한국 퍼지 및 지능시스템 학회 2002, Vol. 12, No. 4, pp. 323-327

집합치 쇼케이적분과 수렴정리에 관한 연구(II)

On set-valued Choquet integrals and convergence theorems(II)

*장이채 · **김태균 · ***전종득
LeeChae Jang · TaeKyun Kim · JongDuek Jeon

*건국대 전산수학과
**공주대 과학교육연구소
***경희대 수학과

요 약

이 논문에서 구간 수의 값을 갖는 함수들의 쇼케이적분을 생각하고자 한다. 이러한 구간 수의 값을 갖는 함수들의 성질들을 조사하여 오토연속인 퍼지측도에 관련된 쇼케이적분에 대한 수렴성 정리를 증명한다.

Abstract

In this paper, we consider Choquet integrals of interval number-valued functions(simply, interval number-valued Choquet integrals). Then, we prove convergence theorem for interval number-valued Choquet integrals with respect to an autocontinuous fuzzy measure.

Keywords : fuzzy measures, autocontinuous, Choquet integrals, Hausdorff metric, convergence theorem.

1. Introduction.

It is well-known that closed set-valued functions had been used repeatedly in many papers [1, 2, 5, 6, 7, 8, 9, 13, 15, 16]. We studied closed set-valued Choquet integrals in [7, 8] and convergence theorems under some sufficient conditions, for examples ; (i) convergence theorems for monotone convergent sequences of Choquet integrably bounded closed set-valued functions(see [7]), (ii) covergence theorems for the upper limit and the lower limit of a sequence of Choquet integrably bounded closed set-valued functions (see [9]).

The aim of this paper is to prove convergence theorem for convergent sequences of Choquet integrably bounded interval number-valued functions in the metric $\triangle_S$ (see Definition 3.4). In section 2, we list various definitions and notations which are used in the proof of convergence theorem and discuss some properties of measurable interval number-valued functions. In section 3, using these definitions and properties, we investigate main results.

2. Definitions and preliminaries.

Definition 2.1 [8, 12] (1) A fuzzy measure on a measurable space $(X, \mathfrak{I})$ is an extended real-valued function $\mu : \mathfrak{I} \rightarrow [0,\infty]$ satisfying

(i) $\mu(\emptyset)=0$,

(ii) $\mu(A) \leq \mu(B)$, whenever $A, B \in \mathfrak{I}$, $A \subset B$.

(2) A fuzzy measure μ is said to be autocontinuous from above[resp.,below] if $\mu(A \cup B_n) \rightarrow \mu(A)$[resp., $\mu(A \sim B_n) \rightarrow \mu(A)$] whenever $A \in \mathfrak{I}$, $\{B_n\} \subset \mathfrak{I}$ and $\mu(B_n) \rightarrow 0$.

(3) If μ is autocontinuous both from above and from below, it is said to be autocontinuous.

Recall that a function $f : X \rightarrow [0,\infty]$ is said to be measurable if $\{x|f(x)>a\} \in \mathfrak{I}$ for all $a \in (-\infty,\infty)$.

Definition 2.2 [12] (1) A sequence $\{f_n\}$ of measurable functions is said to converge to f in measure, in symbols $f_n \rightarrow_M f$ if for every $\varepsilon > 0$,

$$\lim_{n\rightarrow\infty} \mu(\{x| \ |f_n(x)-f(x)| > \varepsilon\}) = 0.$$

(2) A sequence $\{f_n\}$ of measurable functions is said to converge to f in distribution, in symbols $f_n \rightarrow_D f$ if for every $\varepsilon > 0$,

$$\lim_{n\rightarrow\infty} \mu_{f_n}(r) = \mu_f(r) \quad e.c.,$$

where $\mu_f(r) = \mu(\{x|f(x) > r\})$ and "e.c." stands for "except at most countably many values of r".

Definition 2.3 [10,11,12] (1) The Choquet integral of a

접수일자 : 2001년 10월 26일
완료일자 : 2002년 5월 2일

measurable function f with respect to a fuzzy measure μ is defined by

$$(C)\int f d\mu = \int_0^\infty \mu_f(r)\,dr$$

where the integral on the right-hand side is an ordinary one.

(2) A measurable function f is called integrable if the Choquet integral of f can be defined and its value is finite.

Throughout the paper, R^+ will denote the interval $[0,\infty)$,

$$I(R^+) = \{[a,b] \mid a, b \in R^+ \text{ and } a \le b\}.$$

Then a element in $I(R^+)$ is called an interval number. On the interval number set, we define; for each pair $[a,b],[c,d] \in I(R^+)$ and $k \in R^+$,

$$[a,b]+[c,d]=[a+c,b+d],$$
$$[a,b]\cdot[c,d]=[a\cdot c, b\cdot d],$$
$$k[a,b]=[ka,kb],$$
$$[a,b]\le[c,d] \text{ if and only if } a\le c \text{ and } b \le d,$$

Then $(I(R^+), d_H)$ is a metric space, where d_H is the Hausdorff metric defined by

$$d_H(A,B)=\max\{\sup_{x\in A}\inf_{y\in B}|x-y|, \sup_{y\in B}\inf_{x\in A}|x-y|\}$$

for all $A, B \in I(R^+)$. By the definition of the Hausdorff metric, we have immediately the following proposition.

Proposition 2.4 For each pair $[a,b],[c,d]\in I(R^+)$,

$$d_H([a,b],[c,d]) = \max\{|a-c|,|b-d|\}.$$

Let $C(R^+)$ be the class of closed subsets of R^+. Throughout this paper, we consider a closed set-valued function $F: X \to C(R^+)\setminus\{\emptyset\}$ and an interval number-valued function $F: X\to I(R^+)\setminus\{\emptyset\}$. We denote that $d_H - \lim_{n\to\infty} A_n = A$ if and only if $\lim_{n\to\infty} d_H(A_n, A) = 0$, where $A \in I(R^+)$ and $\{A_n\}\subset I(R^+)$.

Definition 2.5 [1,6,7] A closed set-valued function F is said to be measurable if for each open set $O \subset R^+$,

$$F^{-1}(O) = \{x \in X \mid F(x)\cap O \ne \emptyset\} \in \mathfrak{I}.$$

Definition 2.6 [1] Let F be a closed set-valued function. A measurable function $f: X\to R^+$ satisfying

$$f(x)\in F(x) \text{ for all } x \in X$$

is called a measurable selection of F.

We say $f: X \to R^+$ is in $L_c^1(\mu)$ if and only if f is measurable and $(C)\int f\,d\mu < \infty$. We note that "$x\in X$ $\mu-a.e.$" stands for "$x \in X$ μ-almost everywhere". The property $P(x)$ holds for $x\in X$ $\mu-a.e.$ means that there is a measurable set A such that $\mu(A)=0$ and the property $P(x)$ holds for all $x \in A^c$, where A^c is the complement of A.

Definition 2.7 [6,7] (1) Let F be a closed set-valued function and $A \in \mathfrak{I}$. The Choquet integral of F on A is defined by

$$(C)\int_A F d\mu = \{(C)\int_A f d\mu \mid f \in S_c(F)\}$$

where $S_c(F)$ is the family of $\mu-a.e.$ Choquet integrable selections of F, that is,

$$S_c(F) = \{f \in L_c^1(\mu) \mid f(x)\in F(x)\ x\in X\ \mu-a.e.\}$$

(2) A closed set-valued function F is said to be Choquet integrable if $(C)\int F d\mu \ne \emptyset$.

(3) A closed set-valued function F is said to be Choquet integrably bounded if there is a function $g\in L_c^1(\mu)$ such that

$$\|F(x)\| = \sup_{r\in F(x)}|r| \le g(x) \quad \text{for all } x\in X.$$

Instead of $(C)\int_X F d\mu$, we will write $(C)\int F d\mu$. Let us discuss some basic properties of measurable closed set-valued functions. Since $R^+=[0,\infty)$ is a complete separable metric space in the usual topology, using Theorem 8.1.3([1])and Theorem 1.0(2^0)([5]), we have the following two theorems.

Theorem 2.8 [1,5] A closed set-valued function F is measurable if and only if there exists a sequence of measurable selections $\{f_n\}$ of F such that

$$F(x) = \mathrm{cl}\{f_n(x)\} \quad \text{for all } x \in X.$$

Theorem 2.9 [1,5] If F is a measurable closed set-valued function and Choquet integrably bounded, then it is Choquet integrable.

3. Main results.

In this section, we prove convexity of interval number-valued Choquet integrals and discuss the concepts of convergent sequences of measurable interval number-valued functions in the metric Δ_S. Since $(X, \mathfrak{I})$ is a measurable space and R^+ is a separable metric space, Theorem 1.0(2^0)([5]) implies the following theorem. Recall that a measurable closed set-valued function is said to be convex-valued if $F(x)$

is convex for all $x \in X$ and that a set A is an interval number if and only if it is closed and convex.

Theorem 3.1 If F is a measurable closed set-valued function and Choquet integrably bounded, then there exists a sequence $\{f_n\}$ of Choquet integrable functions $f_n : X \to R^+$ such that $F(x) = \mathrm{cl}\{f_n(x)\}$ for all $x \in X$.

Proof. By Theorem 1.0 (2^0) ([5]), there exists a sequence $\{f_n\}$ of measurable functions $f_n : X \to R^+$ such that $F(x) = \mathrm{cl}\{f_n(x)\}$ for all $x \in X$. Since F is Choquet integrably bounded, there is a measurable function $g \in L^1_c(\mu)$ such that

$$\|F(x)\| = \sup\{r | r \in F(x)\} \le g(x), \qquad \text{for all } x \in X$$

Since $f_n(x) \in F(x)$ for all $x \in X$ and all $n = 1, 2, \cdots$, $f_n(x) \le g(x)$ for all $x \in X$. By Proposition 3.2([11]),

$$(C)\int f_n d\mu \le (C)\int g d\mu < \infty, \qquad \text{for all } n = 1, 2, \cdots$$

So, f_n is Choquet integrable for all $n = 1, 2, \cdots$. The proof is complete.

Theorem 3.2 If F is a measurable closed set-valued function and Choquet integrably bounded and if we define

$$f^*(x) = \sup\{r | r \in F(x)\}$$

and

$$f_*(x) = \inf\{r | r \in F(x)\}$$

for all $x \in X$, then f^* and f_* are Choquet integrable selections of F.

Proof. Since F is Choquet integrably bounded, there exists a function $g \in L^1_c(\mu)$ such that $\|F(x)\| \le g(x)$ for all $x \in X$. Theorem 3.1 implies that there is a sequence $\{f_n\}$ of Choquet integrable selections of F such that

$$F(x) = \mathrm{cl}\{f_n(x)\} \text{ for all } x \in X.$$

Then

$$f^*(x) = \sup\{r | r \in F(x)\} = \sup_n f_n(x)$$

and

$$f_*(x) = \inf\{r | r \in F(x)\} = \inf_n f_n(x).$$

Since the supremun and the infimum of a sequence $\{f_n\}$ of measurable functions are measurable, f^* and f_* are measurable. And also, we have

$$0 \le f_*(x) \le f^*(x) = \|F(x)\| \le g(x) \qquad \text{for all } x \in X.$$

Since $g \in L^1_c(\mu)$, f^* and f_* belong to $L^1_c(\mu)$. By the closedness of $F(x)$ for all $x \in X$, $f_*(x) \in F(x)$ and $f^*(x) \in F(x)$ for all $x \in X$. Therefore, f^* and f_* are Choquet integrable selections of F.

Assumption (A) For each pair $f, g \in S_c(F)$, there exists $h \in S_c(F)$ such that $f \sim h$ and $(C)\int g d\mu = (C)\int h d\mu$.

We consider the following classes of interval number-valued functions;

$$\mathfrak{I} = \{F \mid F : X \to I(R^+) \text{ is measurable and Choquet integraby bounded}\}$$

and

$$\mathfrak{I}_1 = \{F \in \mathfrak{I} \mid F \text{ satisfies the assuption(A)}\}.$$

Theorem 3.3 If $F \in \mathfrak{I}_1$, then we have

(1) $cF \in \mathfrak{I}_1$ for all $c \in R^+$,

(2) $(C)\int F d\mu$ is convex,

(3) $(C)\int F d\mu = [(C)\int f_* d\mu, (C)\int f^* d\mu]$.

Proof. (1) The proof of (1) is trivial.

(2) If $(C)\int F\mu$ is a single point set, then it is convex. Otherwise, let $y_1, y_2 \in (C)\int F d\mu$ and $y_1 < y_2$. Then, there exist $f_1, f_2 \in S_c(F)$ such that

$$y_1 = (C)\int f_1 d\mu \text{ and } y_2 = (C)\int f_2 d\mu.$$

Further, let $y \in (y_1, y_2)$ we need to a selection $f \in S_c(F)$ with $y = (C)\int f d\mu$. Since $y \in (y_1, y_2)$, there exists $\lambda_0 \in (0,1)$ such that $y = \lambda_0 y_1 + (1-\lambda_0) y_2$. For above two selections $f_1, f_2 \in S_c(F)$, the assumption (A) implies that there exists $g \in S_c(F)$ such that $f_1 \sim g$ and $(C)\int g d\mu = (C)\int f_2 d\mu$. We define a function $f = \lambda_0 f_1 + (1-\lambda_0) g$ and note that $\lambda_0 f_1 \sim (1-\lambda_0) g$. Since F is interval number-valued, it is convex and hence

$$f(x) = \lambda_0 f_1(x) + (1-\lambda_0) g(x) \in F(x)$$

for $x \in X$ μ-a.e. By Theorem 5.6 [11] and Proposition 3.2 (2)[11],

$$\begin{aligned} y &= \lambda_0 y_1 + (1-\lambda_0) y_2 \\ &= (C)\int \lambda_0 f_1 d\mu + (C)\int (1-\lambda_0) f_2 d\mu \\ &= \lambda_0 (C)\int f_1 d\mu + (1-\lambda_0)(C)\int f_2 d\mu \\ &= \lambda_0 (C)\int f_1 d\mu + (1-\lambda_0)(C)\int g d\mu \\ &= (C)\int \lambda_0 f_1 d\mu + (C)\int (1-\lambda_0) g d\mu \\ &= (C)\int (\lambda_0 f_1 + (1-\lambda_0) g) d\mu \\ &= (C)\int f d\mu \end{aligned}$$

Thus, we have $f \in S_c(F)$ and

$$y = (C)\int f d\mu \in (C)\int F d\mu.$$

The proof of (2) is complete.

(3) We note that $f_* \le f \le f^*$ for all $f \in S_c(F)$. Thus, by Proposition 3.2(2)[11],

$$(C)\int f_* d\mu \le (C)\int f d\mu \le (C)\int f^* d\mu$$

for all $f \in S_c(F)$. Theorem 3.2 implies

$$(C)\int f_* d\mu, (C)\int f^* d\mu \in (C)\int F d\mu.$$

By (2), $(C)\int F d\mu$ is convex in R^+ and hence

$$(C)\int F d\mu = [(C)\int f_* d\mu, (C)\int f^* d\mu].$$

We consider a function Δ_S on $\mathfrak{I}_1$ defined by

$$\Delta_S(F, G) = \sup_{x \in X} d_H(F(x), G(x))$$

for all $F, G \in \mathfrak{I}_1$. Then, it is easily to show that Δ_S is a metric on $\mathfrak{I}_1$.

Definition 3.4 Let $F \in \mathfrak{I}_1$. A sequence $\{F_n\} \subset \mathfrak{I}_1$ converges to F in the metric Δ_S , in symbols, $F_n \rightarrow_{\Delta_S} F$ if $\lim_{n\to\infty} \Delta_S(F_n, F) = 0$.

Theorem 3.5(Convergence Theorem) Let $F, G, H \in \mathfrak{I}_1$ and $\{F_n\}$ be a sequence in $\mathfrak{I}_1$. If a fuzzy measure μ is autocontinuous and if $F_n \rightarrow_{\Delta_S} F$ and $G \le F_n \le H$, then we have

$$d_H - \lim_{n\to\infty} (C)\int F_n d\mu = (C)\int F d\mu.$$

Proof. By Proposition 2.4,

$$d_H(F_n(x), F(x)) = \max\{ |f_{n*}(x) - f_*(x)|, |f_n^*(x) - f^*(x)| \}$$

for all $x \in X$, where

$$f_{n*}(x) = \inf\{r | r \in F_n(x)\},$$
$$f_n^*(x) = \sup\{r | r \in F_n(x)\}$$

for $n = 1, 2, \cdots$,

$$f_*(x) = \inf\{r | r \in F(x)\},$$
$$f^*(x) = \sup\{r | r \in F(x)\}.$$

Since $\Delta_S(F_n, F) \rightarrow 0$ as $n \rightarrow \infty$,

$$\sup_{x\in X} |f_{n*}(x) - f_*(x)| \rightarrow 0$$

and

$$\sup_{x\in X} |f_n^*(x) - f^*(x)| \rightarrow 0.$$

Given any $\varepsilon > 0$, there exist two natural numbers N_1, N_2 such that $|f_{n*}(x) - f_*(x)| < \varepsilon$ for all $n \ge N_1$ and all $x \in X$, and $|f_n^*(x) - f^*(x)| < \varepsilon$ for all $n \ge N_2$ and all $x \in X$. We put $N = \max\{N_1, N_2\}$. Thus for each $n \ge N$,

$$\mu\{x \mid |f_{n*}(x) - f_*(x)| > \varepsilon\} = \mu(\emptyset) = 0$$

and

$$\mu\{x \mid |f_n^*(x) - f^*(x)| > \varepsilon\} = \mu(\emptyset) = 0.$$

Then, clearly we have that for arbitrary $\varepsilon > 0$,

$$\mu\{x \mid |f_{n*}(x) - f_*(x)| \ge \varepsilon\} \rightarrow 0 \text{ as } n \rightarrow \infty$$

and

$$\mu\{x \mid |f_n^*(x) - f^*(x)| \ge \varepsilon\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

That is, $f_{n*} \rightarrow_M f_*$ and $f_n^* \rightarrow_M f^*$ as $n \rightarrow \infty$. It is clearly to show that if $G \le F_n \le H$ then

$$\mu_{g_*}(r) \le \mu_{f_{n*}}(r) \le \mu_{h_*}(r)$$

and

$$\mu_{g^*}(r) \le \mu_{f_n^*}(r) \le \mu_{h^*}(r) \text{ for all } r \in R^+,$$

where

$$g_*(x) = \inf\{r | r \in G(x)\},$$
$$g^*(x) = \sup\{r | r \in G(x)\},$$
$$h_*(x) = \inf\{r | r \in H(x)\},$$
$$h^*(x) = \sup\{r | r \in H(x)\}.$$

Since μ is autocontinuous, by Theorem 3.2[12], we have

$$\lim_{n\to\infty} (C)\int f_{n*} d\mu = (C)\int f_* d\mu$$
$$\text{and } \lim_{n\to\infty} (C)\int f_n^* d\mu = (C)\int f^* d\mu.$$

Therefore,

$$d_H[(C)\int F_n d\mu, (C)\int F d\mu]$$
$$= \max\{|(C)\int f_{n*} d\mu - (C)\int f_* d\mu|, |(C)\int f_n^* d\mu - (C)\int f^* d\mu|\}$$
$$\rightarrow 0$$

as $n \rightarrow \infty$.

ACKNOWLEDGEMENT : This work was supported by grant No. 2001-1-10100-003-1 from the Basic Research Program of the Korea Science and Engineering Foundation.

References.

[1] J. Aubin, Set-valued analysis, 1990, Birkauser Boston.

[2] R. J. Aumann, Integrals of set-valued functions, J. Math. Appl. 12 (1965)1-12.

[3] L. M. Campos and M. J. Bolauos, Characterization and comparision of Sugeno and Choquet integrals, Fuzzy Sets and Systems 52 (1992) 61-67.

[4] G. Choquet, Theory of capacities, Ann. Inst. Fourior, 5(1953) 131-295.

[5] F. Hiai and H. Umegaki, Integrals, conditional expectations, and martingales of multivalued functions, J. Multi. Analysis 7(1977) 149-182.

[6] L. C. Jang, B. M. Kil, Y. K. Kim and J. S. Kwon, Some properties of Choquet integrals of set-valued functions, Fuzzy Sets and Systems 91 (1997) 95-98.

[7] L. C. Jang and J. S. Kwon, On the representation of Choquet integrals of set-valued functions and null sets, Fuzzy Sets and Systems 112(2000)233-239.

[8] L. C. Jang and T. Kim, On set-valued Choquet intgerals and convergence theorems (I), submitted to Fuzzy Sets and Systems.

[9] L. C. Jang and T. Kim, Convexity and additivity of set-valued Choquet intgerals, submitted to Chinese Annals of Mathematics.

[10] T. Murofushi and M. Sugeno, An interpretation of fuzzy measures and the Choquet integral as an integral with respect to a fuzzy measure, Fuzzy Sets and Systems 29 (1989) 201-227.

[11] T. Murofushi and M. Sugeno, A theory of Fuzzy measures: representations, the Choquet integral, and null sets, J. Math. Anal. and Appl. 159 (1991) 532-549.

[12] T. Murofushi, M. Sugeno and M. Suzaki, Autocontinuity, convergence in measure, and convergence in distribution, Fuzzy Sets and Systems 92 (1997) 197-203.

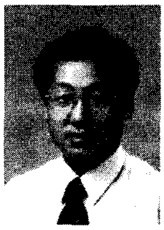
[13] P. Pucci and G. Vitillaro, A representation theorem for Aumann integrals, J. Math.Anal. and Appl. 102 (1984) 86-101.

[14] Z. Wang and G.J. Klir, Fuzzy measure theory, Plenum Press (1992)

[15] C. Wu, D. Zhang, C. Guo, and C. Wu, Fuzzy number fuzzy measures and fuzzy integrals.(I). Fuzzy integrals of functions with respect to fuzzy number fuzzy measures, Fuzzy Sets and Systems 98 (1998) 355-360.

[16] D. Zhang and Z. Wang, On set-valued fuzzy integrals, Fuzzy Sets and Systems 56 (1993) 237-241.

저 자 소 개



장이채(Lee Chae Jang)

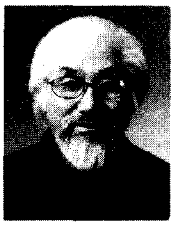
1979년 2월 : 경북대 수학과(이학사)
1981년 2월 : 경북대 대학원 수학과 (이학석사)
1987년 2월 : 경북대 대학원 수학과 (이학박사)
1987년 6월-1998년 6월 : 미국 Cincinnati 대(교환교수)
1987년 3월-현재 : 건국대학교 컴퓨터응용과학부 전산수학 전공교수

관심분야 : 해석학, 퍼지측도와 쇼케이적분, 퍼지이론과 지식공간, 수학교육 등
E-mail : leechae.jang@kku.ac.kr

김태균(Tae Kyun Kim)

1987년 2월 : 경북대 천연섬유학과(농학사)
1989년 2월 : 경북대 대학원 수학과 (이학석사)
1994년 3월 : 일본 九州대학교 대학원 (이학박사)
1999년 3월-2000년 12월 : Simon Fraser Univ.(CECM), Visiting Scolar
2001년 4월-현재 : 공주대학교, 전임연구교수

관심분야 : 정수론, 쇼케이적분, 퍼지이론과 지식공간, 수학교육, 수리물리 등
E-mail : taekyun64@hotmail.net



전종득(Jong Duek Jeon)

1964년 : 경희대 수학과(이학사)
1968년 : 경희대 대학원 수학과(이학석사)
1983년 : 고려대 대학원 수학과(박사과정 수료)
1995년 : 순천향대학교 대학원 수학과(이학박사)
1988년 12월-1989년 12월 : 미국 Kent 주립대(교환교수)
1980년 3월-현재 : 경희대학교 이학부 수학전공교수
관심분야 : 퍼지측도론, 신경망, 인공지능 등
E-mail : jdjeon@nms.kyunghee.ac.kr