Journal of the Korean Statistical Society (2002), **31**: *3, pp 369–378*

On the Strong Laws for Weighted Sums of AANA Random Variables[†]

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Abstract

Strong laws of large numbers for weighted sums of asymptotically almost negatively associated(AANA) sequence are proved by our generalized maximal inequality for AANA random variables at a crucial step.

Keywords. Strong law of large numbers, negatively associated, asymptotically almost negatively associated, weighted sums.
AMS 2000 subject classifications. Primary 60F05.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{X_n, n \geq 1\}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, P)$. A finite family $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated(NA) if for any disjoint subsets $A, B \subset \{1, \cdots, n\}$ and any real coordinatewise nondecreasing functions $f : R^A \to R$ and $g : R^B \to R$,

$$\text{Cov}(f(X_i : i \in A), \quad g(X_j : j \in B)) \leq 0.$$

An infinite family of random variables is negatively associated (NA) if every finite subfamily is negatively associated (NA). This concept was introduced by Joag-Dev and Proschan (1983).

By inspecting the proof of Matula (1992) maximal inequality for the negatively associated random variables, we see that one can also allow positive correlations provided they are small. Primarily motivated by this, one can introduce

Received December 2001; accepted May 2002.
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the followings: A sequence $\{X_n, n \geq 1\}$ of random variables is called asymptotically almost negatively associated(AANA) if there is a nonnegative sequence $q(m) \to 0$ such that

$$\begin{aligned}&\mathrm{Cov}(f(X_m), g(X_{m+1}, \cdots, X_{m+k}))\\&\quad \leq q(m)(\mathrm{Var}(f(X_m))\mathrm{Var}(g(X_{m+1}, \cdots, X_{m+k})))^{1/2} \qquad (1.1)\end{aligned}$$

for all $m, k \geq 1$ and for all coordinatewise increasing continuous functions f and g whenever the right side of (1.1) is finite. This definition was introduced by Chandra and Ghosal (1996a, b).

The family of AANA sequences contains NA(in particular, independent) sequence(with $q(m) = 0$, $\forall\ m \geq 1$) and some more sequences of random variables which are not much deviated from being negatively associated.

The condition roughly means that asymptotically the future is almost negatively associated with the present. For an example of AANA random variables which is not negatively associated, consider $X_n = Y_n + \alpha_n Y_{n+1}$ where $Y_1, Y_2, \cdots$ are *iid* $N(0,1)$ and $\alpha_n \to 0, \alpha_n > 0$(see Chandra and Ghosal, 1996b). Let $\{X_n, n \geq 1\}$ be a sequence of random variables and $\{a_n\}$ be a sequence of positive weights and put $b_n = \sum_{k=1}^n a_k$ and $S_n = b_n^{-1} \sum_{k=1}^n a_k(X_k - EX_k)$. Under various dependence conditions the almost sure convergences of $\{S_n\}$ to zero have been established by many authors(*e.g.* Birkel, 1992; Kim and Baek, 1999; Liu *et al.*, 1999; Matula, 1996; Peligrad and Gut, 1999), especially Chandra and Ghosal (1996a, b) derived the Marcinkiewicz strong law of large numbers and the almost sure convergence of weighted averages on the AANA random variables.

In this paper we derive the maximal inequality for the weighted sums of asymptotically almost negatively associated (AANA) random variables and apply it to obtain the strong laws of large numbers for weighted sums of them. We also show that some results for NA random variables remain true if the assumption of NA random variables is relaxed to AANA random variables with $\sum_{m=1}^{\infty} q^2(m) < \infty$.

2. Results

From the definition of AANA random variable we obtain easily the following property.

Lemma 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of asymptotically almost negatively* *associated(AANA) random variables and $\{f_n, n \geq 1\}$ a sequence of increasing continuous functions. Then $\{f_n(X_n), n \geq 1\}$ is also a sequence of AANA random variables.*

Theorem 2.1. (Chandra, Ghosal, 1996a) *Let $\{X_n, n \geq 1\}$ be a sequence of mean zero, square integrable AANA random variables such that (1.1) holds for $1 \leq m < k+m < n$ and for all coordinatewise increasing continuous functions f and g whenever the right hand side of (1.1) is finite. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$ and $\sigma_k^2 = EX_k^2$, $k \geq 1$. Then*

$$P\left\{\max_{1\leq k\leq n} |S_k| \geq \epsilon\right\} \leq 2\epsilon^{-2}(A + (1+A^2)^{\frac{1}{2}})^2 \sum_{k=1}^{n} \sigma_k^2$$

where $S_n = \sum_{i=1}^{n} X_i$.

Theorem 2.2. *Let $\{a_n, n \geq 1\}$ be a positive sequence of real numbers and $\{b_n, n \geq 1\}$ a sequence of nondecreasing positive real numbers. Let $\{X_n, n \geq 1\}$ be a sequence of mean zero, square integrable AANA random variables such that (1.1) holds for $1 \leq m < k+m < n$ and for all coordinatewise increasing continuous functions f and g whenever the right hand side of (1.1) is finite. Let $A^2 = \sum_{m=1}^{n-1} q^2(m)$ and $\sigma_k^2 = EX_k^2$, $k \geq 1$. Then*

$$P\left\{\max_{1\leq k\leq n} \left|\frac{\sum_{i=1}^{k} a_i X_i}{b_k}\right| \geq \epsilon\right\} \leq 8\epsilon^{-2}(A + (1+A^2)^{\frac{1}{2}})^2 \sum_{k=1}^{n} \left(\frac{a_k^2 \sigma_k^2}{b_k^2}\right). \tag{2.1}$$

Proof. First note that $a_1X_1/b_1, \cdots$, are mean zero, square integrable AANA random variables according to Lemma 2.1 and that they also have the same $q(\cdot)$. Without loss of generality setting $b_0 = 0$, we get

$$\begin{aligned}
\sum_{j=1}^{k} a_j X_j &= \sum_{j=1}^{k} b_j \cdot \frac{a_j X_j}{b_j} \\
&= \sum_{j=1}^{k} \left(\sum_{i=1}^{j} (b_i - b_{i-1}) \frac{a_j X_j}{b_j}\right) \\
&= \sum_{i=1}^{k} (b_i - b_{i-1}) \sum_{i\leq j\leq k} \frac{a_j X_j}{b_j}.
\end{aligned} \tag{2.2}$$

Hence, since $b_k^{-1}\sum_{j=1}^k(b_j-b_{j-1})=1$ it follows from (2.2) that

$$\left\{\left|\frac{\sum_{j=1}^k a_jX_j}{b_k}\right|\geq\epsilon\right\}\subset\left\{\max_{1\leq i\leq k}\left|\sum_{i\leq j\leq k}\frac{a_jX_j}{b_j}\right|\geq\epsilon\right\}. \tag{2.3}$$

From (2.3) we have

$$\begin{aligned}\left\{\max_{1\leq k\leq n}\left|\frac{\sum_{j=1}^k a_jX_j}{b_k}\right|\geq\epsilon\right\} &\subset \left\{\max_{1\leq k\leq n}\max_{1\leq i\leq k}\left|\sum_{i\leq j\leq k}\frac{a_jX_j}{b_j}\right|\geq\epsilon\right\}\\ &= \left\{\max_{1\leq i\leq k\leq n}\left|\sum_{j\leq k}\frac{a_jX_j}{b_j}-\sum_{j<i}\frac{a_jX_j}{b_j}\right|\geq\epsilon\right\}\\ &\subset \left\{\max_{1\leq k\leq n}\left|\sum_{1\leq j\leq k}\frac{a_jX_j}{b_j}\right|\geq\frac{\epsilon}{2}\right\}. \end{aligned}\tag{2.4}$$

By the Kolmogorov-type inequality of AANA random variables(see Theorem 2.1) from (2.4) the desired result (2.1) follows.

From Theorem 2.2 we can get the following generalized inequality. □

Theorem 2.3. *Let $\{a_n,n\geq 1\}$ be a sequence of positive real numbers and $\{b_n,n\geq 1\}$ a sequence of nondecreasing positive real numbers. Let $\{X_n,n\geq 1\}$ be a sequence of mean zero, square integrable AANA random variables such that (1.1) holds for $1\leq m<k+m<n$ and for all coordinatewise increasing continuous functions f and g whenever the right hand side of (1.1) is finite. Let $A^2=\sum_{m=1}^{n-1}q^2(m)$ and $\sigma_k^2=EX_k^2$, $k\geq 1$. Then, for $\epsilon>0$, and for any positive integer $m<n$,*

$$\begin{aligned}P&\left\{\max_{m\leq k\leq n}\left|\frac{\sum_{i=1}^k a_iX_i}{b_k}\right|\geq\epsilon\right\}\\ &\leq 32\epsilon^{-2}(A+(1+A^2)^{\frac{1}{2}})^2\left(\sum_{j=m+1}^n\frac{a_j^2\sigma_j^2}{b_j^2}+\sum_{j=1}^m\frac{a_j^2\sigma_j^2}{b_m^2}\right). \end{aligned}\tag{2.5}$$

Proof. By Theorem 2.2 we have

$$P\left\{\max_{m\leq k\leq n}\left|\frac{\sum_{j=1}^k a_jX_j}{b_k}\right|\geq\epsilon\right\}$$

$$\leq P\left\{\left|\frac{\sum_{j=m+1}^{m} a_j X_j}{b_m}\right| \geq \frac{\epsilon}{2}\right\} + P\left\{\max_{m+1\leq k\leq n}\left|\frac{\sum_{j=m+1}^{k} a_j X_j}{b_k}\right| \geq \frac{\epsilon}{2}\right\}$$

$$\leq P\left\{\max_{1\leq k\leq m}\left|\frac{\sum_{j=1}^{k} a_j X_j}{b_m}\right| \geq \frac{\epsilon}{2}\right\} + P\left\{\max_{m+1\leq k\leq n}\left|\frac{\sum_{j=m+1}^{k} a_j X_j}{b_k}\right| \geq \frac{\epsilon}{2}\right\}$$

$$\leq 32\epsilon^{-2}(A + (1+A^2)^{\frac{1}{2}})^2\left(\sum_{j=1}^{m}\frac{a_j^2\sigma_j^2}{b_m^2} + \sum_{j=m+1}^{n}\frac{a_j^2\sigma_j^2}{b_j^2}\right).$$

□

Theorem 2.4. *Let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers and $\{b_n, n \geq 1\}$ a sequence of positive nondecreasing real numbers. Let $\{X_n, n \geq 1\}$ be a sequence of mean zero, square integrable AANA random variables such that (1.1) holds for $1 \leq m < k+m < n$ and for all coordinatewise increasing continuous functions f and g whenever the right-hand side of (1.1) is finite. Assume*

$$\sum_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right)^2\sigma_n^2 < \infty, \tag{2.6}$$

$$\sum_{k=1}^{\infty} q^2(k) < \infty, \tag{2.7}$$

where $\sigma_k^2 = EX_k^2 < \infty$, $k \geq 1$. Then, for any $0 < r < 2$,

(A) E $\sup_n(|S_n|/b_n)^r < \infty$
(B) $0 < b_n \uparrow \infty$ implies $S_n/b_n \to 0$ *a.s.* as $n \to \infty$, where $S_n = \sum_{i=1}^{n} a_i X_i$, $n \geq 1$.

Proof. Let $B^2 = \sum_{k=1}^{\infty} q^2(k)$.
(A): Note that, for any $0 < r < 2$

$$E\sup_n\left(\frac{|S_n|}{b_n}\right)^r < \infty \Longleftrightarrow \int_1^{\infty} P\left\{\sup_n\frac{|S_n|}{b_n} > t^{\frac{1}{r}}\right\}dt < \infty.$$

By Theorem 2.2 it follows from (2.6) and (2.7) that

$$\begin{aligned}\int_1^{\infty} P\left\{\sup_n\frac{|S_n|}{b_n} > t^{\frac{1}{r}}\right\}dt &\leq 32\int_1^{\infty} t^{-\frac{2}{r}}(B + (1+B^2)^{\frac{1}{2}})^2\sum_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right)^2\sigma_n^2 dt \quad (2.8)\\ &= 32(B + (1+B^2)^{\frac{1}{2}})^2\sum_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right)^2\sigma_n^2\int_1^{\infty} t^{-\frac{2}{r}}dt < \infty.\end{aligned}$$

Hence, the proof of (A) is complete.
(B): By Theorem 2.3, we have

$$P\left\{\max_{m\le k\le n}\left|\frac{\sum_{i=1}^{k} a_iX_i}{b_k}\right|\ge\epsilon\right\} \le 32\epsilon^{-2}(B+(1+B^2)^{\frac{1}{2}})^2\left(\sum_{j=m+1}^{n}\frac{a_j^2\sigma_j^2}{b_j^2}+\sum_{j=1}^{m}\frac{a_j^2\sigma_j^2}{b_m^2}\right). \tag{2.9}$$

But

$$\begin{aligned} &P\left\{\sup_n\left|\frac{\sum_{i=1}^{n} a_iX_i}{b_n}\right|\ge\epsilon\right\}\\ &=\lim_{n\to\infty}P\left\{\max_{m\le j\le n}\left|\frac{\sum_{i=1}^{j} a_iX_i}{b_j}\right|\ge\epsilon\right\} \qquad (2.10)\\ &\le 32\epsilon^{-2}(B+(1+B^2)^{\frac{1}{2}})^2\left(\sum_{j=m+1}^{\infty}\frac{a_j^2\sigma_j^2}{b_j^2}+\sum_{j=1}^{m}\frac{a_j^2\sigma_j^2}{b_m^2}\right)\\ &<\infty \qquad \text{by (2.6) and (2.7).}\end{aligned}$$

Hence, by the Kronecker lemma, we get $\lim_{n\to\infty}P\{\sup_n|(\sum_{i=1}^{n}a_iX_i)/b_n|\ge\epsilon\}=0$, which completes the proof of (B).

□

Remark 2.1. Theorem 2.4 (B) shows that Theorem 2 of Matula (1996) remains true if the assumption of negatively associated random variables is relaxed to AANA random variables with $\sum_{k=1}^{\infty}q^2(k)<\infty$.

Corollary 2.1. *Let $\{a_n, n\ge 1\}$ be a sequence of positive real numbers with $\sup_n a_n^2<\infty$ and let $\{X_n, n\ge 1\}$ be a sequence of mean zero, square integrable AANA random variables such that (1.1) holds for $1\le m<k+m<n$ and for all coordinatewise increasing continuous functions f and g whenever the right-hand side of (1.1) is finite. Let $B^2=\sum_{k=1}^{\infty}q^2(k)$ and $\sup_n\sigma_n^2<\infty$. Then, for $0<t<2$, and for all $\epsilon>0, m\ge 1$,*

$$P\left\{\sup_{n\ge m}\left|\frac{\sum_{i=1}^{n} a_iX_i}{n^{\frac{1}{t}}}\right|\ge\epsilon\right\} \le 32\epsilon^{-2}(B+(1+B^2)^{\frac{1}{2}})^2\frac{2}{2-t}\left(\sup_n\sigma_n^2\right)\left(\sup_n a_n^2\right)m^{(t-2)/t},$$

where $\sigma_n^2 = \mathrm{Var}(X_n)$.

Corollary 2.2. *Let* $\{a_n, n \geq 1\}$ *be a sequence of positive real numbers with* $\sup_n a_n^2 < \infty$ *and let* $\{X_n, n \geq 1\}$ *be a sequence of mean zero, square integrable AANA random variables such that (1.1) holds for* $1 \leq m < k + m < n$ *and for all coordinatewise increasing continuous functions* f *and* g *whenever the right-hand side of (1.1) is finite and* $\sup_n \sigma_n^2 < \infty$ *where* $\sigma_n^2 = \mathrm{Var}(X_n)$. *Assume* $\sum_{k=1}^{\infty} q^2(k) < \infty$. *Then, for* $0 < t < 2$,

(A) $\sum_{i=1}^{n} a_i X_i / n^{\frac{1}{t}} \longrightarrow 0$ *a.s.*, as $n \to \infty$.
(B) $E \sup_n \left(\left| \sum_{i=1}^{n} a_i X_i \right| / n^{\frac{1}{t}} \right)^r < \infty$ for any $0 < r < 2$.

Example 2.1. Let $\{X_n, n \geq 1\}$ be a sequence of mean zero, square integrable AANA random variables such that (1.1) holds for $1 \leq m < k + m < n$ and for all coordinatewise increasing continuous functions f and g whenever the right-hand side of (1.1) is finite and $\sum_{k=1}^{\infty} q^2(k) < \infty$. Assume

$$\sum_{n=1}^{\infty} \frac{\mathrm{Var}(X_n)}{n^4} < \infty.$$

Then, as $n \to \infty$

$$n^{-1} \sum_{k=1}^{n} X_k / k \longrightarrow 0 \quad a.s.$$

Proof. By taking $a_k = 1/k$ and $b_n = n$ from Theorem 2.4 the desired result follows.

□

Example 2.2. Let $\{X_n, n \geq 1\}$ be a sequence of mean zero, square integrable AANA random variables satisfying (1.1) and $\sum_{k=1}^{\infty} q^2(k) < \infty$.
Assume

$$\sum_{n=1}^{\infty} (\log n)^{-2} \frac{\mathrm{Var}(X_n)}{n^2} < \infty.$$

Then, as $n \to \infty$

$$(\log n)^{-1} \sum_{k=1}^{n} X_k / k \longrightarrow 0 \quad a.s.$$

Proof. By taking $a_k = 1/\log k$ and $b_n = n$ from Theorem 2.4 the result follows. □

By taking $a_n = 1$ and $b_n = n^{1/t}$ we use Theorem 2.3 to prove the Marcinkiewicz strong law of large numbers for the AANA random variables as follows:

Theorem 2.5. *Let $\{X_n, n \geq 1\}$ be a sequence of mean zero, square integrable identically distributed AANA random variables such that (1.1) holds for $1 \leq m < k + m < n$ and for all coordinatewise increasing continuous functions f and g whenever the right-hand side of (1.1) is finite and $\sum_{k=1}^{\infty} q^2(k) < \infty$. Assume $E|X_1|^t < \infty$ for some $0 < t < 2$. Then*

$$\sum_{j=1}^{n} X_j / n^{1/t} \longrightarrow 0 \quad a.s.$$

Proof. First note that $\{X_j^+, j \geq 1\}$ and $\{X_j^-, j \geq 1\}$ are AANA random variables with the same $q(\cdot)$. Next, the method of proof is the same as that used in the Marcinkiewicz strong law for the identically distributed NA random variables(see the proof of Theorem 3.2 of Liu *et al.*, 1999). □

Theorem 2.6. *Let $\{b_n, n \geq 1\}$ be a sequence of positive nondecreasing real numbers with $0 < b_n \uparrow \infty$ and let $\{X_n, n \geq 1\}$ be a sequence of mean zero, square integrable identically distributed AANA random variables such that (1.1) holds for $1 \leq m < k+m < n$ and for all coordinatewise increasing continuous functions f and g whenever the right-hand side of (1.1) is finite. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a sequence of real numbers with $\sup_{n \geq 1} \sum_{i=1}^{n} | a_{ni} | < \infty$. Then, as $n \to \infty$*

$$\sum_{i=1}^{n} a_{ni} X_i / b_n \longrightarrow 0 \qquad a.s. \tag{2.11}$$

Proof. Define

$$S_k = \sum_{i=1}^{k} X_i / b_k,$$

$c_{nj} = b_j/b_n(a_{nj} - a_{nj+1})$ for $1 \leq j \leq n-1$ and $c_{nn} = a_{nn}$. Then, we have

$$\sum_{j=1}^{n} a_{nj} X_j / b_n = \sum_{j=1}^{n} c_{nj} S_j, \tag{2.12}$$

$$\sum_{j=1}^{n} | c_{nj} | \leq 2 \sup_{n \geq 1} \sum_{j=1}^{n} | a_{nj} |, \tag{2.13}$$

$$\lim_{n \to \infty} | c_{nj} | = 0 \quad \text{for every fixed } j. \tag{2.14}$$

By (2.13) and (2.14) we easily obtain that, for every sequence $\{d_n\}$ of real numbers with $d_n \to 0$ and $\sum_{j=1}^{n} c_{nj} d_j \to 0$ as $n \to \infty$
Note that $S_n \to 0$ *a.s.* by Theorem 2.4 (B). Combining Theorem 2.4 (B), (2.12) and (2.14) yield the desired result (2.11).

□

Theorem 2.7. *Let* $\{X_n, n \geq 1\}$ *be a sequence of mean zero, square integrable identically distributed AANA random variables such that (1.1) holds for* $1 \leq m < k + m < n$ *and for all coordinatewise increasing continuous functions* f *and* g *whenever the right-hand side of (1.1) is finite and* $\sum_{k=1}^{\infty} q^2(k) < \infty$ *and* $E|X_1|^t < \infty$ *for some* $0 < t < 2$. *Assume that* $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ *is a sequence of real numbers with* $\sup_{n \geq 1} \sum_{i=1}^{n} | a_{ni} | < \infty$. *Then, as* $n \to \infty$

$$\sum_{i=1}^{n} a_{ni} X_i / n^{\frac{1}{t}} \to 0 \quad a.s.$$

Acknowledgements

The authors wish to thank the referees for a very thorough review of this paper. Part of this work appears in the Wonkwang University doctoral dissertation of the second author.

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