International Journal of the Korean Society of Precision Engineering Vol. 3, No. 4, October 2002.

An Interval Approach for Design and Analysis of Mechanical Systems with Uncertainties

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ABSTRACT

This paper addresses the challenges of dealing with uncertainties based on interval analysis. An interval approach is proposed on the basis of Boundary Selection Method (BSM) for treating systems of linear interval equations in the presence of columnwise dependencies. An iterative procedure is developed for the problem solving where uncertainties are characterized in the form of interval quantities. An applied example is used to illustrate effectiveness and usefulness of the proposed approach. This new method can be applied for such circumstances that involve finite element analysis of structures, inverse dynamic analysis of mechanisms, and worst case design studies in the presence of the uncertainties.

Key Words : Uncertainty Modeling, Interval Analysis, Worst Case Design and Analysis, Boundary Selection System

Nomenclature

A, A_1, A_2	A scalar matrix, $A=\{A_{ij}\}$, etc.
A′	A scalar matrix
A	Interval matrix with $\mathbf{A}_{ij} = [\underline{A}_{ij}, \overline{A}_{ij}]$
A^c	Central matrix with $A^c_{ij} = (\underline{A}_{ij} + \overline{A}_{ij})/2$
$\underline{A}$, $\overline{A}$	$\underline{A} = \{\underline{A}_{ij}\}$, $\overline{A} = \{\overline{A}_{ij}\}$
$\underline{a}$, $\overline{a}$	Lower/upper bound denoting the interval number
[a]	Interval number in the form of $[a] = [\underline{a}, \overline{a}]$
b	A vector with scalar elements, i.e. $b=(b_i)^T$
b	Interval vector, i.e. $\mathbf{b} = (\mathbf{b}_i)^T = ([\underline{b}_i, \overline{b}_i])^T$
b^c	Central vector, $b^c_i = (\underline{b}_i + \overline{b}_i)/2$
b_i	Element of vector b
$\underline{b}_i$, $\overline{b}_i$	Lower/upper bound of $\mathbf{b}_i$
$g_1, g_2, \ldots, g_n$	Constants
$h_1, h_2, \ldots, h_n$	Constants
r_k	*k*th row of the vector of the inverse matrix, A^{-1}.
$v_1, v_2, \ldots, v_n$	Constants
$w_1, w_2, \ldots, w_n$	Constants
x	Vector of unknowns, $x=(x_i)^T$
x	Interval vector of unknowns, $\mathbf{x} = ([\underline{x}_i, \overline{x}_i])^T$
$\mathbf{x}^H$	Interval hull of solution vector x.
x_i	*i*th component of vector x.
$\tilde{x}$	A solution vector sought from a selected boundary system
$\underline{x}$, $\overline{x}$	Vector defined as $\underline{x} = (\underline{x}_i)^T, \overline{x} = (\overline{x}_i)^T$
$\alpha, \beta, \delta, \eta$	Parameter shaving uncertainties
[ ]	Denote interval numbers
boldface	Denote vectors and matrices with interval numbered elements

1. Introduction

Engineering design is undertaken in the presence of uncertainties, especially during the early stages of a design cycle. Engineering systems with uncertainties can be modeled with the aid of interval equations. Often such equations are present in the form of linear equations for, as examples, finite element analysis of structures [6, 7] and inverse dynamic analysis of mechanisms [1]. To apply interval equations for such systems design and analysis, it is indispensable to develop an effective method for the problem solving. In this work, we apply the system of linear interval equations, say **Ax** = **b**, for dealing with the worst case analysis of mechanical systems with uncertainties, which in turn contributes significantly to treatment of the extreme design scenarios. Note that the coefficient matrix **A** or the vector **b** is composed of interval numbers.

Current approaches to solving linear interval equations have limitations on dealing with the problems where dependencies are present in the elements of the matrix **A** or the vector **b**. Most of the available methods assumes that no dependency exists between the elements of the system matrices. Too often, however, the elements in the system matrix/vector are dependent on one another. In finite element analysis, for example[7,9], the system stiffness matrix is relevant tightly to design parameters such as the area or mass of a truss bar. The elements in the stiffness matrix are coupled with some design parameters, which in turn leads to the dependencies between the elements of the matrix.

The following sections are organized as follows. First, related work in the literature is reviewed. Then, basic concepts are introduced and illustrated by using a simple example. Based on the illustrative example, our method is presented and applied for the input torque analysis of a window operating mechanism in the face of uncertainties arising from the imprecise friction coefficients, the wind load and the input acceleration. Finally, conclusions are drawn to wrap up this work.

2. Related Work

In the literature, systems of linear interval equations have been tackled in two ways. Chen and Ward [2] addressed two types of solutions called "range" and "domain", resulting from two different interpretations of the linear interval equations. The range solution comes from the traditional meaning of the interval equations where an exact solution is pursued [4]. However, it is generally difficult to obtain the exact solutions of the interval equations. Most of the available methods are focused on finding the interval hull solutions for the given problems [4, 8]. It is well known that the algorithms for obtaining the interval hull solutions such as an interval version of Gaussian algorithm lead to a wider interval solution than the interval hull. On the other hand, the domain solution comes from a different interpretation of the interval equations. Finding a domain solution means to find a solution band such that every point within the band is the solution of the interval equation. In this regard, the domain solution can be well utilized for design applications, while the range solution is more appropriate for the worst case design analysis subject to the given ranges of parametric values.

It is recognized that many methods in the literature do not take into account the dependencies between the elements of the system matrices **A** and **b**. Unfortunately, this is not always the case in practice. Only few works are found to address the dependency issue. Rao and Chen [6] developed an optimization-based systematic approach for finding the domain solution to dependency-oriented problems. However, the solution procedure involves nonlinear mathematical programming. Jansson [5] proposed an iterative solution method for finding the interval hull solution to the systems with limited degree of dependencies. Nevertheless, this method gives only an approximate solution and overestimates the extreme values of the unknowns. This work will present a new approach to an exact interval hull solution while considering dependencies present between the elements in **A** and/or **b**.

3. Boundary Selection Method (BSM)

3.1 Basic Concepts

An interval number, denoted [a], can be defined as follows:

$$[a] = [\underline{a}, \bar{a}] = \left\{ a \subseteq R \mid \underline{a} \le a \le \bar{a} \right\} \tag{1}$$

The values labeled by $\underline{a}$ *and* $\bar{a}$ are called *lower* and *upper* bounds, respectively. An interval vector **x** and

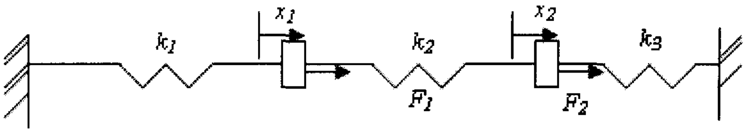

Fig. 1 A Spring-mass Systetem

interval matrix **A** are composed of such interval numbers, i.e.

$$\mathbf{x} = [\underline{x}, \overline{x}] = \left\{ x \in R^n \;\middle|\; \underline{x} \le x \le \overline{x} \right\} \quad (2)$$
$$\mathbf{A} = [\underline{A}, \overline{A}] = \left\{ A \in R^{n\mathrm{x}n} \;\middle|\; \underline{A} \le A \le \overline{A} \right\}$$

The center of an interval matrix $\mathbf{A}^c$ is defined by

$$\mathbf{A}^c = \left(\underline{A} + \overline{A}\right)/2 \quad (3)$$

The center of a vector can be defined similarly. A linear interval system is defined as a family of linear equations given by $A\,x = b$ with $A \in \mathbf{A}$ and $b \in \mathbf{b}$. Correspondingly, the solution set can be defined by

$$\{ x \in R^n \mid Ax = b, A \in A \text{ and } b \in b \} \quad (4)$$

The boundary system is implied to any set of linear equations $Ax = \dot{b}$ where A_{ij} is either $\underline{A}_{ij}$ or $\overline{A}_{ij}$ for all i, j and b_i is either $\underline{b}_i$ or $\overline{b}_i$ for all i. There exist different ways to interpret the interval solution to the linear system. This will be illustrated through an example problem presented below.

3.2 Boundary Selection Method: An Illustration

To illustrate, consider the problem of finding the static equilibrium positions for a spring-mass system shown in Fig.1. The governing equations are derived below:

$$(k_1+k_2)x_1 - k_2 x_2 = F_1 \quad (5a)$$
$$- k_2 x_1 + (k_2+k_3)\, x_2 = F_2 \quad (5b)$$

Let it be assumed that uncertainties are present in the spring coefficient k_1 and the applied forces F_1 and F_2. These uncertain parameters are expressed in the form of interval quantities, e.g., say $[k_1] = [\underline{k}_1, \overline{k}_1] = [0.8, 1.2]$, $[F_1] = [\underline{F}_1, \overline{F}_1] = [0.52, 1.42]$ and $[F_2] = [0.9, 1.3]$. Other parameters are still considered as certain, e.g., say $k_2 = k_3 = 1$. Thus the interval form of Eq. (5) is given as follows.

$$[1.8, 2.2]\, x_1 - x_2 = [0.52,\ 1.42] \quad (6a)$$
$$- x_1 + 2\, x_2 = [0.9,\ 1.3] \quad (6b)$$

Equation (6) represents a set of linear interval equations, since it involves interval quantities. Due to the difficulty in obtaining the exact solution, solving the equations is usually meant to finding a smallest interval vector $\mathbf{x}^H$ to contain the solution set. This solution vector is known as the interval hull solution to the internal equations.

Different explanations of the interval solutions to Ax = b can be found in the literature in terms of range and domain solutions. For the example problem, the range solution is intended to find all the possible displacements subject to the parametric variation ranges prescribed. It is nothing more than the traditional definition of the solution to the interval equations. The domain solution, on the other hand, is defined as an interval vector x such that $Ax \subseteq b$. This definition is useful for dealing with a design problem since finding a complete solution set may not be necessary. In this work, it is focused on the traditional definition of the interval solutions.

In this example, the worst case design analysis requires to obtain the extreme displacements subject to the presence of uncertainties. This can be done by finding the interval hull solution. Huddak [4] proposed an algorithm for finding an exact interval hull solution, showing how to select a proper boundary system. His approach (i.e. BSM) is based on the claim that all extreme points of the convex hull of the solution set x are the solutions of the boundary selection systems. With the BSM, the interval hull can be found by solving the boundary systems (i.e. the linear equations) 2n times while the following two conditions must be satisfied:

a) The matrix **A** is inverse-stable; and
b) The problem is a single orthant one.

An interval matrix **A** is said to be inverse-stable if the inverse of any two matrices A_1, $A_2 \in$ **A** have the same signs, as the signs are compared elementwise. If the solution complex lies in one orthant of R^n, then it is a single orthant problem. Further discussions on BSM can be found in Appendix. If either condition is not held, modification to the BSM is necessary. Towards this end, this work develops a practical yet efficient procedure for this study, which will be addressed later.

By applying the BSM for this example problem, the interval hull solution to the present problem can be identified. Assume it is known that the signs of x_i are all positive. Thus, we have

$$z = \text{sign}(x) = (1,1)^T \tag{7}$$

Also it can be easily shown that the signs of all the elements of the inverse matrix A $\in$ **A** are positive. Here, **A** represents the coefficient matrix for Eq. (6). Thus, this target system satisfies the two conditions necessary for application of BSM. The remaining procedure is presented below (also see Appendix).

Step 1: Find the minimum of x_1.

$$y = \text{sign}(r_1)^T = (1,1)^T;$$
$$P = zy^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad A = \begin{bmatrix} 2.2 & -1 \\ -1 & 2 \end{bmatrix} \tag{8}$$

Since P_{11} is positive and $y_i = +1$, $\overline{A_1}$ should be chosen and $b_i = \underline{b}_1$ should be used, respectively. The selected boundary system becomes b ={0.52, 0.9}^{T}. Solving the resulting equation turns out that **x** = $(0.57,0.73)^T$ or x_{1min}=0.57.

Step 2: Find the maximum of x_1.

Likewise, the selected boundary system can be generated and the result is given as

$$A = \begin{bmatrix} 1.8 & -1 \\ -1 & 2 \end{bmatrix}; \quad b=\{1.42,1.3\} \tag{9}$$

The resulting solution is obtained as x = $(1.59, 1.44)^T$ or $x_{1\max}$=1.59.

Using the same procedure to find the limits of x_2, the interval hull solution is finally given as follows:

$$\mathbf{x}^H = ([0.57, 1.59], [0.73, 1.44])^T \tag{10}$$

Figure 2 shows the comparison of the exact solution set with the interval hull solution.

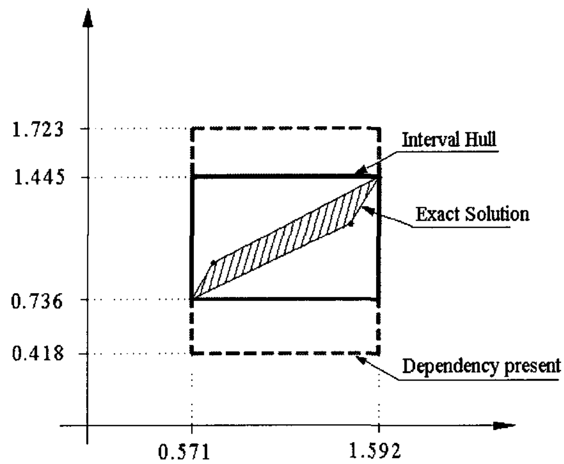

Fig. 2 Exact solution vs. interval hull solution

4. Approach to Linear Interval Systems

4.1 Problem with Dependencies

In the above problem solving, an underlying assumption is implicitly made that the elements of the system matrices **A** and **b** are perfectly independent of one another. The matrices associated with Eq. (5) satisfy this assumption.

Because each uncertain parameter k_1, F_1 or F_2 appears only once in the system matrices. To show how dependencies will affect the solution process, an equivalent set of equations is used as the variant for Eq. (5) when solving the example problem, i.e.

$$(k_1+k_2)x_1 - k_2 x_2 = F_1 \tag{11a}$$
$$k_1 x_1 + \quad k_3 x_2 = F_1 + F_2 \tag{11b}$$

Note that Eq. (11b) is obtained by simply adding Eqs. (5a) and (5b). This result is given below:

$$[1.8, 2.2]\, x_1 - x_2 = [0.52, 1.42] \quad (12a)$$
$$[0.8, 1.2]\, x_1 + x_2 = [1.42, 2.72] \quad (12b)$$

Thus, BSM can be applied for finding the interval hull solution. It can be easily shown that the sign of inverse of A can be determined as

$$\text{sign}\,(A^{-1}) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (13)$$

Also it is known that the signs of the unknowns are the same as before. To get the minimum of x_2, it proceeds

$$z = (1,1)^T;\ y = \text{sign}\,(r_2^T) = (1,-1)^T;$$
$$P = zy^T = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad (14)$$

The following equations show the selected boundary system, i.e.

$$1.2\, x_1 + x_2 = 1.42 \quad (15a)$$
$$2.2\, x_1 - x_2 = 1.42 \quad (15b)$$

from which the minimum of $x_2 = 0.42$ can be found.

Using the same procedure, other extreme values can also be identified. As the result, the interval hull solution can be determined correspondingly to the system of equation (4), i.e. $\mathbf{x}^H$ = ([0.57, 1.59], [0.42, 1.72])^T. Comparing this solution with the previous one, it can be seen that broader solution ranges are produced. Obviously, different solution sets are found for the same problem. This discrepancy results from different choices of the system equation form. As a matter of fact, in Eq. (12) the uncertain parameters k_1 and F_1 appear two times, respectively. This means, for example, A_{11} and A_{21} are not independent of each other. Due to this dependency, some of the boundary systems cannot be valid in fact. In the boundary system (15a) and (15b), the value of 1.42 is chosen for both b_1 and b_2. However, this is inconsistent with Eq. (11) where b_1=F_1 and b_2=F_1+F_2. Apparently, this value can not be held simultaneously for b_1 and b_2, subject to Eq. (11).

Neglecting the dependencies present in the system matrices will result in overestimation of the extreme values, because it would allow for more sets of the boundary systems. Thus, the dependencies present in the systems must be taken into account in the problem solving. For this purpose, two different strategies can be used for dealing with the dependencies. The first one is to incorporate the existence of dependencies into a specific solution process. For example, it is possible to choose a boundary system parameterwise but not elementwise. However, this strategy may not be practical since the supporting theoretical background is not available, allowing for choosing the proper boundary system for a specific application. The second one, which is more realistic, is to transform the problem into an equivalent system of equations with no dependencies. For example, Eq. (11) can be converted into the variant form of Eq. (5), the latter does not involve any dependency.

However, for a more complex problem, ransformation may not be always feasible, even if it is theoretically possible. In this work, it is intended to develop a generalized approach to effectively treat the problem of dependencies. Our approach will be illustrated by the example problem and then will be generalized to a general case. The proposed method follows the strategy of problem transformation so as to remove the dependencies. To make this possible, each uncertain parameter should appear only once in the system equations. For a certain class of problems, this can be accomplished by introducing the artificial variables. As for the example problem, k_1 and F_1 appear twice in Eq. (11). Correspondingly, two artificial variables should be introduced and are defined as

$$x_3 = k_1 x_1 \quad (16)$$
$$x_4 = F_1 \quad (17)$$

Accordingly, the original set of equations is rearranged in the form of

$$k_2\, x_1 - k_2 x_2 - x_3 - x_4 = 0 \quad (18a)$$
$$k_3 x_2 + x_3 - x_4 = F_2 \quad (18b)$$
$$k_1\, x_1 \quad - x_3 = 0 \quad (18c)$$
$$x_4 = F_1 \quad (18d)$$

Note that Eqs. (18a) and (18b) correspond to Eqs. (11a) and (11b), respectively while the additional equations (18c) and (18d) result from Eqs. (16) and (17). As can be seen, the uncertain parameters k_1, F_1 and F_2 thus appear only once in Eq. (18). Although the system size is increased by 2, this form is suited for application of any available methods valid for independent problems. Note that the transformed system in terms of Eq. (18) is equivalent to its original form. This method will be further discussed in the next section.

4.2 Problem Transformation

The foregoing discussion can be generalized into a systematic method. This method is valid for any problem having columnwise linear dependencies in the system matrices **A** and **b**. For the system of linear equations **A**x = **b**, the columnwise linear dependencies can be characterized in the form of the following matrices:

$$\mathbf{A}=\begin{bmatrix} A_{11}, A_{12}, & \cdots\cdots & A_{1i-1}, A_{1i}^0+g_1[\alpha]+h_1[\beta], & \ldots, & A_{1n} \\ A_{21}, A_{22}, & \cdots\cdots & A_{2i-1}, A_{2i}^0+g_2[\alpha]+h_2[\beta], & \ldots, & A_{2n} \\ \vdots & & & & \\ A_{n1}, A_{n2}, & \cdots\cdots & A_{ni-1}, A_{ni}^0+g_n[\alpha]+h_n[\beta], & \ldots, & A_{nn} \end{bmatrix} \quad (19)$$

and

$$\mathbf{b}=\begin{Bmatrix} b_1^0+v_1[\delta]+w_1[\eta] \\ \vdots \\ b_i^0+v_i[\delta]+w_i[\eta] \\ \vdots \\ b_n^0+v_n[\delta]+w_n[\eta] \end{Bmatrix} \quad (20)$$

In the above equations, uncertain parameters are denoted and highlighted with the square bracket. To remove the dependencies, an additional set of artificial variables is added to facilitate implementation of the method:

$$x_{n+1} = \alpha x_i \quad (21)$$

$$x_{n+2} = \beta x_i \quad (22)$$

$$x_{n+3} = \delta \quad (23)$$

$$x_{n+4} = \eta \quad (24)$$

Thus, the original system of equations is transformed appropriately into a corresponding system of equations in the form of **A**' **x**' = **b**', where

$$\mathbf{A}'=\begin{bmatrix} A_{11}, A_{12}, \ldots, A_{1i}^0, \ldots, A_{1n}, r_1, s_1, -v_1, -w_1 \\ A_{21}, A_{22}, \ldots, A_{2i}^0, \ldots, A_{2n}, r_2, s_2, -v_2, -w_2 \\ \vdots \\ A_{n1}, A_{n2}, \ldots, A_{ni}^0, \ldots, A_{nn}, r_n, s_n, -v_n, -w_n \\ 0,\ 0,\ \ldots, [\alpha], \ldots, 0, -1,\ 0,\ 0,\ 0 \\ 0,\ 0,\ \ldots, [\beta], \ldots, 0,\ 0, -1,\ 0,\ 0 \\ 0,\ 0,\ \ldots,\ 0, \ldots, 0, -1,\ 0,\ 1,\ 0 \\ 0,\ 0,\ \ldots,\ 0, \ldots, 0, -1,\ 0,\ 0,\ 1 \end{bmatrix} \quad (25)$$

$$\mathbf{b}' = \{b_1^0, b_2^0, \ \ldots \ b_n^0,\ 0,\ 0,\ [\delta],\ [\eta]\}^T \quad (26)$$

$$\mathbf{x}' = \{x_1, x_2, \ \ldots \ x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}\}^T \quad (27)$$

In the above derivation, it is assumed that elements only on the ith column have linear dependencies. Nevertheless the method is equally applicable to treat the same dependencies simultaneously appearing in other individual columns while the elements across different columns are independent. It is clear that after the transformation, the uncertain parameters appear only once in the system of equations.

5. Illustrative Example

To demonstrate effectiveness and usefulness, the proposed approach is applied for the inverse dynamic analysis of a window operating mechanism, shown in Fig. 3. This mechanism is adapted from Erdman and Sandor [3]. In designing the window mechanism, one of the key concerns is the magnitude of the input torque necessary for opening or closing the window. If the input torque is too large, it makes difficult to open or close the window. On the other hand, if this torque is too small, the window may not be able to stay in the open position when subjected to external disturbances (e.g. wind). Hence, it is necessary to analyze the input torque requirement and determine its appropriate magnitude through the design. However, this is not easy because some necessary design parameters may not be known precisely due to the presence of uncertainties. For example, the friction coefficients may vary with the weather; or the wind load may alter from time to time. For this reason, these parameters are considered as uncertain and can be characterized by interval quantities. Interval analysis allows for finding the extreme values of the input torque that can reach within the variation ranges of the uncertain

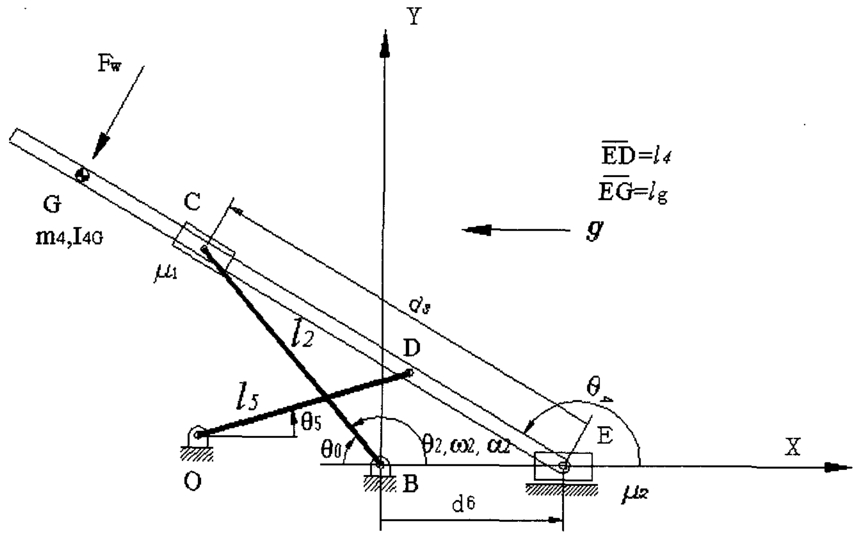

Fig. 3 Window operation mechanism

parameters. The analysis result contributes significantly to the worst case design study.

In this example, four interval parameters - friction coefficients μ_1, μ_2, wind load F_w, input acceleration of the input link (l_2) α_2 - are considered. The aim is to find the extreme input torque required to initiate opening the window from various starting positions. As the window moves from a stationary position, the input velocity is set to zero in the analysis. For simplicity, only link l_4 is considered to have mass. The governing equation is given in the following:

$$\begin{bmatrix} 1 & 0 & S_4+[\mu_1]C_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -c_4+[\mu_1]S_4 & 0 & 0 & 0 & 0 & 0 \\ l_2s_2 & -l_2c_2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -S_4-[\mu_1]C_4 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & C_4-[\mu_1]S_4 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & d_3 & -l_4S_4 & l_4C_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & [-\mu_2] & 0 \\ 0 & 0 & 0 & l_5S_5 & -l_5C_5 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} F_{BX} \\ F_{BY} \\ F_C \\ F_{DX} \\ F_{DY} \\ F_{EX} \\ F_{EY} \\ T \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ [F_W]S_4+m_4k_X[\alpha_2] \\ -[F_W]C_4+m_4k_Y[\alpha_2]+m_4g \\ -(l_g)[F_W]-m_4l_Gs_4g \\ Q[\alpha_2] \\ 0 \end{Bmatrix} \tag{28}$$

with

$$Q = m_4k_yl_gC_4 + l_4k_4 - m_4k_xl_gS_4 \tag{29}$$

where C_4, S_4 stand for, respectively, $\cos\theta_4$ and $\sin\theta_4$; and the kinematic influence coefficients k_x, k_y are given by

$$k_x = \ddot{x}_G/\alpha_2;\ \ k_y = \ddot{y}_G/\alpha_2 \tag{30}$$

The four interval parameters are highlighted with the square bracket in Eq. (28). Note that the interval parameter μ_1 appears four times on the left hand side while the parameters F_w and α_2 appear three times each on the right hand side. As can be seen, they appear in only one column. Accordingly, the method developed earlier in the form of Eqs. (19) and (20) can be followed. By introducing three additional artificial unknowns of μF_c, F_w and α_2, the system of equations can be rearranged in the form of

$$\begin{bmatrix} 1 & 0 & S_4 & 0 & 0 & 0 & 0 & 0 & C_4 & 0 & 0 \\ 0 & 1 & -C_4 & 0 & 0 & 0 & 0 & 0 & S_4 & 0 & 0 \\ l_2S_2 & -l_2C_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -S_4 & 1 & 0 & 1 & 0 & 0 & -C_4 & -S_4 & -m_4k_X \\ 0 & 0 & S_4 & 0 & 1 & 0 & 1 & 0 & -S_4 & C_4 & -m_4k_Y \\ 0 & 0 & d_3 & -l_4S_4 & l_4C_4 & 0 & 0 & 0 & 0 & l_G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -[\mu_2] & 0 & 0 & 0 & -Q \\ 0 & 0 & 0 & l_5S_5 & -l_5C_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & [\mu_1] & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} F_{BX} \\ F_{BY} \\ F_C \\ F_{DX} \\ F_{DY} \\ F_{EX} \\ F_{EY} \\ T \\ \mu_1F_C \\ F_W \\ \alpha_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ m_4g \\ 0 \\ -m_4l_gS_4g \\ 0 \\ 0 \\ 0 \\ [F_W] \\ [\alpha_2] \end{Bmatrix} \tag{31}$$

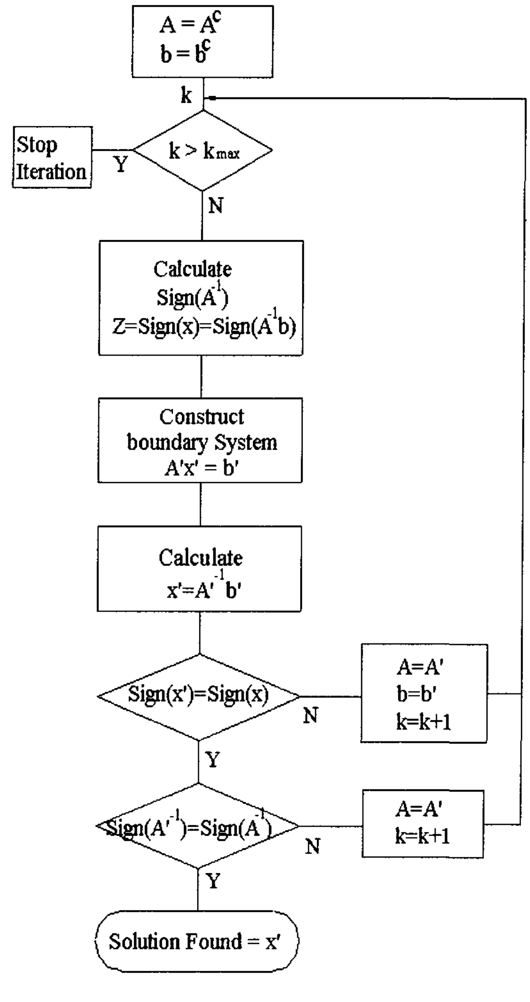

Fig. 4 The Flowchart of BSM

In Eq. (31), each interval parameter appears only once as expected. To this stage, BSM can be used for the problem solving; however, it still requires some modifications to accommodate this example problem. It is needed to exactly predict the inverse of the selected boundary system and the signs of all the unknowns. This can be accomplished by adopting the inverse of the central matrix A^c *for predicting the signs of the inverse* matrix while the signs of the unknowns x can be obtained by solving the central system A^c x = b^c. The predicted signs should be correct if a) the matrix **A** is inverse-stable and b) the solution set of the interval equation lies in a single orthant. To choose a proper boundary system, these two conditions must be held; otherwise, it leads to improper selection of the boundary system. To check whether a boundary system is chosen correctly, the signs of inverses and the solution vectors should be compared. This process is illustrated in Fig. 4.

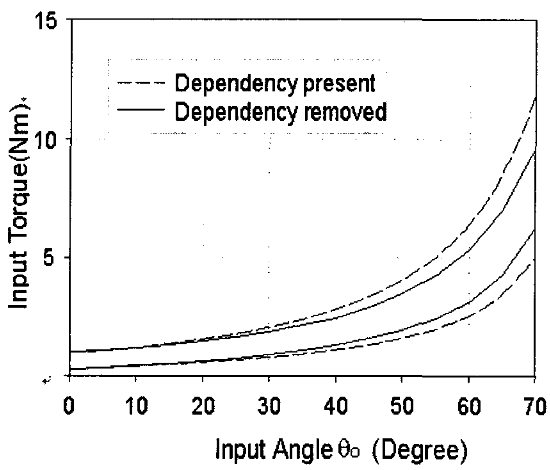

Fig. 5 Results of the Example Problem

Table 1 Numerical Data

Parameter	Symbol	Value
Length (m)	l_2 l_4 l_{EG}	0.15 0.0981 0.25
Position (m)	(O_{5x}, O_{5y})	(- 0.1, 0.015)
Mass (kg)	m_4	4.0
Inertia(kg-m^2)	I_{4G}	0.0833
Friction Coefficient	μ_1, μ_2	[0.2,0.4],[0.2,0.4]
Wind Load (N)	F_w	[0,5]
Input Acceleration	α	[- 1,- 0.5]
Gravitational acceleration	g	9.78

Our method is an iteration-based solution procedure. In iteration k, A_{k-1} and x_{k-1} are used for the signs of A and x while A_k and b_k are set for the next iteration of the selected boundary system. Solving the boundary system equation A_k x = b_k, the solution x_k can be obtained. If sign (A_k^{-1}) = sign (A_{k-1}^{-1}) together with sign (x_k) = sign (x_{k-1}), it tells that a correct boundary system is chosen and the iteration process can be terminated. Otherwise, A_i and/or x_i will be used to replace A_i^{-1} and/or x_{i-1} and the iteration process continues.

Figure 5 shows the maximum and minimum values of input torque for the example problem, where input data necessary for this analysis is given in Table 1. In Fig. 5, the dotted line represents solutions with respect to BSM by neglecting the dependencies and the solid line corresponding to our method by taking the dependencies into consideration. It can be seen that the problem of dependency leads to overestimation of the extreme values. The results were checked through the exhaustive calculations. It can be concluded that our method worked well for this example problem.

6. Conclusions

In this work, a new method has been presented for effectively finding an exact interval hull solution to the system of linear interval equations. Particularly, the method allows for dealing with the columnwise linear dependency present in the elements of **A** and/or **b**. In our method, the boundary selection method (BSM), originally developed for solving the systems with no dependency, has been explored in depth and extended to the interval problems involving dependencies with the system matrices. This is accomplished by first converting the original system into the one with no dependency and then applying the BSM for the problem solving. This new method enables to attain an exact interval hull solution when dependencies between elements in **A** and/or **b** are present. More specifically, this work has shown how to appropriately transform the dependent problem into an independent one by introducing the artificial unknowns. An efficient method has been developed for expanding applicability of the BSM to such problems that either of the underlying conditions on which the BSM stands fails to satisfy. The example problem has clearly demonstrated effectiveness of the proposed method for applications. The iterative procedure illustrated in Fig.4 is quite effective for solution convergence with only 1 or 2 iterations. This new method can be applied for such situations as finite element analysis of structures, inverse dynamic analysis of mechanisms, and worst case engineering design where uncertainties are present.

References

1. L. Chen, S.S. Rao, and K. Ku, "A Fuzzy Kinetostatic Analysis of Imprecisely-Defined Mechanisms," ASME J. Mech. Design, 120, pp. 377-380, 1998.
2. R. Chen, and A.C, Ward, "Generalizing Interval Matrix Operations for Design," ASME J. Mech. Design, 119, pp. 65-72, 1997.
3. A. G. Erdman, and G. N. Sandor, Mechanism Design: Analysis and Synthesis, Vol. 1, Chapter 8, Prentice Hall, 1994.
4. D. Huddak, "On the Determination of the Interval Hull of Linear Interval Equations," Computing, 46, pp. 253-263, 1991.
5. C. Joasson, "Interval Linear Systems with Symmetric Matrices, Skew-Symmeric Matrices and Dependencies in the Right Hans Side," Computing 46, pp. 265-274, 1991.
6. S. S. Rao, and L. Chen, "Numerical Solution of Fuzzy Linear Equations in Engineering Analysis," Int. J. Numer. Meth. Engng. 42, pp. 829-846, 1998.

7. S.S. Rao. and P. Sawyer, "Fuzzy Finite Element Approach for the Analysis of Imprecisely Defined Systems," AIAA Journal, 33 (12), pp. 2364-2370, 1995.
8. Jiri Ron, and Vladik Kreinovich, "Computing Exact Copnonentwise Bounds on Solutions of Linear Systems with Interval Data is NP-Hard," SIAM J. Matrix Ana. Appl. 16 (2), pp. 415-420, 1995.
9. O. Kaung Lim, and Byung Woo Lee, "Structural Optimization Using Stochastic Finite Element Method," Trans KSME, 18(8), pp. 1920-1929, 1994.

Appendix: Boundary Selection Systems

This is to illustrate choice of a boundary selection system for generating the extreme point of the convex hull $\mathbf{x}^H$.

Let $z = \text{sign}(\tilde{x})$, $y = \text{sign}(r_k^T)$, $P = \{p_{ij}\} = zy^T = \{z_i y_j\}$, where r_k points out the kth row of the vector of A^{-1}. Here, $\tilde{x}$, A represent the solution vector and the system matrix of the selected system, respectively.

To obtain the minimal point of the kth coordinate, choose the boundary points according to
$A_{ij} = \underline{A}_{ij}$ (if $p_{ij} = -1$) or $\overline{A}_{ij}$ (if $p_{ij} = +1$)
$b_i = \underline{b}_i$ (if $y_i = +1$) or $\overline{b}_i$ (if $y_i = -1$)

To obtain the maximum, use the scheme according to
$A_{ij} = \underline{A}_{ij}$ (if $p_{ij} = 1$) or $\overline{A}_{ij}$ (if $p_{ij} = -1$)
$b_i = \underline{b}_i$ (if $y_i = -1$) or $\overline{b}_i$ (if $y_i = +1$)

Thus, $\mathbf{x}^H$ can be found by solving n by n the scalar linear system $2n$ times.