N_y)$. A two-sided situation arises if (1.2) is replaced by

$$P_\theta \{\mu - \delta(\mu) \in \text{CI} \ \text{ or } \ \mu + \delta(\mu) \in \text{CI}\} \leq \beta \qquad \text{for every } \theta \in \Theta.$$

In this case the appropriate CI has the form $\big[L_1(X_1,\ X_2, \cdots,\ X_{N_x},\ Y_1,\ Y_2, \cdots,\ Y_{N_y}),\ L_2(X_1,\ X_2, \cdots,\ X_{N_x},\ Y_1,\ Y_2, \cdots,\ Y_{N_y})\big]$ where L_1 and L_2 are some measurable functions of stopped random variables.

Various sequential estimation procedures with β-protection have been studied. Wijsman (1982, 1983) treated the mean of a normal with known variance, and of μ/σ for unknown mean μ and variance σ^2 and Juhlin (1985) studied the mean of a scale parameter exponential distribution. Furthermore Wijsman (1986) studied the one-parameter problem in a rather general setting and it was generalized to vector valued parameters by Fakhre-Zakeri (1989). Kim (1990, 1997) studied the mean problem when nuisance parameters are present.

Throughout this paper, let $\Phi(x)$ be the distribution of the standard normal and z_α be its upper α point. $\overline{X}_n$ and $S^2_{x,n}$ ($\overline{Y}_n$ and $S^2_{y,n}$) are the sample mean and the sample variance of X's (Y's) respectively. Denote the standardized X's and Y's by

$$Z_{x,i} = \frac{X_i - \mu_x}{\sigma_x}, \qquad Z_{y,i} = \frac{Y_i - \mu_y}{\sigma_y}.$$

$\overline{Z}_{x,n}$ and $S'^2_{x,n}$ ($\overline{Z}_{y,n}$ and $S'^2_{y,n}$) are the sample mean and the sample variance of Z_x's (Z_y's).

2. Sequential Procedure

We want to find an one-sided confidence interval with both coverage probability $\geq \alpha$ and β-protection at $\mu - \delta(\mu)$. In order to derive a sequential procedure, temporarily we assume that $\delta(x)$ is a constant function, *i.e.*, $\delta(x) = d > 0$, and pretend first that there is a fixed sample size interval CI of the form

$$\text{CI} = \big[\overline{X}_m - \overline{Y}_n - \rho d,\ \infty\big) \tag{2.1}$$

for m observations on X's and n observations on Y's where ρ $(0 < \rho < 1)$ is still to be chosen. Then by the two requirements (1.1) and (1.2), for any given $0 < \alpha$, $\beta < 1$,

$$P_\theta\left\{\overline{X}_m - \mu_x - (\overline{Y}_n - \mu_y) \leq \rho d\right\} \geq 1 - \alpha$$

and

$$P_\theta\left\{\overline{X}_m - \mu_x - (\overline{Y}_n - \mu_y) \leq -(1-\rho) d\right\} \leq \beta.$$

If both σ_x and σ_y were known, we get

$$\sigma_x^2/m + \sigma_y^2/n \leq [d/(z_\alpha + z_\beta)]^2 . \tag{2.2}$$

Regarding m and n as continuous variables then, the pair $(m^*,\ n^*)$ which satisfies (2.2) and for which the total number of observations $T = m + n$ is minimized is given by

$$m^* = \sigma_x(\sigma_x + \sigma_y)\left[(z_\alpha + z_\beta)/d\right]^2, \quad n^* = \sigma_y(\sigma_x + \sigma_y)\left[(z_\alpha + z_\beta)/d\right]^2 . \tag{2.3}$$

Thus, the minimal total sample size is

$$T = m^* + n^* = \left[(\sigma_x + \sigma_y)(z_\alpha + z_\beta)/d\right]^2 . \tag{2.4}$$

When σ_x and σ_y are unknown we can not find with fixed sample size procedure any interval CI of the form (2.1). So we shall propose sequential procedures determining m and n as random variables where the goal can be achieved. The procedure consists of (i) a sampling scheme, and (ii) a stopping rule and a terminal decision rule. Now let $S_{x,m}^2$ and $S_{y,n}^2$ be the estimates of σ_x^2 and σ_y^2 respectively.

For a sampling scheme we take n_0 (≥ 2) observations on the two populations to begin with. Then if at any stage we have taken m observations on X's and n observations on Y's, we take the next observation

(a) on X's if $m/n \leq S_{x,m}^2/S_{y,n}^2$,

(b) on Y's if $m/n > S_{x,m}^2/S_{y,n}^2$. (2.5)

For stopping rules we can give four more or less similar stopping rules, easily motivated from (2.3) and (2.4) by replacing the unknown σ_x^2 and σ_y^2 by $S_{x,m}^2$ and $S_{y,n}^2$ respectively, and d by $\delta(\overline{X}_m,\ \overline{Y}_n)$ (refer to Eisele, 1990). We give the following one of the four more or less similar stopping rules.

Stop with the first m, n ($\geq n_0$) such that N_x observations on X's and N_y observations on Y's have been taken with

$$\begin{aligned} N_x &= N_x(c) = \inf\{m \geq n_0 : \ m \geq c^2 S_{x,m}(S_{x,m} + S_{y,n})/\delta^2(\overline{X}_m, \ \overline{Y}_n)\}, \\ N_y &= N_y(c) = \inf\{n \geq n_0 : \ n \geq c^2 S_{y,n}(S_{x,m} + S_{y,n})/\delta^2(\overline{X}_m, \ \overline{Y}_n)\}, \\ N &= N_x + N_y, \end{aligned} \quad (2.6)$$

where the $c > 0$ is still to be chosen.

Using the above stopping rules (2.6) we propose the sequential confidence interval for $\mu = \mu_x - \mu_y$ as

$$\mathrm{CI} = \left[\overline{X}_{N_x} - \overline{Y}_{N_y} - \rho\delta(\overline{X}_{N_x}, \ \overline{Y}_{N_y}), \ \infty\right) \quad (2.7)$$

with ρ $(0 < \rho < 1)$ to be chosen.

We shall make the following assumption about the imprecision function $\delta(x, y)$.

Assumption. $\delta(x, \ y) = \delta^*(x - y)$, $\delta^* : \mathbb{R} \to \mathbb{R}^+$ such that

(a) $0 < \delta^*(x) < L$ for all $x \in \mathbb{R}$ for some $L < \infty$ and $\delta^*(x) \to 0$ as $x \to -\infty$.

(b) δ^* is differentiable and $0 < \delta^{*\prime}(x) < M$ for all $x \in \mathbb{R}$, $M < \infty$.

(c) $\delta^*(x + y)/\delta^*(y) \to 1$ as $y \to 0$ uniformly in x.

The next theorem shows that our sequential procedure satisfies the two requirements (1.1) and (1.2). Throughout this paper, we define

$$\alpha(c, \ \theta) = P_\theta \{\mu \notin \mathrm{CI}\}$$

and

$$\beta(c, \ \theta) = P_\theta \{\mu - \delta(\mu_x, \ \mu_y) \in \mathrm{CI}\}$$

where $\mu = \mu_x - \mu_y$.

Theorem 2.1. *For any given α, β, ρ $(0 < \alpha, \ \beta, \ \rho < 1)$ there exists $c_0 > 0$ such that with the sampling scheme (2.5), the stopping time (2.6) and the confidence interval (2.7) for the difference of two means, $\mu = \mu_x - \mu_y$, the two requirements (1.1) and (1.2) are achieved for all $c > c_0$.*

Before proving the theorem, we need lemmas.

Lemma 2.2. *N_x and N_y go to ∞ a.s. as $c \to \infty$ uniformly in $\theta \in \Theta_1 = \{\theta : \min(\sigma_x, \ \sigma_y) \geq d\delta(\mu_x, \ \mu_y)\}$ for some $d > 0$ where N_x and N_y are as in (2.6).*

Proof. We rewrite the stopping time N_x in (2.6) as follows:

$$N_x(c) = \inf\left\{ m \geq n_0 : \ m \geq \frac{c^2 \sigma_x S'_{x,m}(\sigma_x S'_{x,m} + \sigma_y S'_{y,n})}{\delta^2(\mu_x + \sigma_x \overline{Z}_{x,m}, \ \mu_y + \sigma_y \overline{Z}_{y,n})} \right\}.$$

We define a new stopping time N'_x as follows:

$$N'_x(c) = \inf\left\{ m \geq n_0 : \ m \geq c^2 \min(\sigma_x^2 S'^2_{x,m}, \ \sigma_x \sigma_y S'_{x,m} S'_{y,n})/L^2 \right\}.$$

Then $N'_x(c) \leq N_x(c)$ *a.s.* for all c and $\theta \in \Theta$ and $N'_x(c) \to \infty$ *a.s.* as $c \to \infty$. So $N_x(c) \to \infty$ *a.s.* as $c \to \infty$ uniformly in $\theta \in \Theta_1 = \{\theta : \ \min(\sigma_x, \ \sigma_y) \geq d\delta(\mu_x, \ \mu_y)\}$ from the assumption about $\delta(x, \ y)$. Similarly $N_y(c) \to \infty$ *a.s.* as $c \to \infty$ uniformly in Θ_1. □

Lemma 2.3. *As* $c \to \infty$ $N_x \delta^2(\mu_x, \ \mu_y)/\left[c^2 \sigma_x(\sigma_x + \sigma_y)\right] \to 1$ *a.s. and* $N_y \delta^2(\mu_x, \ \mu_y)/\left[c^2 \sigma_y(\sigma_x + \sigma_y)\right] \to 1$ *a.s. uniformly in* $\Theta_1 = \{\theta : \ \min(\sigma_x, \ \sigma_y) \geq d\delta(\mu_x, \ \mu_y)\}$ *for some* $d > 0$.

Proof. By definition of N_x the following two inequalities hold.

$$\begin{aligned}
&\frac{\delta^2(\mu_x, \ \mu_y)}{\delta^2(\overline{X}_{N_x}, \ \overline{X}_{N_y})} S'_{x,N_x} \frac{\sigma_x S'_{x,N_x} + \sigma_y S'_{y,N_y}}{\sigma_x + \sigma_y} \\
&\quad \leq N_x \delta^2(\mu_x, \ \mu_y)/\left[c^2 \sigma_x(\sigma_x + \sigma_y)\right] \\
&\quad \leq \frac{\delta^2(\mu_x, \ \mu_y)}{c^2 \sigma_x(\sigma_x + \sigma_y)} + \frac{\delta^2(\mu_x, \ \mu_y)}{\delta^2(\overline{X}_{N_x - 1}, \ \overline{Y}_{N_y - 1})} S'_{x,N_x - 1} \frac{\sigma_x S'_{x,N_x - 1} + \sigma_y S'_{y,N_y - 1}}{\sigma_x + \sigma_y}.
\end{aligned}$$

By Lemma 2.2, SLLN, and the assumption on $\delta(x, \ y)$, we can easily show that $N_x \delta^2(\mu_x, \ \mu_y)/\left[c^2 \sigma_x(\sigma_x + \sigma_y)\right] \to 1$ *a.s.* as $c \to \infty$ uniformly in Θ_1. Similar argument shows that $N_y \delta^2(\mu_x, \ \mu_y)/\left[c^2 \sigma_y(\sigma_x + \sigma_y)\right] \to 1$ *a.s.* as $c \to \infty$ uniformly in Θ_1. □

Proof of Theorem 2.1. First we shall show that there exists $d > 0$ such that $\alpha(c, \ \theta) \leq \alpha$, $\beta(c, \ \theta) \leq \beta$ for all θ if $\max(\sigma_x, \ \sigma_y) < d\delta(\mu_x, \ \mu_y)$. Using the confidence interval (2.7), $\alpha(c, \ \theta)$ and $\beta(c, \ \theta)$ can be written

$$\begin{aligned}
\alpha(c, \ \theta) &= P_\theta\left\{\sigma_x \overline{Z}_{x,N_x} - \sigma_y \overline{Z}_{y,N_y} > \rho\delta^*(\mu_x - \mu_y + \sigma_x \overline{Z}_{x,N_x} - \sigma_y \overline{Z}_{y,N_y})\right\} \\
&= P_\theta\left\{\sigma_x \overline{Z}_{x,N_x} - \sigma_y \overline{Z}_{y,N_y} > \rho\delta^*(\mu_x - \mu_y + \sigma_x \overline{Z}_{x,N_x} - \sigma_y \overline{Z}_{y,N_y}), \right. \\
&\qquad \left. \sigma_x \overline{Z}_{x,N_x} - \sigma_y \overline{Z}_{y,N_y} \geq 0\right\} \\
&\leq P_\theta\left\{\sigma_x \overline{Z}_{x,N_x} - \sigma_y \overline{Z}_{y,N_y} > \rho\delta^*(\mu_x - \mu_y)\right\} \qquad (2.8)
\end{aligned}$$

and

$$\begin{aligned}\beta(c,\ \theta) &= P_\theta\{\sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y} \le \rho\delta^*(\mu_x - \mu_y + \sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y}) \\ &\qquad -\delta^*(\mu_x - \mu_y)\} \\ &\le P_\theta\{\sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y} \le \rho\delta^*(\mu_x - \mu_y + \sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y}) \\ &\qquad -\delta^*(\mu_x - \mu_y),\ \sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y} \ge 0\} \\ &\quad + P_\theta\{\sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y} \le \rho\delta^*(\mu_x - \mu_y + \sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y}) \\ &\qquad -\delta^*(\mu_x - \mu_y),\ \sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y} < 0\}. \qquad (2.9)\end{aligned}$$

If $\sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y} \ge 0$, write $\delta^*(\mu_x - \mu_y + \sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y}) = \delta^*(\mu_x - \mu_y) + \delta^{*\prime}(V_N)(\sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y}) \le \delta^*(\mu_x - \mu_y) + M(\sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y})$ for some V_N in the neighborhood of $\mu_x - \mu_y$. If $\sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y} < 0$, then $\delta^*(\mu_x - \mu_y + \sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y}) \le \delta^*(\mu_x - \mu_y)$. Therefore (2.9) can be bounded above either by

$$\begin{aligned}&P_\theta\left\{\sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y} \ge (1-\rho)\delta^*(\mu_x - \mu_y)/(\rho M - 1)\right\} \\ &+P_\theta\left\{\sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y} \le -(1-\rho)\delta^*(\mu_x - \mu_y)\right\} \quad \text{if } \rho M - 1 > 0,\end{aligned}$$

or by

$$P_\theta\left\{\sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y} \le -(1-\rho)\delta^*(\mu_x - \mu_y)\right\} \quad \text{if } \rho M - 1 \le 0.$$

Let

$$t = \begin{cases} \min\{\rho,\ 1-\rho\}, & \text{if } \rho M - 1 \le 0, \\ \min\{\rho,\ 1-\rho,\ (1-\rho)/(\rho M - 1)\}, & \text{if } \rho M - 1 > 0. \end{cases}$$

Then for all $c > 0$ and θ,

$$\alpha(c,\ \theta) \le P_\theta\left\{\left|\sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y}\right| > t\delta^*(\mu_x - \mu_y)\right\} \qquad (2.10)$$

and

$$\beta(c,\ \theta) \le P_\theta\left\{\left|\sigma_x\overline{Z}_{x,N_x} - \sigma_y\overline{Z}_{y,N_y}\right| > t\delta^*(\mu_x - \mu_y)\right\}. \qquad (2.11)$$

Define $\varepsilon = \min(\alpha,\ \beta)$. Since $\overline{Z}_{x,m}$ and $\overline{Z}_{y,n}$ go to zero *a.s.* as m and $n \to +\infty$ uniformly in θ, there exists a constant $a > 0$ such that $P_\theta\left\{\left|\overline{Z}_{x,N}\right| > a/2\right\} \le \varepsilon/2$ and $P_\theta\left\{\left|\overline{Z}_{y,N}\right| > a/2\right\} \le \varepsilon/2$ for all θ no matter what the stopping time N is. If we choose $d = t/a$, then (2.10) and (2.11) are less than ε for all θ if $\sigma_x < d\delta^*(\mu_x - \mu_y)$ and $\sigma_y < d\delta^*(\mu_x - \mu_y)$.

Next we will prove that $\alpha(c,\ \theta)$ and $\beta(c,\ \theta) \to 0$ as $c \to \infty$ uniformly in θ if $\sigma_x \geq d\delta^*(\mu_x - \mu_y)$ and $\sigma_y \geq d\delta^*(\mu_x - \mu_y)$. By (2.10) and (2.11) both $\alpha(c,\ \theta)$ and $\beta(c,\ \theta)$ are bounded by

$$P_\theta\left\{\left|\sigma_x \overline{Z}_{x,N_x}\right| > (t/2)\delta^*(\mu_x - \mu_y)\right\} + P_\theta\left\{\left|\sigma_y \overline{Z}_{y,N_y}\right| > (t/2)\delta^*(\mu_x - \mu_y)\right\}. \quad (2.12)$$

Using Lemmas 2.2 and 2.3 and Anscombe's theorem, (2.12) converges to zero as $c \to \infty$.

Finally either for $\sigma_x < d\delta^*(\mu_x - \mu_y)$, $\sigma_y \geq d\delta^*(\mu_x - \mu_y)$ or for $\sigma_x \geq d\delta^*(\mu_x - \mu_y)$, $\sigma_y < d\delta^*(\mu_x - \mu_y)$, using $P_\theta\left\{\left|\overline{Z}_{x,N_x}\right| > a\right\} < \varepsilon$ for all θ and Anscombe's theorem, we can easily prove that both $\alpha(c,\ \theta)$ and $\beta(c,\ \theta)$ are less than ε as $c \to \infty$. □

Theorem 2.4. *For any given $c > 0$, $\alpha(c,\ \theta) \to 1 - \Phi(\rho c)$ and $\beta(c,\ \theta) \to 1 - \Phi\big(c(1-\rho)\big)$ as $\mu_x - \mu_y \to -\infty$.*

Proof. By a straightforward two-sample extension of Anscombe's theorem

$$N^{1/2}(\overline{X}_{N_x} - \mu_x,\ \overline{Y}_{N_y} - \mu_y) \xrightarrow{\mathcal{L}} N_2(0,\ \Sigma) \quad \text{as } N_x \text{ and } N_y \to \infty, \quad (2.13)$$

where $\Sigma = \begin{pmatrix} \sigma_x(\sigma_x + \sigma_y) & 0 \\ 0 & \sigma_y(\sigma_x + \sigma_y) \end{pmatrix}$.

We can easily show that $N_x \to \infty$ and $N_y \to \infty$ *a.s.* as $\mu_x - \mu_y \to -\infty$ by the assumption on $\delta(x,\ y)$. We rewrite the $\alpha(c,\ \theta)$ as follows:

$$\alpha(c,\ \theta) = P_\theta\left\{\frac{N^{1/2}(\overline{X}_{N_x} - \mu_x - \overline{Y}_{N_y} + \mu_y)}{\sigma_x + \sigma_y} \geq \frac{\rho\delta(\overline{X}_{N_x},\ \overline{Y}_{N_y})}{\sigma_x + \sigma_y} N^{1/2}\right\}.$$

Then from (2.13) $N^{1/2}(\overline{X}_{N_x} - \mu_x - \overline{Y}_{N_y} + \mu_y)/(\sigma_x + \sigma_y) \xrightarrow{\mathcal{L}} N(0,\ 1)$ and $\delta(\overline{X}_N,\ \overline{Y}_N)N^{1/2}/(\sigma_x + \sigma_y) \to c$ *a.s.* as $\mu_x - \mu_y \to -\infty$ by the assumption on $\delta(x,\ y)$. Therefore $\alpha(c,\ \theta) \to 1 - \Phi(\rho c)$ as $\mu_x - \mu_y \to -\infty$. Similarly we can prove that $\beta(c,\ \theta) \to 1 - \Phi\big(c(1-\rho)\big)$. □

Theorem 2.5. *Set $N_0 = [c(\sigma_x + \sigma_y)/\delta(\mu_x, \mu_y)]^2$. Then $(N - N_0)/N_1^{1/2} \xrightarrow{\mathcal{L}} N(0,\ 1)$ as $c \to \infty$ where $N_1 = 2N_0\{1 + 2[(\sigma_x + \sigma_y)\delta^{*\prime}(\mu_x - \mu_y)/\delta^*(\mu_x - \mu_y)]^2\}$.*

Proof. By a straightforward two-sample extension of Anscombe's theorem

$$N^{1/2}(\overline{X}_{N_x} - \mu_x,\ \overline{Y}_{N_y} - \mu_y) \xrightarrow{\mathcal{L}} N_2(0,\ \Sigma_1) \quad \text{as } c \to \infty, \quad (2.14)$$

where $\Sigma_1 = \begin{pmatrix} \sigma_x(\sigma_x+\sigma_y) & 0 \\ 0 & \sigma_y(\sigma_x+\sigma_y) \end{pmatrix}$ and

$$N^{1/2}(S^2_{x,N_x}-\sigma^2_x,\ S^2_{y,N_y}-\sigma^2_y) \xrightarrow{\mathcal{L}} N_2(0,\ \Sigma_2) \quad \text{as } c\to\infty \tag{2.15}$$

where $\Sigma_2 = \begin{pmatrix} 2\sigma^3_x(\sigma_x+\sigma_y) & 0 \\ 0 & 2\sigma^3_y(\sigma_x+\sigma_y) \end{pmatrix}$. Since $N/N_0 \to 1$ *a.s.* as $c\to\infty$, (2.14) and (2.15) lead to

$$N_0^{1/2}(\overline{X}_{N_x}-\mu_x,\ \overline{Y}_{N_y}-\mu_y) \xrightarrow{\mathcal{L}} N_2(0,\ \Sigma_1) \quad \text{as } c\to\infty \tag{2.16}$$

and

$$N_0^{1/2}(S^2_{x,N_x}-\sigma^2_x,\ S^2_{y,N_y}-\sigma^2_y) \xrightarrow{\mathcal{L}} N_2(0,\ \Sigma_2) \quad \text{as } c\to\infty. \tag{2.17}$$

We can reduce (2.17) to

$$N_0^{1/2}(S_{x,N_x}-\sigma_x,\ S_{y,N_y}-\sigma_y) \xrightarrow{\mathcal{L}} N_2(0,\ \Sigma_3) \quad \text{as } c\to\infty \tag{2.18}$$

where $\Sigma_3 = \begin{pmatrix} \sigma_x(\sigma_x+\sigma_y)/2 & 0 \\ 0 & \sigma_y(\sigma_x+\sigma_y)/2 \end{pmatrix}$. Applying the Taylor theorem to (2.18), we get

$$N_0^{1/2}\left[(S_{x,N_x}+S_{y,N_y})^2-(\sigma_x+\sigma_y)^2\right] \xrightarrow{\mathcal{L}} N\left(0,\ 2(\sigma_x+\sigma_y)^4\right) \tag{2.19}$$

as $c\to\infty$. A result analogous to (2.16) and (2.19) will hold with N_x-1 and N_y-1 instead of N_x and N_y , respectively.

Now $N_x \geq c^2 S_{x,N_x}(S_{x,N_x}+S_{y,N_y})/\delta^2(\overline{X}_{N_X},\ \overline{Y}_{N_y})$ *a.s.* and $N_y \geq c^2 S_{y,N_y}$ $(S_{x,N_x}+S_{y,N_y})/\delta^2(\overline{X}_{N_X},\ \overline{Y}_{N_y})$ *a.s.* So we have

$$\begin{aligned} &P\left\{(N-N_0)/(2N_0)^{1/2} \leq x\right\} \\ &\leq P\left\{\frac{\delta^2(\mu_x,\ \mu_y)}{\delta^2(\overline{X}_{N_x},\ \overline{Y}_{N_y})}\frac{N_0^{1/2}\left[(S_{x,N_x}+S_{y,N_y})^2-(\sigma_x+\sigma_y)^2\right]}{\sqrt{2}(\sigma_x+\sigma_y)^2}\right. \\ &\qquad \left.+\frac{c^2}{\delta^2(\overline{X}_{N_x},\ \overline{Y}_{N_y})}\frac{(\sigma_x+\sigma_y)^2}{(2N_0)^{1/2}}\left[1-\frac{\delta^2(\overline{X}_{N_x},\ \overline{Y}_{N_y})}{\delta^2(\mu_x,\ \mu_y)}\right] \leq x\right\}. \end{aligned}$$

Since $\overline{X}_{N_x}\to\mu_x$ and $\overline{Y}_{N_y}\to\mu_y$ *a.s.* as $c\to\infty$, it follows from (2.19) that

$$\frac{\delta^2(\mu_x,\ \mu_y)}{\delta^2(\overline{X}_{N_x},\ \overline{Y}_{N_y})}\frac{N_0^{1/2}\left[(S_{x,N_x}+S_{y,N_y})^2-(\sigma_x+\sigma_y)^2\right]}{\sqrt{2}(\sigma_x+\sigma_y)^2} \xrightarrow{\mathcal{L}} N(0,\ 1) \text{ as } c\to\infty.$$

Observe that

$$1-\frac{\delta^2(\overline{X}_{N_x},\ \overline{Y}_{N_y})}{\delta^2(\mu_x,\ \mu_y)}=-\frac{2\delta^*(U_{(N_x,N_y)})\delta^{*\prime}(U_{(N_x,N_y)})(\overline{X}_{N_x}-\overline{Y}_{N_y}-\mu_x+\mu_y)}{\delta^{*2}(\mu_x-\mu_y)}$$

where $\left|U_{(N_x,N_y)}-(\mu_x-\mu_y)\right|\le\left|\overline{X}_{N_x}-\overline{Y}_{N_y}-\mu_x+\mu_y\right|$. As $c\to\infty$, $U_{(N_x,N_y)}\to\mu_x-\mu_y$ *a.s.* Therefore, by (2.16), we have

$$\frac{c^2}{\delta^2(\overline{X}_{N_x},\ \overline{Y}_{N_y})}\frac{(\sigma_x+\sigma_y)^2}{(2N_0)^{1/2}}\left(1-\frac{\delta^2(\overline{X}_{N_x},\ \overline{Y}_{N_y})}{\delta^2(\mu_x,\ \mu_y)}\right)$$
$$\xrightarrow{\mathcal{L}} N\left(0,\ 2(\sigma_x+\sigma_y)^2\left[\delta^{*\prime}(\mu_x-\mu_y)/\delta^*(\mu_x-\mu_y)\right]^2\right)$$

as $c\to\infty$. Hence

$$\limsup_{c\to\infty}\ P\{(N-N_0)/N_1^{1/2}\le x\}\le\Phi(x). \tag{2.20}$$

It is easy to obtain that

$$N_x\le c^2S_{x,N_x-1}(S_{x,N_x-1}+S_{y,N_y-1})/\delta^2(\overline{X}_{N_x-1},\ \overline{Y}_{N_y-1})+n_0\quad a.s.$$

and

$$N_y\le c^2S_{y,N_y-1}(S_{x,N_x-1}+S_{y,N_y-1})/\delta^2(\overline{X}_{N_x-1},\ \overline{Y}_{N_y-1})+n_0\quad a.s.,$$

so that

$$P\left\{\frac{N-N_0}{N_1^{1/2}}\le x\right\}\ge P\left\{N_1^{-1/2}\left(\frac{c^2(S_{x,N_x-1}+S_{y,N_y-1})^2}{\delta^2(\overline{X}_{N_x-1},\overline{Y}_{N_y-1})}-N_0\right)+\frac{2n_0}{N_1^{1/2}}\le x\right\},$$

and hence

$$\liminf_{c\to\infty}P\{(N-N_0)/N_1^{1/2}\le x\}\ge\Phi(x). \tag{2.21}$$

Combining (2.20) and (2.21), we get the desired result of the theorem. □

References

Anscombe, F. J. (1952). "Large sample theory of sequential estimation", *Proceedings of Cambridge Philosophical Society*, **48**, 600–607.

Chow, Y. S. and Robbins, H. (1965). "On the asymptotic theory of fixed width confidence intervals for the means", *The Annals of Mathematical Statistics*, **36**, 456–462.

Eisele, J. R. (1990). "An adaptive biased coin design for the Behrens-Fisher problem", *Sequential Analysis*, **9**, 343–359.

Fahker-Zakeri (1989). "Sequential confidence sets with beta-protection in multiparameter families", *Sequential Analysis*, **8**, 1–26.

Ghosh, M. and Mukhopadhyay, N. (1980). "Sequential point estimation of the difference of two normal mean", *The Annals of Statistics*, **8**, 221–225.

Juhlin, K. D. (1985). *Sequential and Non-sequential Intervals with Guaranteed Coverage Probability and Beta Protection*, Ph.D. Dissertation, University of Illinois.

Kim, S. L. (1990). "Sequential confidence intervals for the mean with β-protection in a certain parameter space", *Journal of the Korean Statistical Society*, **19**, 113–121.

Kim, S. L. (1997). "Sequential confidence interval with β-protection for a linear function of two normal means", *Journal of the Korean Statistical Society*, **26**, 309–317.

Robbins, H., Simons, G., and Starr, N. (1967). "Sequential analogue of the Behrens-Fisher problem", *The Annals of Mathematical Statistics*, **38**, 1384–1391.

Wijsman, R. A. (1981). "Confidence sets based on sequential test", *Communications in Statistics-Theory and Method A*, **10**, 2137–2147.

Wijsman, R. A. (1982). "Sequential confidence sets; Estimation-oriented versus Test-oriented construction", *Statistical Decision Theory and Related Topics III, Vol. 2*, 435–450.

Wijsman, R. A. (1983). "Sequential estimation and test-oriented confidence intervals for μ/σ in a normal population", *Sequential Analysis*, **2**, 149–174.

Wijsman, R. A. (1986). "Sequential confidence intervals with beta-protection in one-parameter families", *Adaptive Statistical Procedures and Related Topics*, 96–107.

Wolfowitz, J. (1950). "Minimax estimates of the mean of a normal distribution with unknown variance", *The Annals of Mathematical Statistics*, **21**, 218–230.