Unknown Inputs Observer Design Via Block Pulse Functions

Pius Ahn

Abstract: Unknown inputs observer(UIO) which is achieved by the coordinate transformation method has the differential of system outputs in the observer and the equation for unknown inputs estimation. Generally, the differential of system outputs in the observer can be eliminated by defining a new variable. But it brings about the partition of the observer into two subsystems and need of an additional differential of system outputs still remained to estimate the unknown inputs. Therefore, the block pulse function expansions and its differential operation which is a newly derived in this paper are presented to alleviate such problems from an algebraic form.

Keywords: block pulse functions, UIO, BPF differential operation, algebraic approach

I. Introduction

Over the last decade there have been much interests in unknown inputs observer(UIO) design since it is directly applicable to decentralized observer, process supervision, and instrument failure detection. Many researchers who have interests in UIO design procedure proposed various approaches. They are geometric approach[1], singular value decomposition method[2], and direct design method[3]. Recently, the coordinate transformation method[4], [5], [6] is mostly used in UIO design procedure by reason that it gives systematic guideline than the other methods. But, in this method, there are some problems that the derived observer and the equation for unknown input estimation have the differential of the system outputs. In this case, an additional differentiator is needed to have an information of differentiated system outputs and it would be disadvantageous in the economic and practical design viewpoints. To alleviate these problems, this paper proposes an algebraic method which does not need any differential of the system outputs to design an observer and estimate the unknown inputs. It is achieved by using block pulse function(BPF)[8] expansions and its differential operation which are newly derived in this paper on the basis of BPF's orthogonal and disjoint properties. Kraus and Schaufelberger[7] proposed the differential operation of block pulse function, but it is achieved by approximating the calculus of variations without any interest of BPF's orthogonal property. This paper is constituted as follows. In Section II, BPF is explained and its differential operation is derived. The coordinate transformation method for UIO design is introduced in section III-1 and the application of BPF expansions are presented in section III-2. Lastly, numerical examples are presented to check the experimental results for the proposed method in section IV.

II. Derivation of BPF's differential operation

1. Brief review of BPF

Block pulse functions set $\mathbf{\Phi(t)} = [\phi_1(\mathbf{t})\phi_2(\mathbf{t})\dots\phi_m(\mathbf{t})]^T$ is a set of piecewise constant function and defined in the time interval $[0, t_f)$ as follows.

$$\phi_i(t) = \begin{cases} 1, & \text{for } (i-1)\frac{t_f}{m} \le t < i\frac{t_f}{m} \\ 0, & \text{otherwise} \end{cases} \qquad for,\ i = 1, 2, \cdots, m \tag{1}$$

In the equation (1), m is the BPF's expansion number. BPF has the following orthogonal and disjoint properties.

$$Orthogonal\ property: \int_0^{t_f} \phi_i(t)\phi_j(t)dt = \begin{cases} \frac{t_f}{m},\ i = j \\ 0,\ i \ne j \end{cases} \tag{2}$$

$$Disjoint\ property: \phi_i(t)\phi_j(t) = \begin{cases} \phi_i(t), i = j \\ 0, i \ne j \end{cases} \tag{3}$$

If an arbitrary function $f(t)$ is absolutely integrable in the interval $[0, t_f)$, it can be approximated by using BPF expansions as follows.

$$f(t) \simeq \sum_{i=1}^{m} \mathbf{F}_i\phi_i(t) \tag{4}$$

$\mathbf{F}_i$ is a coefficient of the i-th block pulse function. In the equation (4), the i-th BPF's coefficient $\mathbf{F}_i$ of an arbitrary function $f(t)$ is determined as follows.

$$\begin{aligned} \mathbf{F}_i = \tfrac{m}{t_f}\textstyle\int_0^{t_f} f(t)\phi_i(t)dt &= \tfrac{m}{t_f}\textstyle\int_{(i-1)\frac{t_f}{m}}^{i\frac{t_f}{m}} f(t)dt \\ &\simeq \tfrac{1}{2}\left[f(i\tfrac{t_f}{m}) + f((i-1)\tfrac{t_f}{m})\right] \\ & for, i = 1, 2, \cdots, m \end{aligned} \tag{5}$$

The approximation of the forward integral of BPF is

$$\int_0^t \phi_i(\tau)d\tau \simeq \frac{t_f}{2m}\phi_i(t) + \frac{t_f}{m}\sum_{j=i+1}^{m}\phi_j(t) \tag{6}$$

2. BPF's differential operation

Similar to (4), if a differential of $f(t)$ is absolutely integrable in the interval $[0, t_f]$, it can be also approximated as (7) by using a BPF expansions [9] as follows.

$$\dot{f}(t) \simeq \sum_{i=1}^{m} \bar{\mathbf{F}}_i\phi_i(t) \tag{7}$$

Manuscript received: Jun. 20, 2001., Accepted: Mar. 30, 2002.
Pius Ahn: Office of Electric-Electro Eng., Small Business Corporation, 24-3, Yeouido-dong, Yeongdeunpo-gi, Seoul, 150-718, Korea. (piusahn@sbc.or.kr)

The interest of (7) is how to obtain a coefficient $\bar{\mathbf{F}}_i$. Many researchers resolved the problem by approximating a calculus of variation viewpoints such as (8).

$$\begin{aligned} \dot{f}(t) &= \lim_{\Delta t \to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t} \\ &= \frac{m}{t_f} \sum_{i=1}^{m} [f(i+1)\frac{t_f}{m} - f(i\frac{t_f}{m})] \end{aligned} \quad (8)$$

where, $\Delta t = \frac{t_f}{m}$ and $i = 1, 2, \cdots, m-1$

But, in this paper, a coefficient $\bar{\mathbf{F}}_i$ is obtained on the basis of BPF's orthogonal and disjoint properties. To obtain a recursive algorithm for the BPF's coefficients of $\dot{f}(t)$ described by $\mathbf{F}$ and $f(0)$ which are $f(t)'s$ BPF coefficients and their initial value respectively, let us integrate $f(t)$ in the interval $[0, t)$.

$$\int_0^t \dot{f}(\tau)d\tau = f(t) - f(0) \quad (9)$$

Equation (9) can be represented as (10) by using BPF expansions

$$\begin{aligned} &\sum_{i=1}^{m} \bar{\mathbf{F}}_i \int_o^t \phi_i(\tau)d\tau \\ &= \sum_{i=1}^{m} \mathbf{F}_i\phi_i(t) - \sum_{i=1}^{m} f(0)\phi_i(t) \\ &where, \dot{f}(t) \simeq \sum_{i=1}^{m} \bar{\mathbf{F}}_i\phi_i(t) \end{aligned} \quad (10)$$

Substituting (6) in (10), we can obtain a equation as following,

$$\begin{aligned} &\sum_{i=1}^{m} \bar{\mathbf{F}}_i[\frac{t_f}{2m}\phi_i(t) + \frac{t_f}{m}\sum_{j=i+1}^{m} \phi_i(t)] \\ &= \sum_{i=1}^{m} \mathbf{F}_i\phi_i(t) - \sum_{i=1}^{m} f(0)\phi_i(t) \end{aligned} \quad (11)$$

Expanding (11) such as (12)

$$\begin{aligned} &\frac{t_f}{m} \times \{\bar{\mathbf{F}}_1 \left(\tfrac{1}{2}\phi_1(t) + \phi_2(t) + \cdots + \phi_m(t)\right) \\ &+\bar{\mathbf{F}}_2 \left(\tfrac{1}{2}\phi_2(t) + \phi_3(t) + \cdots + \phi_m(t)\right) \\ &\cdots + \bar{\mathbf{F}}_i \left(\tfrac{1}{2}\phi_i(t) + \phi_{i+1}(t) + \cdots + \phi_m(t)\right) \\ &+\bar{\mathbf{F}}_{i_1} \left(\tfrac{1}{2}\phi_{i+1}(t) + \phi_{i+2}(t) + \cdots + \phi_m(t)\right) \\ &\cdots + \bar{\mathbf{F}}_m \left(\tfrac{1}{2}\phi_m(t)\right)\} \\ &= \mathbf{F}_1\phi_1(t) + \mathbf{F}_2\phi_2 + \cdots + \mathbf{F}_m\phi_m(t) \\ &-f(0)[\phi_1(t) + \phi_2(t) + \cdots + \phi_m(t)] \end{aligned} \quad (12)$$

By multiplying $\phi_1(t), \phi_2(t), \cdots, \phi_m(t)$ to (12) sequentially and applying BPF's disjoint property, (12) can be deduced as (13)

$$\begin{aligned} &\frac{t_f}{2m}\bar{\mathbf{F}}_1 = \mathbf{F}_1 - f(0) \\ &\frac{t_f}{m}(\bar{\mathbf{F}}_1 + \tfrac{1}{2}\bar{\mathbf{F}}_2) = \mathbf{F}_2 - f(0) \\ &\vdots \\ &\frac{t_f}{m}(\bar{\mathbf{F}}_1 + \bar{\mathbf{F}}_2 + \cdots + \bar{\mathbf{F}}_{i-1} + \tfrac{1}{2}\bar{\mathbf{F}}_i) = \mathbf{F}_i - f(0) \\ &\frac{t_f}{m}(\bar{\mathbf{F}}_1 + \bar{\mathbf{F}}_2 + \cdots + \bar{\mathbf{F}}_i + \tfrac{1}{2}\bar{\mathbf{F}}_{i+1}) = \mathbf{F}_{i+1} - f(0) \\ &\vdots \\ &\frac{t_f}{m}(\bar{\mathbf{F}}_1 + \bar{\mathbf{F}}_2 + \cdots + \bar{\mathbf{F}}_{m-1} + \tfrac{1}{2}\bar{\mathbf{F}}_m) = \mathbf{F}_m - f(0) \end{aligned} \quad (13)$$

Then arranging (13) for $\bar{\mathbf{F}}_i$, we can obtain (14).

$$\begin{aligned} &\bar{\mathbf{F}}_1 = \frac{2m}{t_f}[\mathbf{F}_1 - f(0)] \\ &\bar{\mathbf{F}}_{i+1} = \frac{2m}{t_f}[\mathbf{F}_{i+1} - F_i] - \bar{\mathbf{F}}_i \\ &for,\ i = 1, 2, \cdots, m-1 \end{aligned} \quad (14)$$

In (14), the $\dot{f}(t)'s$ i-th BPF coefficient $\bar{\mathbf{F}}_i$ can be obtained by using $\mathbf{F}_i$ and $f(0)$, recursively. And (15) is the generalized form of (14).

$$\begin{aligned} &\bar{\mathbf{F}}_i = \frac{2m}{t_f}[\mathbf{F}_i + (2 \times (-1)^i \sum_{j=1}^{i-1}(-1)^j\mathbf{F}_j) + (-1)^i f(0)] \\ &where, \dot{f}(t) \simeq \sum_{i=1}^{m} \bar{\mathbf{F}}_i\phi_i(t) \\ &for,\ i = 1, 2, \cdots, m \end{aligned} \quad (15)$$

III. UIO design procedure

1. Coordinate transformation method

Let us consider a linear time-invariant dynamic system represented as follows.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Dd(t) \\ y(t) &= Cx(t) \end{aligned} \quad (16)$$

where, $x \in R^n, u \in R^p, y \in R^m$ and $d \in R^q$ are the state, input, outputs and the unknown input vectors of the system, respectively. In (16), the ranks of the matrix D and C are $\rho(D) = q$ and $\rho(C) = m$, respectively. It is well known that if $m \geq q$ is satisfied, then there exists a similar transformation matrix T_1 such as $T_1^{-1}D = \begin{bmatrix} 0 \\ I_q \end{bmatrix}$ and (16) can be represented as follows.

$$\begin{aligned} \dot{x}_1^*(t) &= A_{11}^*x_1^*(t) + A_{12}^*x_2^*(t) + B_1^*u(t) \\ \dot{x}_2^*(t) &= A_{21}^*x_1^*(t) + A_{22}^*x_2^*(t) + B_2^*u(t) + d(t) \\ y(t) &= C_1^*x_1^*(t) + C_2^*x_2^*(t) \end{aligned} \quad (17)$$

where, $x = T_1x^* = T_1 \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$, $x_1^* \in R^{(n-q)}$, $x_2^* \in R^q$, $T_1^{-1}AT_1 = \begin{bmatrix} A_{11}^* & A_{12}^* \\ A_{21}^* & A_{22}^* \end{bmatrix}$, $T_1^{-1}B = \begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix}$, and $CT_1 = \begin{bmatrix} C_1^* & C_2^* \end{bmatrix}$

By defining a variable such as $z(t) = \dot{x}_2^*(t) - B_2^*u(t) - d(t) - A_{22}^*x_2^*(t)$ in (17), (18) can be obtained.

$$\begin{aligned} &\begin{bmatrix} C_1^* & C_2^* & \vdots & -I_m & 0 \\ A_{21}^* & 0 & \vdots & 0 & -I_q \end{bmatrix} \cdot \begin{bmatrix} x^*(t) \\ y^*(t) \end{bmatrix} = 0 \\ &where, y^*(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} and\ z(t) = A_{21}^*x_1^*(t) \end{aligned} \quad (18)$$

It is obvious that the matrix pencil $\begin{bmatrix} C_1^* & C_2^* & \vdots & -I_m & 0 \\ A_{21}^* & 0 & \vdots & 0 & -I_q \end{bmatrix}$ in (18) always guarantees the full row rank $m + q$. Thus, the nonsingular transformation matrix $T_2 \in R^{(m+q)\times(m+q)}$ which satisfies (19) exists at any time.

$$T_2 \cdot \left[\begin{array}{cc:cc} C_1^* & C_2^* & -I_m & 0 \\ A_{21}^* & 0 & 0 & -I_q \end{array}\right] = \left[\begin{array}{cc:cc} M_1 & 0 & N_1 & N_2 \\ M_2 & I_q & N_3 & N_4 \end{array}\right] \quad (19)$$

Using a transformation matrix T_2, (18) can be represented as (20) and (21).

$$(M_1 + N_2 A_{21}^*)x_1^*(t) + N_1 y(t) = 0 \quad (20)$$

$$x_2^*(t) = -(M_2 + N_4 A_{21}^*)x_1^*(t) - N_3 y(t) \quad (21)$$

Let us substitute $x_2^*(t)$ in (17) by (21) and define a variable such as $\tilde{y}(t) = -N_1 y(t)$ in (20). As the result of the previous, we can deduce the following $(n-q)$th order dynamic system (22) which consists of $x_1^*(t)$ only.

$$\begin{aligned} \dot{x}_1^*(t) &= A^o x_1^*(t) + B_1^* u(t) - A_{12}^* N_3 y(t) \\ \tilde{y}(t) &= (M_1 + N_2 A_{21}^*)x_1^*(t) \quad (22) \\ where,\ A^o &= A_{11}^* - A_{12}^* M_2 - A_{12}^* N_4 A_{21}^* \end{aligned}$$

If the dynamic system (16) is observable, then (22) also guarantees the observability. It means that if $\rho \begin{pmatrix} -sI_n + A \\ C \end{pmatrix} = n$, then $\begin{pmatrix} -sI_{n-q} + A^o \\ M_1 + N_2 A_{21}^* \end{pmatrix} = n - q$ for $\forall s \in \mathcal{C}$, $Re(s) \geq 0$.

Proof: $\rho \begin{pmatrix} -sI_n + A \\ C \end{pmatrix} = \rho \begin{pmatrix} -sI_{n-q} + A_{11}^* & A_{12}^* \\ C_1^* & C_2^* \\ A_{21}^* & 0 \end{pmatrix}$

by pre-multiplying $-A_{12}^* N_{22}$ to the 3-rd row and adding to the 1-st row

$$= \rho \begin{pmatrix} -sI_{n-q} + A_{11}^* - A_{12}^* N_{22} A_{21}^* & A_{12}^* \\ C_1^* & C_2^* \\ A_{21}^* & 0 \end{pmatrix}$$

$$= \rho \begin{pmatrix} -sI_{n-q} + A_{11}^* - A_{12}^* N_{22} A_{21}^* & A_{12}^* \\ M_1 & 0 \\ M_2 & I_q \end{pmatrix}$$

by pre-multiplying $-A_{12}^*$ to the 3-rd row and adding to the 1-st row

$$= \rho \begin{pmatrix} -sI_{n-q} + A_{11}^* - A_{12}^* N_4 A_{21}^* - A_{12}^* M_2 & 0 \\ M_1 & 0 \\ M_2 & I_q \end{pmatrix}$$

$$= \rho \begin{pmatrix} -sI_{n-q} + A_{11}^* - A_{12}^* N_4 A_{21}^* - A_{12}^* M_2 & 0 \\ M_1 + N_2 A_{21}^* & I_q \\ M_2 A_{21}^* & I_q \end{pmatrix}$$

$$= q + \rho \begin{pmatrix} -sI_{n-q} + A^o \\ M_1 + N_2 A_{21}^* \end{pmatrix} = n$$

Therefore, $\begin{pmatrix} -sI_{n-q} + A^o \\ M_1 + N_2 A_{21}^* \end{pmatrix} = n - q$ for $\forall s \in \mathcal{C}$, $Re(s) \geq 0$. ∎

2. Algebraic UIO design procedure

It is obvious that (22) is a convenient form to design a Luenberger-type observer. To design an algebraic observer for (22), let us represent (22) by using a BPF expansions.

$$\begin{aligned} \textstyle\sum_{i=1}^m \bar{\mathbf{X}}_{1_i}^* \phi_i(t) &= A^o \textstyle\sum_{i=1}^m \mathbf{X}_{1_i}^* \phi_i(t) \\ +B_1^* \textstyle\sum_{i=1}^m \mathbf{U}_i \phi_i(t) &- A_{12}^* N_3 \textstyle\sum_{i=1}^m \mathbf{Y}_i \phi_i(t) \quad (23) \\ \textstyle\sum_{i=1}^m \tilde{\mathbf{Y}}_i^* \phi_i(t) &= (M_1 + N_2 A_{21}^*) \textstyle\sum_{i=1}^m \mathbf{X}_{1_i}^* \phi_i(t) \end{aligned}$$

$$\begin{aligned} \dot{x}_1^*(t) &\simeq \textstyle\sum_{i=1}^m \bar{\mathbf{X}}_{1_i}^* \phi_i(t) \\ x_1^*(t) &\simeq \textstyle\sum_{i=1}^m \mathbf{X}_{1_i}^* \phi_i(t) \\ u(t) &\simeq \textstyle\sum_{i=1}^m \mathbf{U}_i \phi_i(t) \quad (24) \\ y(t) &\simeq \textstyle\sum_{i=1}^m \mathbf{Y}_i \phi_i(t) \\ \tilde{y}(t) &\simeq \textstyle\sum_{i=1}^m \tilde{\mathbf{Y}}_i \phi_i(t) \end{aligned}$$

Algebraic Luenberger-type observer for (23) can be designed as following

$$\begin{aligned} \textstyle\sum_{i=1}^m \bar{\mathbf{W}}_i \phi_i(t) &= A^o \textstyle\sum_{i=1}^m \mathbf{W}_i \phi_i(t) + B_1^* \textstyle\sum_{i=1}^m \mathbf{U}_i \phi_i(t) \\ &-A_{12}^* N_3 \textstyle\sum_{i=1}^m \mathbf{Y}_i \phi_i(t) + L\left(\textstyle\sum_{i=1}^m \tilde{\mathbf{Y}}_i \phi_i(t)\right. \\ &\left.-(M_1 + N_2 A_{21}^*) \textstyle\sum_{i=1}^m \mathbf{W}_i \phi_i(t)\right) \quad (25) \end{aligned}$$

Determine an error function as following (26).

$$e(t) = \sum_{i=1}^m \mathbf{W}_i(t)\phi_i(t) - \sum_{i=1}^m \mathbf{X}_{1_i}^* \phi_i(t) \quad (26)$$

From (23) and (25), (27) is derived directly.

$$\begin{aligned} \dot{e}(t) &= \textstyle\sum_{i=1}^m \bar{\mathbf{W}}_i(t)\phi_i(t) - \textstyle\sum_{i=1}^m \bar{\mathbf{X}}_{1_i}^* \phi_i(t) \\ &= [A^o - L(M_1 + N_2 A_{21}^*)]e_i(t) \end{aligned} \quad (27)$$

It is easy to know that if the observer gain matrix L in (27) is chosen to have the eigen values of the matrix $[A^o - L(M_1 + N_2 A_{21}^*)]$ is negative assignment, proposed algebraic observer can be converged to the actual state of (23) at $t \to \infty$. By using a relation $\sum_{i=1}^m \tilde{\mathbf{Y}}_i \phi_i(t) = -N_1 \sum_{i=1}^m \mathbf{Y}_i \phi_i(t)$, (25) can be represented as (28).

$$\begin{aligned} \textstyle\sum_{i=1}^m \bar{\mathbf{W}}_i(t)\phi_i(t) &= F \textstyle\sum_{i=1}^m \mathbf{W}_i(t)\phi_i(t) \\ +G \textstyle\sum_{i=1}^m \mathbf{U}_i(t)\phi_i(t) &+ H \textstyle\sum_{i=1}^m \mathbf{Y}_i(t)\phi_i(t) \\ where,\ F = [A^o - L(M_1 &+ N_2 A_{21}^*)],\ G = B_1^* \\ ,and\ H &= -(A_{12}^* N_3 + L N_1) \end{aligned} \quad (28)$$

Now, the main problem of the algebraic observer equation (28) is how to obtain the BPF coefficients, $\mathbf{W} = [\mathbf{W}_1, \mathbf{W}_2, \cdots, \mathbf{W}_m]$. We note that the recursive algorithm which is deduced by using a proposed BPF differential operation has been summarized as follows.

step 1 : Obtain the first BPF expansion equation from (28)

$\bar{\mathbf{W}}_1 = F\mathbf{W}_1 + G\mathbf{U}_1 + H\mathbf{Y}_1$

step 2 : Adopting (14) for $\bar{\mathbf{W}}_1$

$\bar{\mathbf{W}}_1 = \frac{2m}{t_f}[\mathbf{W}_1 - \omega(0)]$

where, $\omega(0)$ is an initial value of the observer

step 3 : The first BPF coefficient $\mathbf{W}_1$ which is obtained from step 1 and step 2 sets to be equal.

$$\mathbf{W}_1 = \left(I - \frac{t_f}{2m}F\right)^{-1} \times \left[\omega(0) + \frac{t_f}{2m}G\mathbf{U}_1 + \frac{t_f}{2m}H\mathbf{Y}_1\right] \quad (29)$$

step 4 : Sum of the i-th and (i+1)th BPF coefficient of (28)

$\bar{\mathbf{W}}_{i+1} + \bar{\mathbf{W}}_i$
$= F(\mathbf{W}_{i+1} + \mathbf{W}_i) + G(\mathbf{U}_{i+1} + \mathbf{U}_i) + H(\mathbf{Y}_{i+1} + \mathbf{Y}_i)$

step 5 : Redo like step 2 for (14)

$\bar{\mathbf{W}}_{i+1} + \bar{\mathbf{W}}_i = \frac{2m}{t_f}[\mathbf{W}_{i+1} - \mathbf{W}_i]$

step 6 : Redo like step 3 for $\mathbf{W}_{i+1}$ by using step 4 and step 5

$$\begin{aligned}\mathbf{W}_{i+1} &= \left(I - \tfrac{t_f}{2m}F\right)^{-1} \times \left[\left(I + \tfrac{t_f}{2m}F\right)\mathbf{W}_i + \right.\\ &\left. + \tfrac{t_f}{2m}G(\mathbf{U}_{i+1} + \mathbf{U}_i) + \tfrac{t_f}{2m}H(\mathbf{Y}_{i+1} + \mathbf{Y}_i)\right] \\ &for, i = 1, 2, \cdots, m-1\end{aligned} \quad (30)$$

Therefore, the BPF coefficients $\mathbf{W} = [\mathbf{W}_1, \mathbf{W}_2, \cdots, \mathbf{W}_m]$, can be obtained recursively from (29) and (30). Finally, from (21), estimated states of the system (16) can be obtained by state reconstruction such as (31).

$$\hat{x}(t) = T_1 \times \left[\begin{array}{c} \sum_{i=1}^m \mathbf{W}_i\phi_i(t) \\ -(M_2 + N_4 A_{21}^*)\sum_{i=1}^m \mathbf{W}_i\phi_i(t) - N_3\sum_{i=1}^m \mathbf{Y}_i\phi_i(t)\end{array}\right] \quad (31)$$

From (17), if $\zeta(t) \simeq \sum_{i=1}^m \mathbf{W}_i\phi_i(t)$ in (31), an equation for unknown input estimation is deduced as follows.

$$\begin{aligned}d(t) &= -(M_2 + N_4A_{21}^*)\dot{\zeta}(t) - N_3\dot{y}(t) \\ &-(A_{21}^* - A_{22}^*(M_2 + N_4A_{21}^*))\zeta(t) \\ &+A_{22}^*N_3y(t) - B_2^*u(t)\end{aligned} \quad (32)$$

In (32), unknown input estimation can be achieved by additional differential of system outputs. By using a BPF expansions and its differential operation (14), it can be eliminated as an algebraic form (33).

$$\begin{aligned}\hat{d}(t) &= -(M_2 + N_4A_{21}^*)\sum_{i=1}^m \bar{\mathbf{W}}_i\phi_i(t) \\ &-N_3\sum_{i=1}^m \bar{\mathbf{Y}}_i\phi_i(t) \\ &-(A_{21}^* - A_{22}^*(M_2 + N_4A_{21}^*))\sum_{i=1}^m \mathbf{W}_i\phi_i(t) \\ &+A_{22}^*N_3\sum_{i=1}^m \mathbf{Y}_i\phi_i(t) - B_2^*\sum_{i=1}^m \mathbf{U}_i\phi_i(t)\end{aligned}$$

$$\begin{aligned}\hat{d}(t) &= Q\sum_{i=1}^m \mathbf{W}_i\phi_i(t) + R\sum_{i=1}^m \mathbf{Y}_i\phi_i(t) \\ &+S\sum_{i=1}^m \mathbf{U}_i\phi_i(t) - N_3\sum_{i=1}^m \bar{\mathbf{Y}}_i\phi_i(t)\end{aligned}$$

$$\begin{aligned}where, Q &= [-(M_2 + N_4A_{21}^*)F - A_{21}^* \\ &+A_{22}^*(M_2 + N_4A_{21}^*)], \\ R &= -[(M_2 + N_4A_{21}^*)H - A_{22}^*N_3], \\ and\ S &= -[(M_2 + N_4A_{21}^*)G + B_2^*]\end{aligned} \quad (33)$$

In (33), it is easy to know that the differential of system output can be from (34).

$$\begin{aligned}\bar{\mathbf{Y}}_1 &= \tfrac{2m}{t_f}[\mathbf{Y}_1 - y(0)] \\ \bar{\mathbf{Y}}_{i+1} &= \tfrac{2m}{t_f}[\mathbf{Y}_{i+1} - \mathbf{Y}_i] - \bar{\mathbf{Y}}_i \\ &for, i = 2, 3, \cdots, m-1\end{aligned} \quad (34)$$

IV. Examples

1. Example 1

Let us consider the following linear time-invariant dynamical systems with unknown inputs[3], [5]. For the convenience of the simulation, $u(t)$ is omitted without loss of generality.

$$\dot{x}(t) = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} d(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x(t) \quad (35)$$

From the previous UIO design procedure, the coordinate transformation matrices T_1 and T_2 are determined as follows.

$$T_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\ and\ T_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (36)$$

The observer gain is chosen such as $L = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and the derived algebraic observer equation is as follows.

$$\sum_{i=1}^m \bar{\mathbf{W}}_i\phi_i(t) = -4\sum_{i=1}^m \mathbf{W}_i\phi_i(t) + \begin{bmatrix} 0 & -3 \end{bmatrix}\sum_{i=1}^m \mathbf{Y}_i\phi_i(t) \quad (37)$$

Derived recursive formula for the algebraic observer equation (37) are as follows

$$\begin{aligned}\mathbf{W}_1 &= \begin{bmatrix} 0 & -0.125 \end{bmatrix}\mathbf{Y}_1 \\ \mathbf{W}_{i+1} &= 0.667\mathbf{W}_i + \begin{bmatrix} 0 & -0.125 \end{bmatrix}\mathbf{Y}_i \\ &for, i = 1, 2, \cdots, m-1\end{aligned} \quad (38)$$

And an algebraic equation for unknown inputs estimation is as follows.

$$\hat{d}(t) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}\sum_{i=1}^m \mathbf{W}_i\phi_i(t) + \begin{bmatrix} 2 & 5 \\ 0 & 0 \end{bmatrix}\sum_{i=1}^m \mathbf{Y}_i\phi_i(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\sum_{i=1}^m \bar{\mathbf{Y}}_i\phi_i(t) \quad (39)$$

In this example1, we assume that $d(t) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ and choose $t_f = 10sec.$ and $m = 100$.

Figure 3 and 4 are the comparison of the unknown inputs estimation between the additional differentiator is used in MATLAB SIMULINK and the proposed algebraic method. In the figures, we show that the proposed algebraic method is advantageous and effective than the other non-algebraic estimation.

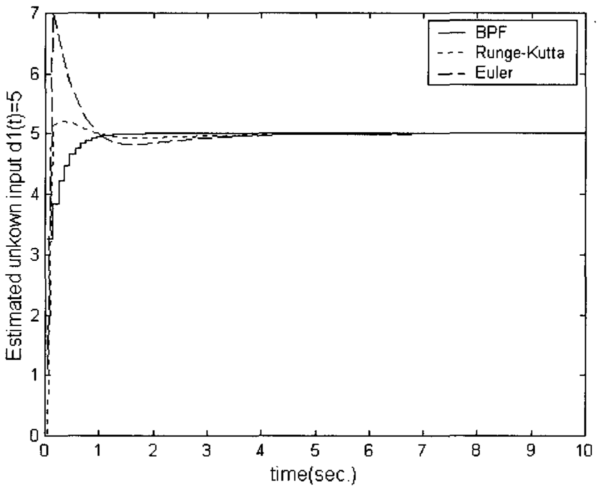


Fig. 1. Actual and estimated state $x_1(t)$.

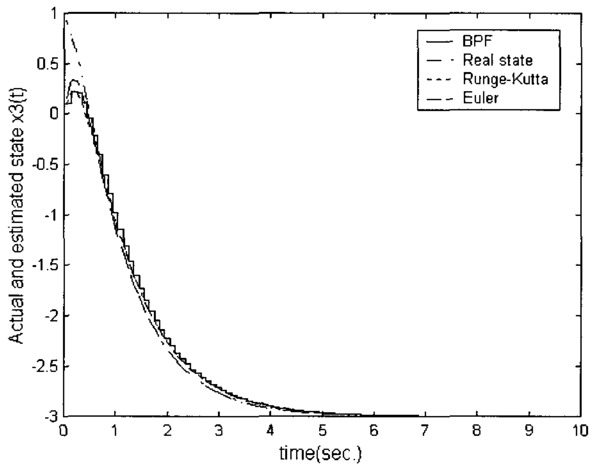


Fig. 2. Actual and estimated state $x_3(t)$.

2. Example 2

Considered linear time invariant dynamical system[5] is as follows.

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, and\, C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{40}$$

Transformation matrices and observer gain matrix are as follows.

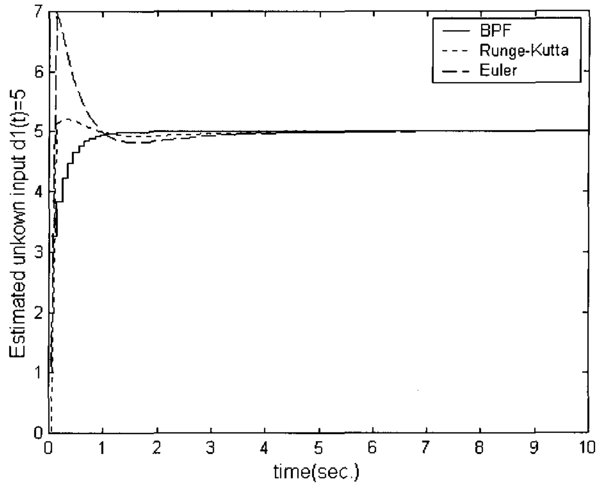


Fig. 3. Estimated unknown input $d_1(t) = 5$.

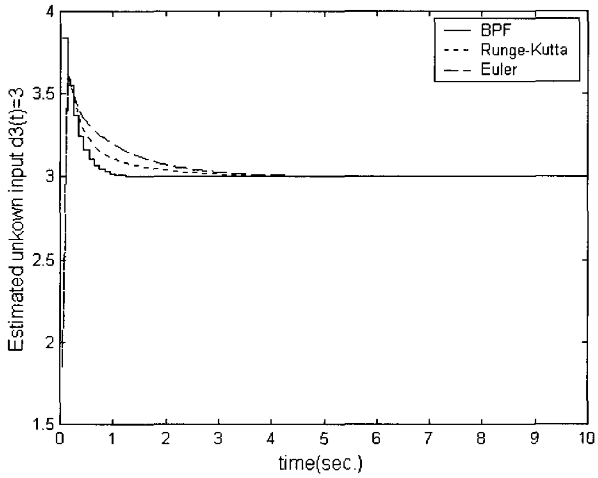


Fig. 4. Estimated unknown input $d_2(t) = 3$.

$$T_1 = T_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, L = \begin{bmatrix} 0 & 5 \\ 0 & 1 \end{bmatrix} \tag{41}$$

By using a proposed algebraic observer design procedure, derived algebraic observer equation is as follows.

$$\begin{aligned} \textstyle\sum_{i=1}^{m} \bar{\mathbf{W}}_i \phi_i(t) = & \begin{bmatrix} -6 & 0 \\ 0 & -1 \end{bmatrix} \textstyle\sum_{i=1}^{m} \mathbf{W}_i \phi_i(t) \\ & + \begin{bmatrix} 0 & 5 \\ 0 & 1 \end{bmatrix} \textstyle\sum_{i=1}^{m} \mathbf{Y}_i \phi_i(t) \end{aligned} \tag{42}$$

$$\hat{x}(t) = \begin{bmatrix} -\sum_{i=1}^{m} \mathbf{W}_{1_i} \phi_i(t) + \sum_{i=1}^{m} \mathbf{Y}_{1_i} \phi_i(t) \\ \sum_{i=1}^{m} \mathbf{W}_{2_i} \phi_i(t) \\ \sum_{i=1}^{m} \mathbf{W}_{1_i} \phi_i(t) \end{bmatrix} \tag{43}$$

Equation (44) is a derived unknown input estimation equation as an algebraic form.

$$\begin{aligned} \hat{d}(t) = & \begin{bmatrix} 5 & 1 \end{bmatrix} \textstyle\sum_{i=1}^{m} \mathbf{W}_i \phi_i(t) \\ & + \begin{bmatrix} 1 & -5 \end{bmatrix} \textstyle\sum_{i=1}^{m} \mathbf{Y}_i \phi_i(t) - \sum_{i=1}^{m} \bar{\mathbf{Y}}_{1_i} \phi_i(t) \end{aligned} \tag{44}$$

In this example 2, we assume that $d(t) = 3$ and choose $t_f = 3sec.$ and $m = 100$.

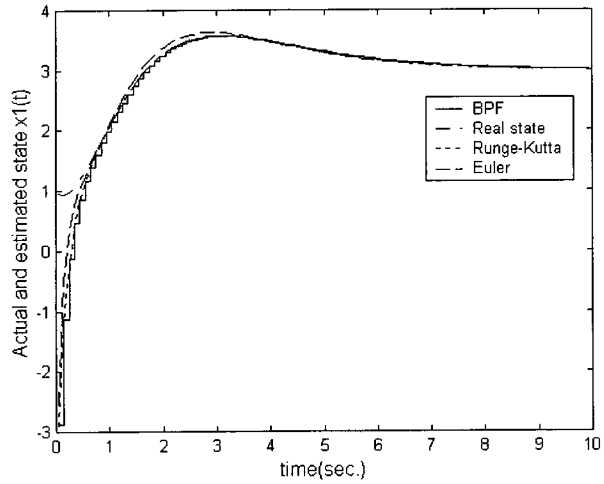


Fig. 5. Actual and estimated state $x_1(t)$.

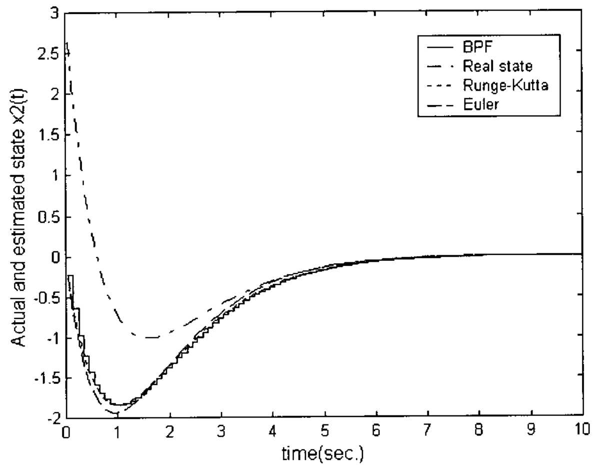


Fig. 6. Actual and estimated state $x_2(t)$.

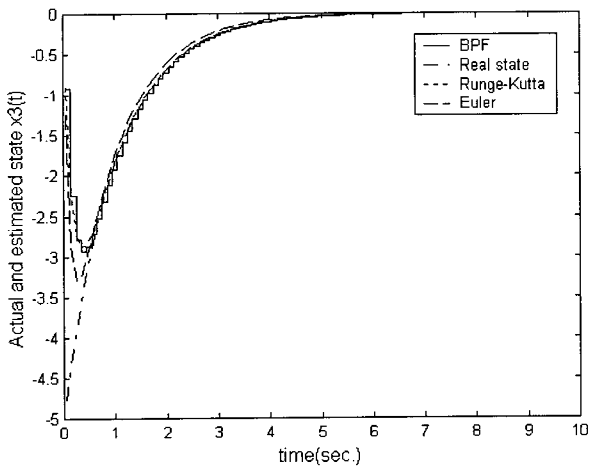


Fig. 7. Actual and estimated state $x_3(t)$.

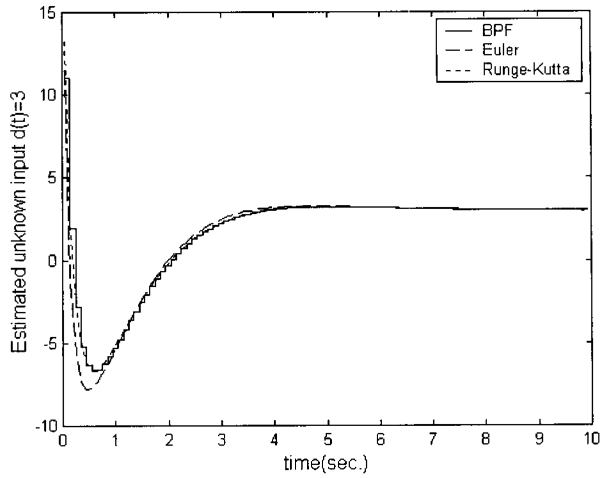


Fig. 8. Estimated unknown input $d(t) = 3$.

V. Conclusions

This paper deals with an algebraic approach to design an unknown inputs observer(UIO) by using a block pulse function(BPF) expansions and its differential operation. In the UIO design procedure, some coordinate transformation method is adopted to decouple the unknown inputs from the system. And the system in which the unknown inputs are decoupled is used directly to design an algebraic Luenberger-type observer to estimate the state vectors. But, if one wants to obtain the unknown inputs estimation from the system, it needs additional differential of the system outputs like the other methods [3], [4], [5], [6]. Therefore, this paper introduces an improved BPF's differential operation which can alleviate such problem in an algebraic form. Simulation for arbitrary numerical examples which are carried out using a MATLAB are presented to check on the validity of the proposed method. The least square error of the estimated states and unknown inputs for the actual values are presented in table 1 and 2. It is obvious that the proposed method in this paper is advantageous for the unknown input estimation than the other method from the table 1 and 2.

Table 1. Comparison of estimation errors for each method (Example 1).

	Proposed BPF	Runge-Kutta	Euler
$x_1(t)$	1.1136×10^{-2}	1.1256×10^{-2}	1.4158×10^{-2}
$x_3(t)$	1.1136×10^{-2}	1.1256×10^{-2}	1.0216×10^{-2}
$d_1(t)$	2.3316×10^{-2}	4.9827×10^{-2}	5.0072×10^{-2}
$d_2(t)$	1.1199×10^{-2}	1.5273×10^{-2}	1.6683×10^{-2}

Table 2. Comparison of estimation errors for each method (Example 2).

	Proposed BPF	Runge-Kutta	Euler
$x_1(t)$	4.5648×10^{-2}	4.6358×10^{-2}	3.8955×10^{-2}
$x_2(t)$	6.7226×10^{-2}	6.7110×10^{-2}	7.0168×10^{-2}
$x_3(t)$	4.5648×10^{-2}	4.6358×10^{-2}	3.8780×10^{-2}
$d(t)$	3.1983×10^{-1}	3.2719×10^{-1}	3.4957×10^{-1}

References

[1] S. P. Bhattacharyya, "Observer design for linear systems with unknown inputs," *IEEE Trans. Automatic Control* , vol. 23, pp. 483–484, 1978.

[2] F. W. Fairman, S. S. Mahil and L. Luk , "Disturbance decoupled observer design via singular value decomposition," *IEEE Trans. Automatic Control*, vol. 29, pp. 84–86, 1984.

[3] F. Yang and R. W. Wilde, "Observers for linear systems with unknown inputs," *IEEE Trans. Automatic Control* , vol. 33, pp. 677–681, 1988.

[4] Y. Guan and M. Saif, "A novel approach to the design of unknown input observer," *IEEE Trans. Automatic Control* , vol. 36, pp. 632–635,1991.

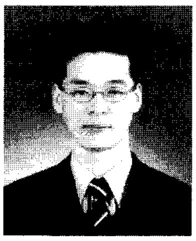
[5] M. Hou and P. C. Muller, "Design of observers for linear systems with unknown inputs," *IEEE Trans. Automatic Control*, vol. 37, pp. 871–875,1992.

[6] M. Darouach, M. Zasadzinski and S. J. Xu , "Full-order observers for linear systems with unknown inputs ," *IEEE Trans. Automatic Control*, vol. 39, pp. 606–609,1994.

[7] F. Kraus and W. Schaufelberger, "Identification with block-pulse functions, modulating functions and differential operator ," *Int. Journal of Control* , vol. 51, pp. 931-942, 1990.

[8] Z. H. Jiang and W. Schaufelberger, *Block-pulse functions and their and their applications in control systems,* Springer-Verlag , 1990.

[9] P. Ahn, M. K. Kim and D. S. Ahn , "A novel approach to unknown input observer design via block pulse function's differential operation ," *Proc. of IFAC World Congress*, vol. D, pp. 225–260, 1999.



Pius Ahn

He received the B.S., M.S. and Ph.D degrees in electrical engineering from Sung Kyun Kwan University, Korea, in 1992, 1994 and 1998, respectively. He was a post doctor at KAERI(Korea Atomic Force Research Institute), Teajon, from 1998 to 1999. After then he worked in MIDAS Korea Co., Ltd. as a CTO from 2000 to 2001. Since 2002, he has been an associate consultant with the office of electric-electro engineering, Small Business Corporation. His research interests are decentralized control theory, application of orthogonal functions for algebraic controller design and device-net for on-line SCADA system construction.