New Direct Kinematic Formulation of 6 D.O.F Stewart-Gough Platforms Using the Tetrahedron Approach

Se-Kyong Song and Dong-Soo Kwon

Abstract: The paper presents a single constraint equation of the direct kinematic solution of 6-dof (Stewart-Gough) platforms. Many research works have presented a single polynomial of the direct kinematics for several 6-dof platforms. However, the formulation of the polynomial has potential problems such as complicated formulation procedures and discrimination of the actual solution from all roots. This results in heavy computational burden and time-consuming task. Thus, to overcome these problems, we use a new formulation approach, called the *Tetrahedron Approach*, to easily derive a single constraint equation, not a polynomial one, of the direct kinematics and use two well-known numerical iterative methods to find the solution of the single constraint equation. Their performance and characteristics are investigated through a series of simulation.

Keywords: direct kinematics, tetrahedron, Stewart-Gough platform and parallel manipulator

I. Introduction

Many of robotic manipulators have used 6-dof (Stewart-Gough) platforms which generally possess low inertia effect, high rigidity, high local dexterity and compact size. The platform consists of two rigid plates (the moving and the base platforms) and six actuating links that provide up to 6-dof for the moving platform with respect to the base platform. There exist various types of the platform according to arrangements of the connecting joints on the moving and the base platforms [1]. Recently, in much of research, they have been widely called as 3-6, 4-5, 4-6 and 6-6 platforms.

The direct kinematics of the platform is to determine the posture of the moving platform relative to a reference frame fixed in the base platform when the lengths of the six actuating links are given. As far as the authors are aware, the direct kinematics solution (six generalized coordinates in Cartesian space) is not possible to be expressed in an unique closed form except for some highly specialized structure such as the 3-6 platform with a 3-2-1 type [2], because the direct kinematics is highly complicated due to the existence of multi-closed kinematic loops between the six links and the moving and the base platforms. In many previous works [3-8], the direct kinematics has been formulated through matrix manipulations of the multi-closed kinematic equations. This may increase the formulation complexity.

The direct kinematics still has challenging problems such as complicated formulation procedures and heavy computational burden; it is difficult to perform derivation of the direct kinematics in view of kinematic analysis and real-time computation in determining the direct kinematic solution in view of practical implementation.

In spite of these problems, the direct kinematics is inevitable in the following application fields: tracking control [9], calibration/accuracy analysis [10,11] and model-based control [12]. Especially, in the teleoperation and haptic applications, the computational burden of solving the direct kinematics degrades the control bandwidth and results in deteriorating the transparency due to the delay of the feedback transmission [13,14].

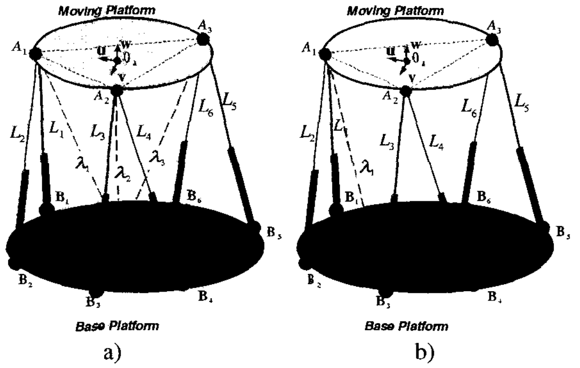

Fig. 1. Two tetrahedron configurations of the 3-6 Stewart-Gough platform composed of three tetrahedrons with a) three unknown variables and b) one unknown variable.

A number of formulation methods have been proposed for the direct kinematics in the literature. Generally, the conventional direct kinematics can be categorized in the following two approaches for different purposes: the *polynomial-based* and the *numerical-iterative* approaches [2]. The *polynomial-based* approach is a method for reduction of the resulting constraint equations into a univariate high-order polynomial by the elimination method [15]. All roots of the polynomial can be found by using a root-solver such as *NSolve* in Mathematica™ [16]. Since each real root corresponds to an assembly configuration of the mechanism, one can get the physical insight on all possible kinematic configurations. Many research works have been presented for the *polynomial-based* approach; for the 3-6 platform, Griffis *et. al* [5], Innocenti *et. al* [6] and Nanua *et. al* [17] presented a 16th-order polynomial in the result of the direct kinematics for the 3-6 platform. For the 4-5 platform, Lin *et. al* obtained a 16th-order polynomial [18]. For the 4-6 platform, Chen and Song obtained a 16th-order polynomial [4]. Innocenti presented a 32th-order polynomial

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by using five closed-loop equations [7]. For the 6-6 platform, Sreenviasan *et. al* [3] has shown that the direct kinematics presented a 16th-order closed-form polynomial under the geometric condition that the moving and the base platforms are a similar plane. However, the *polynomial-based* approach requires extremely complicated formulation procedures. Furthermore, since several numerical algorithms are available to determine all roots of a polynomial [19,20], the determination of the actual solution among all roots is still a challenging task. Root finding of a high-order polynomial is very sensitive to the accuracy of the coefficients of polynomial [21]. Thus, this approach is much slower than numerical methods based on numerical iteration such as the Newton-Raphson (*NR*) method [22]. Thus the *polynomial-based* approach may be appropriate for the design problem determining all the roots of the polynomial rather than the actual solution.

Unlike the *polynomial-based* approach, the *numerical-iterative* approach for numerical iteration has potentially been known to be a method better suitable for real-time computation of the direct kinematics solution [22]. For the 3-6 platform, the three-dimensional *NR* method has been widely employed for the practical application of the 3-6 platform. Liu *et. al* [8] and Ku [23] proposed numerical procedures for the three-dimensional *NR* method using three unknown angle variables instead of a 16th-order closed-form polynomial. However, the *NR* method has several potential problems such as evaluation of the partial derivative matrix and calculation of its inverse matrix.

Consequently, the *numerical-iterative* approach may be more practical than the *polynomial-based* approach. The main objectives of this research are to propose an efficient formulation approach and a numerical scheme for robustness and real-time computation of the direct kinematics. This research will focus on the 3-6 platform in whose structure the advantage of the proposed formulation approach is well exhibited.

This paper is organized as follows. In Section 2, we briefly review the *Tetrahedron Approach* that we proposed. Section 3 presents the direct kinematic analysis for deriving a single constraint equation using only one unknown length variable. In Section 4, the characteristics of the two numerical iterative methods are investigated for solving the single constraint equation. Section 5 describes the numerical performance of two numerical methods concerned with this paper.

II. Tetrahedron approach for the direct kinematics

We proposed the *Tetrahedron Approach* for easy derivation the direct kinematics of 6-dof parallel manipulators with a tetrahedron structure [2]. In the *Tetrahedron Approach*, the formulation procedure is simply reduced to a process of first identifying a tetrahedron based on the geometric structure of the linkages, and then using it as a basis for identifying and piling up next tetrahedrons. From the viewpoint of the procedure in solving the forward kinematics, the *Tetrahedron Approach* differs from the previous works that use tetrahedrons [24,25].

The concepts mentioned above were proposed as the *Tetrahedron Proposition* and the *Tetrahedron Theorem* [2]. The

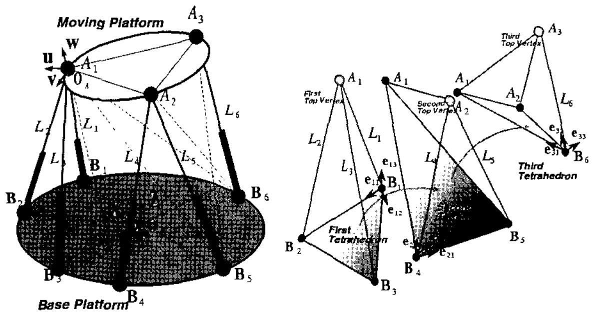

Fig. 2. The 3-6 platform with 3-2-1 type and its tetrahedron configurations.

Tetrahedron Proposition is defined to uniquely identify a tetrahedron based on the geometrical relationship between the moving and the base platforms. The *Tetrahedron Theorem* is defined to be a sufficient and necessary condition that there exists a unique closed-form solution of the direct kinematics and is also used as a formulation guideline to perform the *Tetrahedron Approach* for solving the direct kinematics by using the *Tetrahedron Proposition*.

Terminologies and notations used in this paper are defined as follows: firstly, assume that there exist two vectors among the six lines that compose a tetrahedron, as depicted in a tetrahedron of Fig. 1. A *base* is defined as the plane with two vectors that can be expressed with respect to a known reference coordinate. Three lines that lie on the base are called *base lines* and their vectors are called *base vectors*. The three lines that connect the base to a vertex are defined as *space lines* and their vectors are called *space vectors*. The vertex rising above the base from the three space lines is defined as a *top vertex*. A set of three mutual-orthogonal unitary vectors formed from two base vectors is called a *tetrahedron coordinate*. A tetrahedron that satisfies the *Tetrahedron Proposition* is called a *directional tetrahedron*.

[B]=**[X,Y,Z]** and **[M]**=**[u,v,w]** are defined as a base frame fixed on the base platform and a moving frame attached on the moving platform, respectively. O_A and O_B are the origins of the moving and the base platforms, respectively. **H** is the position vector of the moving platform with respect to the base frame. L_i is i-th link length. B_i and A_i indicate the i-th joints on the moving and the base platforms. r and R are the radius of the moving and the base platforms. The moving frame **[M]** can be expressed with the three column vectors of the orientation matrix with *roll*(α), *pitch*(β) and *yaw*(γ) angles with respect to the base frame. The three mutual-orthogonal column vectors are denoted by **u**, **v** and **w** in order: **[R]**=**[u,v,w]**. The orientation matrix **[R]** is identical to the moving frame **[M]**.

The *Tetrahedron Proposition* allows the successive selection of one of the binary choices of the tetrahedron coordinates that are yielded in the process of piling up directional tetrahedrons for solving the direct kinematics. Selecting one of the binary choices is possible under considering a configuration singularity and mechanical constraints such as linkage interference and spherical joint limitation. If additional space lines need to satisfy the *Tetrahedron Proposition*, this deficiency can be supplemented with unknown variables or extra sensors.

Once three top vertices are found through the *Tetrahedron Approach*, the direct kinematic solution is obtained from the geometric constraints of the moving platform. If the deficiency does not occur, the solution is determined in a unique closed form. Otherwise, the solution can be found by using either the *polynomial-based* or the *numerical-iterative* approach.

2.1. Unique closed-form solution

The great advantage of the *Tetrahedron Approach* is shown in some parallel manipulators that satisfy the *Tetrahedron Theorem*. In this case, we can obtain a unique closed-form solution of the direct kinematics [2].

Hunt and Primrose presented that the 3-6 platform with a 3-2-1 type shown in Fig. 2 has eight solutions by using the Bezout theorem [1]. Bruyninckx [24] and Ryu and Cho [26] obtained eight solutions through complicated formulation procedures. However, by using the *Tetrahedron Approach*, we can obtain a unique closed-form solution directly. As shown in Fig. 2, the geometric structure of the 3-6 platform can be broken into three directional tetrahedrons that all satisfy the *Tetrahedron Proposition*. Thus the three top vertices ($\mathbf{A}_1$,$\mathbf{A}_2$,$\mathbf{A}_3$) are be uniquely determined. This condition satisfies the *Tetrahedron Theorem*. Consequently, the position and orientation of the moving platform can be represented in the following unique closed form by geometry of the moving platform:

$$\textit{Position:}\ \mathbf{H}=[x,\,y,\,z]^T=\mathbf{A}_1, \tag{1a}$$

$$\textit{Orientation:}\ \mathbf{u}=\frac{(2\mathbf{A}_1-\mathbf{A}_2-\mathbf{A}_3)}{3r},\ \mathbf{v}=\frac{(\mathbf{A}_2-\mathbf{A}_3)}{\sqrt{3}r},\ \mathbf{w}=\mathbf{u}\times\mathbf{v}. \tag{1b}$$

III. Derivation of a single constraint equation with one unknown variable

This Section proposed a new formulation approach for deriving a single constraint equation of the direct kinematics of the 3-6 platform.

In view of tetrahedron configurations, it is possible to pile up several tetrahedron configurations of the 3-6 platform that are composed of three tetrahedrons, like two cases shown in Fig. 1. We dramatically reduced the formulation complexity of the direct kinematics by which the three position vectors ($\mathbf{A}_1$,$\mathbf{A}_2$,$\mathbf{A}_3$) are expressed in the base frame and their lengths are established as unknown variables, as shown in the left side of Fig. 1 [27]. However, using three unknown variables requires the three-dimensional Newton-Raphson (*NR*) method for solving three constraint equations. The three-dimensional *NR* method has some problems such as evaluation of the 3x3 partial derivative matrix and computation of its inverse matrix at each iterative step.

Therefore, in order to avoid using the three-dimensional *NR* method, we reconfigure the 3-6 platform into a different tetrahedron configuration with only one single variable, as shown in the right side of Fig. 1. As a result, three constraint equations are reduced into a single constraint equation, not a polynomial one. To the best of the author knowledge, the derivation of a single constraint equation of the direct kinematics has been not yet presented in the literature.

When one unknown length variable (λ_1) is added to identify the first tetrahedron that includes four vertices (A_1,$\mathbf{B}_1$, $\mathbf{B}_2$,$\mathbf{B}_3$),

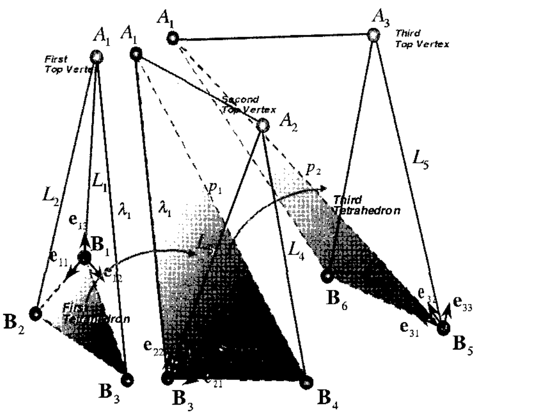

Fig. 3. Tetrahedron configurations with an single unknown variable.

the three top vertices ($\mathbf{A}_1$,$\mathbf{A}_2$,$\mathbf{A}_3$) can be readily obtained as functions of λ_1 by using the *Tetrahedron Approach*. The formulation procedure for deriving a single constraint equation is different to one of the *polynomial-based* approach for deriving a single polynomial.

As shown in Fig. 3, in the first tetrahedron, the space vector ($\mathbf{L}_1$) and the top vertex ($\mathbf{A}_1$) are uniquely obtained with respect to a tetrahedron coordinate [$\mathbf{e}_{11}$,$\mathbf{e}_{12}$,$\mathbf{e}_{13}$]:

$$\mathbf{L}_1=L_{11}\mathbf{e}_{11}+L_{12}\mathbf{e}_{12}+L_{13}\mathbf{e}_{13},\quad \mathbf{A}_1=\mathbf{L}_1+\mathbf{B}_1, \tag{2}$$

where

$$\mathbf{e}_{11}=\frac{\mathbf{B}_{21}}{\|\mathbf{B}_{21}\|},\quad \mathbf{e}_{12}=\frac{-(\mathbf{B}_{31}\cdot\mathbf{e}_{11})\mathbf{e}_{11}+\mathbf{B}_{31}}{\|-(\mathbf{B}_{31}\cdot\mathbf{e}_{11})\mathbf{e}_{11}+\mathbf{B}_{31}\|},$$
$$\mathbf{e}_{13}=\mathbf{e}_{11}\times\mathbf{e}_{12}=\mathbf{Z}.$$

The components of $\mathbf{L}_1$ are given by:

$$L_{11}=\frac{L_1^{\,2}+B_{21}^{\,2}-L_2^{\,2}}{2B_{21}},$$
$$L_{12}=\frac{L_1^{\,2}+B_{31}^{\,2}-\lambda_1^{\,2}-2L_{11}(\mathbf{B}_{31}\cdot\mathbf{e}_{11})}{2(\mathbf{B}_{31}\cdot\mathbf{e}_{12})}, \tag{3}$$
$$L_{13}^{\,2}=L_1^{\,2}-L_{11}^{\,2}-L_{12}^{\,2}.$$

In Eq. (3), L_{12} and L_{13} are functions of λ_1^2.

Next, the second tetrahedron coordinate [$\mathbf{e}_{21}$,$\mathbf{e}_{22}$,$\mathbf{e}_{23}$] is defined to identify the second tetrahedron that includes four vertices (A_1,A_2,$\mathbf{B}_3$,$\mathbf{B}_4$):

$$\mathbf{e}_{21}=\frac{\mathbf{B}_{43}}{\|\mathbf{B}_{43}\|},\quad \mathbf{e}_{22}=\frac{-(\boldsymbol{\lambda}_1\cdot\mathbf{e}_{21})\mathbf{e}_{21}+\boldsymbol{\lambda}_1}{\|-(\boldsymbol{\lambda}_1\cdot\mathbf{e}_{21})\mathbf{e}_{21}+\boldsymbol{\lambda}_1\|},$$
$$\mathbf{e}_{23}=\mathbf{e}_{21}\times\mathbf{e}_{22}, \tag{4}$$

where

$$\boldsymbol{\lambda}_1=\mathbf{L}_1-\mathbf{B}_{13}.$$

The third tetrahedron coordinate [$\mathbf{e}_{31}$,$\mathbf{e}_{32}$,$\mathbf{e}_{33}$] is defined to identify the third tetrahedron that includes four vertices (A_1,A_3,$\mathbf{B}_5$,$\mathbf{B}_6$) :

$$\mathbf{e}_{31}=\frac{\mathbf{B}_{65}}{\|\mathbf{B}_{65}\|},\quad \mathbf{e}_{32}=\frac{-(\mathbf{p}_2\cdot\mathbf{e}_{31})\mathbf{e}_{31}+\mathbf{p}_2}{\|-(\mathbf{p}_2\cdot\mathbf{e}_{31})\mathbf{e}_{31}+\mathbf{p}_2\|},$$

$$\mathbf{e}_{33} = \mathbf{e}_{31} \times \mathbf{e}_{32}, \tag{4b}$$

where

$$\mathbf{p}_2 = \mathbf{L}_1 - \mathbf{B}_{15}.$$

$\mathbf{p}_2$ is a base vector in the third tetrahedron. Then two space vectors ($\mathbf{L}_3$,$\mathbf{L}_5$) and two top vertices ($\mathbf{A}_2$,$\mathbf{A}_3$) are obtained with respect to two tetrahedron coordinates, respectively:

$$\mathbf{L}_3 = L_{31}\mathbf{e}_{21} + L_{32}\mathbf{e}_{22} + L_{33}\mathbf{e}_{23}, \quad \mathbf{A}_2 = \mathbf{L}_3 + \mathbf{B}_3, \tag{5a}$$

$$\mathbf{L}_5 = L_{51}\mathbf{e}_{31} + L_{52}\mathbf{e}_{32} + L_{53}\mathbf{e}_{33}, \quad \mathbf{A}_3 = \mathbf{L}_5 + \mathbf{B}_5, \tag{5b}$$

where

$$L_{31} = \frac{L_3^2 + B_{43}^2 - L_4^2}{2B_{43}},$$

$$L_{32} = \frac{L_3^2 + \lambda_1^2 - A_{21}^2 - 2L_{31}(\boldsymbol{\lambda}_1 \cdot \mathbf{e}_{21})}{2(\boldsymbol{\lambda}_1 \cdot \mathbf{e}_{22})},$$

$$L_{33}^2 = L_3^2 - L_{31}^2 - L_{32}^2,$$

$$L_{51} = \frac{L_5^2 + B_{65}^2 - L_6^2}{2B_{65}},$$

$$L_{52} = \frac{L_5^2 + P_2^2 - A_{31}^2 - 2L_{51}(\mathbf{p}_2 \cdot \mathbf{e}_{31})}{2(\mathbf{p}_2 \cdot \mathbf{e}_{32})},$$

$$L_{53}^2 = L_5^2 - L_{51}^2 - L_{52}^2.$$

Here, B_{ij} and A_{ij} are norms of the difference of two position vectors ($\mathbf{B}_i - \mathbf{B}_j$) and ($\mathbf{A}_i - \mathbf{A}_j$), respectively. L_{32}, L_{33}, L_{52} and L_{53} are functions of λ_1^2.

This approach needs to include if-else statements because the dot-product signs are determined to be either positive or negative according to the angles between two adjacent vectors. For the first tetrahedron, if-else statements are expressed as follows:

$$\mathbf{L}_1 \cdot \mathbf{B}_{21} = \begin{cases} +L_2 B_{21}, \textit{if} \ \ L_1^2 + B_{21}^2 \geq L_2^2, \\ -L_2 B_{21}, \textit{else} \ L_1^2 + B_{21}^2 < L_2^2, \end{cases} \tag{6a}$$

$$\mathbf{L}_1 \cdot \mathbf{B}_{31} = \begin{cases} +L_{11}\mathbf{B}_{31} \cdot \mathbf{e}_{11} + L_{12}\mathbf{B}_{31} \cdot \mathbf{e}_{12}, \textit{if} \ \ L_1^2 + B_{31}^2 \geq \lambda_1^2, \\ -L_{11}\mathbf{B}_{31} \cdot \mathbf{e}_{11} - L_{12}\mathbf{B}_{31} \cdot \mathbf{e}_{12}, \textit{else} \ L_1^2 + B_{31}^2 < \lambda_1^2. \end{cases} \tag{6b}$$

A single constraint equation can be expressed as follows:

$$G(\lambda_1) = C^2 - (\mathbf{A}_3 - \mathbf{A}_2) \cdot (\mathbf{A}_3 - \mathbf{A}_2) = 0. \tag{7}$$

C is a constraint that is a distance between $\mathbf{A}_3$ and $\mathbf{A}_2$ fixed on the moving platform.

We will introduce two well-known numerical iterative methods for finding the solution of the resulting single constraint equation. They are the one-dimensional *NR* and the Secant (*S*) method. When an error value exist within the prescribed convergence tolerance, the three position vectors ($\mathbf{A}_1$,$\mathbf{A}_2$,$\mathbf{A}_3$) on the moving platform are obtained by which the final calculated value is sequentially substituted into Eqs. (2), (4a) and (4b). Accordingly, the position (**H**) and the orientation [**u**,**v**,**w**] of the moving platform is determined from:

$$\textit{Position: } \mathbf{H} = (\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3)/3, \tag{8a}$$

$$\textit{Orientation: } \mathbf{u} = \frac{\mathbf{A}_1 - \mathbf{H}}{r}, \ \mathbf{v} = \frac{(\mathbf{A}_2 - \mathbf{A}_3)}{\sqrt{3}r}, \ \mathbf{w} = \mathbf{u} \times \mathbf{v}. \tag{8b}$$

IV. Numerical iterative methods for solving the direct kinematics

In this Section, we investigate two numerical iterative methods to find the actual solution of the resulting single constraint equation derived in Section 3. The one-dimensional *NR* method is expressed in a simple form:

$$\lambda_1[n+1] = \lambda_1[n] - \frac{G(\lambda_1[n])}{dG/d\lambda_1}, \tag{9a}$$

$$\| \lambda_1[n+1] - \lambda_1[n] \| < \text{Convergence tolerance}. \tag{9b}$$

$dG/d\lambda_1$ is a derivative in term of λ_1.

Generally, the *NR* method exhibits good convergence under circumstances that initial estimate values lie inside the vicinity of the actual solution and the gradients of constraint equations are moderate near the region. Otherwise, the *NR* method involves potential problems such as oscillating, diverging [19]. Since the single constraint equation (*G*) is highly nonlinear, thus it includes more local minima, compared to the three constraint equations [27]. At such local minima, the inverse of the derivative is diverged. This implies that the numerical performance of the *NR* method is strongly sensitive to initial estimate value and nature of the resulting constraint equation. Additional potential problem in implementing is evaluation of the partial derivative matrix. The derivative of the resulting single constraint equation is extremely difficult to evaluate due to including many nonlinear terms.

In this case, we see that it is appropriate to introduce the *S* method because they need not the derivative in the process of finding the root [19,20,28]. In a finite region where the gradient of the resulting equation is moderate, the derivative can be approximated by the following finite divided difference, that is a linear interpolation:

$$\frac{G(\lambda_1[n]) - G(\lambda_1[n-1])}{\lambda_1[n] - \lambda_1[n-1]} \approx \frac{dG}{d\lambda_1}. \tag{10}$$

Thus this approximation can be substituted into the formula (9) to yield the following one:

$$\lambda_1[n+1] = \lambda_1[n] - G(\lambda_1[n]) \frac{\lambda_1[n] - \lambda_1[n-1]}{G(\lambda_1[n]) - G(\lambda_1[n-1])}. \tag{11}$$

This equation is called the *S* method that requires two initial estimates of $\lambda_1[1]$ and $\lambda_1[0]$. Though the *S* method is also potential problems such as diverging, it is less sensitive than the *NR* method.

In view of the convergence speed, it has been known that the *NR* method is quadratic and the *S* method is approximately 1.62 [28]. The *NR* method is most wisely used due to its property of quadratic convergence. However, it potentially has several shortcomings as mentioned above. In this case, the *S* method is preferable. Generally, the convergence speed of the *S* method is slower than one of the *NR* method in initial iteration steps due to the property of the finite difference approximation. However, as an updated value approaches the actual solution, the difference approximate of the *S* method may exhibit better performance of convergence speed than the derivative of the *NR* method [19,20,28].

Practically, the total computation time of the direct kinematics is determined by the multiplication of the number of iterations for finding the actual solution by the computational burden caused by the number of the calculating terms at each iteration. Thus the *S* method has great advantage of easy implementation and reduction of the number of the calculating terms over the three-dimensional *NR* method that has been used in [27]. However, the convergence speed of the *S* method depends on the location of two initial estimate values that thus needs to be carefully selected in the vicinity of the actual solution.

V. Numerical examples

This section presents comparison between the convergence and performance of the three dimensional *NR* in [27], the one-dimensional *NR* and the *S* method in Section 4.

As addressed in Section 4, the three numerical methods make different numerical performance according to the nature of the resulting equation. At first, we investigate the contour shape of the single constraint equation (*G*) in Eq. (7). Afterward, their convergences are investigated through a series of simulation results under several initial estimate values.

Without loss of generality, the joint positions are given under the assumption that the 3-6 platform is symmetrically arranged in each circular plane of the moving and the base platforms:

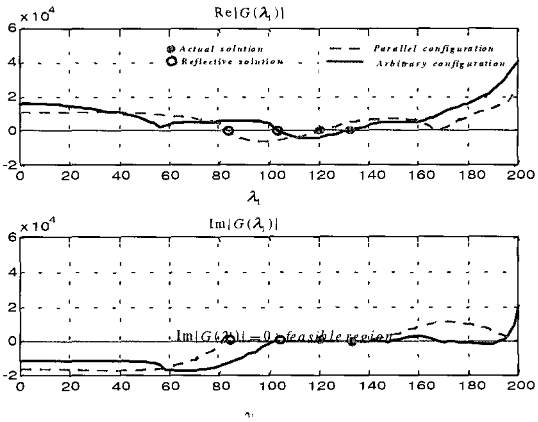

Fig. 4. Real and imaginary contours of the single constraint equation at the *parallel* and *arbitrary* configurations.

$$\mathbf{B}_1 = [R,0,0]^T,\ \mathbf{B}_i = \mathbf{R}(60^\circ i)\mathbf{B}_1, \qquad for\ i = 1,2,3,4,5 \quad (12a)$$

$$\mathbf{A}_1 = [r,0,0]^T,\ \mathbf{A}_j = \mathbf{R}(120^\circ j)\mathbf{A}_1, \qquad for\ j = 1,2,3 \quad (12b)$$

$$R = 80mm,\ r = 40mm.$$

When the moving and the base platforms are *parallel*, the six Cartesian coordinates are given by:

$$\alpha = 0^\circ, \beta = 0^\circ, \gamma = 0^\circ, \mathbf{H} = [0,0,80]^T. \qquad (13)$$

In this condition, lengths of the six links are determined as L_o = 94.114*mm* through the inverse kinematics.

Consider the following *arbitrary* configuration of the platform:

$$\alpha = 10^\circ, \beta = 20^\circ, \gamma = 10^\circ, \mathbf{H} = [20,-20,80]^T. \qquad (14)$$

Assume that the three unknown variables (λ_i) are arbitrary varying in the following wide ranges under considering the mechanical constraints of the 3-6 platform such as spherical joint limitation:

$$\lambda_{min} = L_o \times\ 0.1 < \lambda_1, \lambda_2, \lambda_3 < L_o \times\ 1.9 = \lambda_{max}. \qquad (15)$$

In the above *parallel* and *arbitrary* configurations, the contour of the single constraint equation is displayed as shown in Fig 4. The real and the imaginary plots of the single constraint equation (*G*) are obtained when the unknown variable is varying ($0<\lambda_1<200$). Fig. 4 shows that the single constraint equation is highly nonlinear and includes imaginary values in the entire range. Imaginary contours (Im[$G(\lambda_1)$]) result from the imaginary values of L_{13}, L_{33} and L_{53}. Fig. 4 also shows the existence of two feasible solutions that are the actual and the reflective solutions of the 3-6 platform. One of them may be converged to the actual solution by carefully choosing the initial estimate values.

When initial estimate values are λ_1=120 or 140, the one-dimensional *NR* method doesn't converge after 200 steps, as shown in the above side of Fig. 5. In cases of λ_1= 80, 100 and 160, it diverges. Therefore, it is not practical to employ the one-dimensional *NR* method for the single constraint equation.

In order to overcome this problem, we introduce the *S* method because they are less sensitive to the nature of the single constraint equation. Assume that two initial estimate values can be guessed from the contours of the resulting single constraint equation at first stage. For the *S* method, the several initial estimate values are chosen as $\lambda_1[0]$ = 105, 100, 110, 120, 130 and $\lambda_1[1]$ = 150 over a finite range of λ_1. It is surprising that the *S* method displays no less good performance than the three-dimensional *NR* method for the three constraint equations [26] irrespective of initial estimate values.

Actually, there exist the trade-offs between robustness and convergence speed of several numerical iterative methods such as the *NR*, *S*, False-position methods and *et. al*. Thus, the *S* method needs to be combined with strong points of other numerical schemes, for example the False-position method [19]. It is noted that the proposed formulation approach can be also directly applicable to 6-dof platforms which require one unknown variable for satisfying the *Tetrahedron Theorem*.

An advanced numerical scheme for solving the single constraint equation and applications of the proposed formulation approach are currently underway by the authors and will be reported in an upcoming paper.

VI. Conclusions

We have presented a new efficient formulation approach, based on the *Tetrahedron Approach*, for fast computation of the direct kinematics of the 3-6 platform. We have investigated the characteristics of the well-known two numerical iterative methods to find a numerical scheme that is capable of quickly finding the actual solution of the single constraint equation.

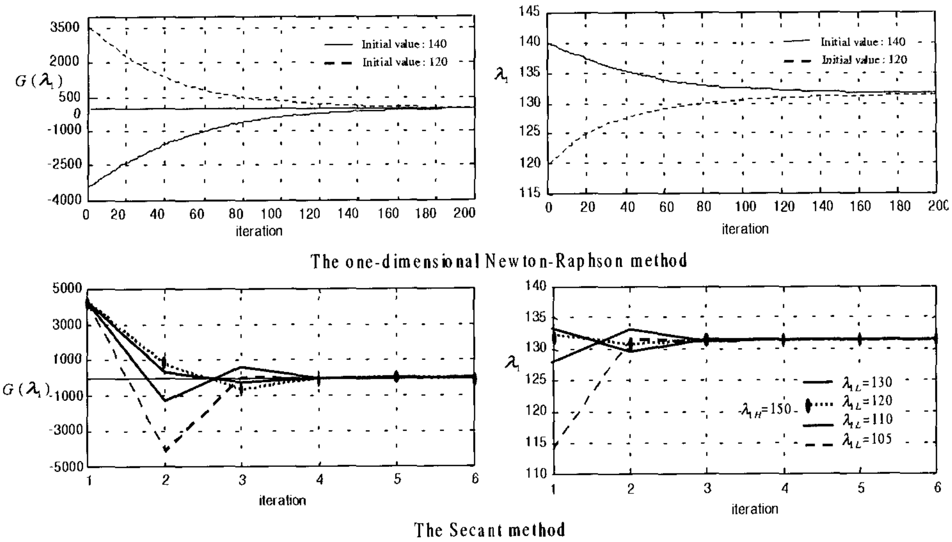

Fig. 5. The iteration convergence of the direct kinematics according to different initial estimate values at the arbitrary configuration.

Even though the Newton-Raphson method has been known to have quadratic convergence speed, it has potential problems such as evaluation of the partial derivative matrix and computation of its inverse matrix at each iterative step. We have shown that the Newton-Raphson method is very sensitive to the nature of the constraint equation and the Secant method has the advantage of good convergence and easy implementation.

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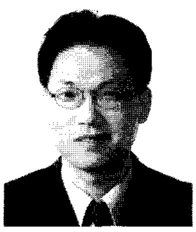

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