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Cubic Equations in General Saddlepoint Approximations

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Abstract

This paper discusses cubic equations in general saddlepoint approximations. Exact roots are found for various cases by trigonometric identities, the root which is appropriate for the general saddlepoint approximations is selected and discussed, and the defective cases in which the general saddlepoint approximations cannot be used are found.

Keywords : Cubic equation, Cumulant generating function, General saddlepoint approximation

1. Introduction

For the approximation to the density f_n of some statistic $V_n(X_1, \ldots, X_n)$, where $X_1, \ldots, X_n$ are n independent identically distributed (iid) random variables with the underlying density f, usually asymptotic theory based on the central limit theorem is used. But very often the central limit theorem requires (very) large sample size for a good approximation. Moreover, these approximations tend to be inaccurate in the tails of the distribution. The Edgeworth expansion can be used to increase the accuracy of the approximation to the density f_n. In general, the Edgeworth expansion provides a good approximation in the center of the density, but it can be inaccurate in the tails when the sample size is moderate to small. Saddlepoint asymptotic techniques overcome this problem and give a good approximation even when the sample size is small. The first few terms (or even just the leading term) often give very accurate approximations in the far tails of the distribution even for very small sample sizes. One Empirical reason for this is that saddlepoint approximations are density-like objects and do not show the polynomial-like waves exhibited by Edgeworth approximations.

Saddlepoint approximations were first introduced into statistics by Daniels (1954). Daniels (1954) used saddlepoint techniques to derive a very accurate approximation to f_n when V_n is the mean of $X_1, \ldots, X_n$. After Daniel's (1954) paper, saddlepoint techniques were applied to several types of problems. For a general review, see Reid (1988).

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Saddlepoint approximations to the tail probability or cumulative distribution function of the sample mean were derived by Lugannani and Rice (1980) and Daniels (1987). The approximations given by them are very accurate.

Easton and Ronchetti (1986) derived a saddlepoint approximation for the density f_n of a general statistic $V_n(X_1, \ldots, X_n)$. The saddlepoint approximation utilizes the cumulant generating function of V_n, but in general it is unknown. Easton and Ronchetti (1986) approximated the cumulant generating function of V_n using the first four cumulants of V_n.

Combining the methods of Easton and Ronchetti (1986) and Daniels (1987), Na (1998) derived a saddlepoint approximation for the cumulative distribution function of V_n.

The general saddlepoint approximations by Easton and Ronchetti (1986) and Na (1998) require the solution of cubic equations if the cumulant generating function of V_n is approximated using the first four cumulants of V_n. In addition, the existence and selection of the root which is appropriate for the general saddlepoint approximations should be checked. But Easton and Ronchetti (1986) and Na (1998) do not give a detailed explanation of these points.

This paper discusses cubic equations in general saddlepoint approximations. Exact roots are found by trigonometric identities. So the numerical solution of cubic equations is not necessary. The existence and selection of the root which is appropriate for the general saddlepoint approximations are examined. In Section 2, general saddlepoint approximations to densities and cumulative distribution functions are introduced and the necessity of solving cubic equations is explained. In Section 3, exact roots are found for various cases, the selection of the appropriate root for the saddlepoint approximations is discussed, and the defective cases in which the general saddlepoint approximations cannot be used are found. A conclusion is given in Section 4.

2. General Saddlepoint approximations

Easton and Ronchetti (1986) derived the following saddlepoint approximation for the density f_n of a general statistic V_n by applying the saddlepoint technique to the Fourier inversion of the characteristic function of V_n.

$$f_n(x) = \left[\frac{n}{2\pi R_n''(t_0)}\right]^{1/2} \exp[n\{R_n(t_0) - t_0 x\}]\{1 + O(n^{-1})\} \tag{1}$$

where t_0 is determined as a solution to

$$R_n'(t_0) = x, \tag{2}$$

$R_n'(t_0)$ and $R_n''(t_0)$ are the first two derivatives of $R_n(t) = K_n(nt)/n$ at t_0, and K_n is the cumulant generating function of V_n.

Using the method of Daniels (1987) for the saddlepoint approximation to the cumulative distribution function of a sample mean, Na (1998) obtained the following approximation to the cumulative distribution function of V_n.

$$\Pr\{V_n \le x\} = \begin{cases} \Phi(w) + \phi(w)\left\{\frac{1}{w} - \frac{1}{\zeta} + O(n^{-3/2})\right\}, & x \ne E(V_n) \\ \frac{1}{2} - \frac{R_n^{(3)}(0)}{6\sqrt{2\pi(R_n''(0))^3}}\{1 + O(n^{-3/2})\}, & x = E(V_n) \end{cases} \tag{3}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the density and cumulative distribution function of the standard normal distribution, respectively, $R_n^{(3)}(0)$ is the third derivative of $R_n(t)$ at 0,

$$w = [2n\{t_0 x - R_n(t_0)\}]^{1/2}\,\mathrm{sgn}(t_0),$$

and

$$\zeta = t_0\{nR_n''(t_0)\}^{1/2}.$$

The problem in (1) and (3) is that in general K_n is unknown. Easton and Ronchetti (1986) approximated K_n by

$$\widetilde{K}_n(t) = k_{1n}t + \frac{k_{2n}}{2!}t^2 + \frac{k_{3n}}{3!}t^3 + \frac{k_{4n}}{4!}t^4$$

where k_{1n}, k_{2n}, k_{3n}, and k_{4n} are (approximations to) the mean, the variance, and the third and fourth cumulants of V_n. Replacing K_n by $\widetilde{K}_n$ in (1), f_n is approximated by

$$\widetilde{f}_n(x) = \left[\frac{n}{2\pi\widetilde{R}_n''(t_0)}\right]^{1/2} \exp[n\{\widetilde{R}_n(t_0) - t_0 x\}]$$

where t_0 is determined as a solution to

$$\widetilde{R}_n'(t_0) = x, \tag{4}$$

$\widetilde{R}_n'(t_0)$ and $\widetilde{R}_n''(t_0)$ are the first two derivatives of $\widetilde{R}_n(t) = \widetilde{K}_n(nt)/n$ at t_0. Na (1998) also replaced K_n by $\widetilde{K}_n$ in (3) and obtained similar result for the cumulative distribution function of V_n.

Since K_n is convex, the solution of the equation (2) is unique and real. But the cubic equation (4) can have multiple real solutions and only the solution t_0 with $\widetilde{R}_n''(t_0) > 0$ can be used for the saddlepoint approximations. In the next section, the exact solutions of the cubic equation (4) are given, the selection of the appropriate root for the general saddlepoint approximations is discussed, and the defective cases in which the solution t_0 with $\widetilde{R}_n''(t_0) > 0$ does not exist are found. The formula

$$(a+b)^3 - 3ab(a+b) = a^3 + b^3$$

or

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

can be used to solve cubic equations. But it is tedious to find exact roots of a cubic equation by these formulae. Instead, the use of trigonometric identities is easier and straightforward. Birkhoff and Mac Lane (1977) gives a brief explanation of cubic equations.

3. Cubic Equations

Let $g(t)=\widetilde{R}_n'(t)-x$. Then t_0 is determined as a solution to $g(t)=0$ and $\widetilde{R}_n''(t_0)>0$ is equivalent to $g'(t_0)>0$. The function $g(t)$ is written as

$$g(t)=\frac{k_{4n}}{3!}\left\{(nt)^3+3\frac{k_{3n}}{k_{4n}}(nt)^2+6\frac{k_{2n}}{k_{4n}}(nt)+6\frac{k_{1n}-x}{k_{4n}}\right\}.$$

Now, by making the substitution $nt=y-(k_{3n}/k_{4n})$, $g(t)$ is rewritten as

$$g(t)=\frac{k_{4n}}{3!}(y^3+py-q)=\frac{k_{4n}}{3!}\left\{\left(nt+\frac{k_{3n}}{k_{4n}}\right)^3+p\left(nt+\frac{k_{3n}}{k_{4n}}\right)-q\right\}$$

where

$$p=3\left\{-\left(\frac{k_{3n}}{k_{4n}}\right)^2+2\frac{k_{2n}}{k_{4n}}\right\}$$

and

$$q=2\left\{-\left(\frac{k_{3n}}{k_{4n}}\right)^3+3\frac{k_{3n}k_{2n}}{k_{4n}^2}-3\frac{k_{1n}-x}{k_{4n}}\right\}.$$

Case 1 : $p=q=0$

The cubic equation $g(t)=0$ has a triple root $t_0=-k_{3n}/(nk_{4n})$ and $g'(t_0)=0$. In this case, the solution t_0 with $\widetilde{R}_n''(t_0)>0$ does not exist, and so the saddlepoint approximations by Easton and Ronchetti (1986) and Na (1998) cannot be used.

Case 2 : $p=0$ and $q\neq 0$

The cubic equation $g(t)=0$ has two conjugate imaginary roots

$$n^{-1}\{q^{1/3}e^{2\pi i/3}-(k_{3n}/k_{4n})\},\ n^{-1}\{q^{1/3}e^{4\pi i/3}-(k_{3n}/k_{4n})\}$$

where $i=\sqrt{-1}$, and a real root

$$t_0=n^{-1}\{q^{1/3}-(k_{3n}/k_{4n})\}. \tag{5}$$

The value $g'(t_0)$ is positive (negative) if k_{4n} is positive (negative). Hence, if k_{4n} is positive, (5) is the solution t_0 with $\widetilde{R}_n''(t_0)>0$. If k_{4n} is negative, the solution t_0 with $\widetilde{R}_n''(t_0)>0$ does not exist, and so the saddlepoint approximations by Easton and Ronchetti (1986) and Na (1998) cannot be used.

If $p \neq 0$, set $y = 2(|p|/3)^{1/2} z$. Then

$$g(t) = \frac{k_{4n}}{3!} 2\left(\frac{|p|}{3}\right)^{3/2} \{4z^3 + 3 \operatorname{sgn}(p) z - C\}$$

where

$$z = \frac{1}{2}\left(\frac{3}{|p|}\right)^{1/2}\left(nt + \frac{k_{3n}}{k_{4n}}\right) \quad \text{and} \quad C = \frac{q}{2}\left(\frac{3}{|p|}\right)^{3/2}.$$

Case 3 : $p > 0$

The function $g(t)$ is strictly increasing (decreasing) if k_{4n} is positive (negative). The cubic equation $g(t) = 0$ is equivalent to $4z^3 + 3z = C$. Using the identity

$$4 \sinh^3 u + 3 \sinh u = \sinh 3u = \sinh(3u - 2l\pi i)$$

for any integer l, the solutions are

$$z = \sinh\left(\frac{1}{3} \sinh^{-1} C\right) \cos \frac{2l\pi}{3} + i \cosh\left(\frac{1}{3} \sinh^{-1} C\right) \sin \frac{2l\pi}{3}.$$

The real root is

$$z_0 = \sinh\left(\frac{1}{3} \sinh^{-1} C\right) = \frac{(C + \sqrt{C^2+1})^{1/3} - (C + \sqrt{C^2+1})^{-1/3}}{2}.$$

Hence, the cubic equation $g(t) = 0$ has two conjugate imaginary roots and a real root

$$t_0 = n^{-1}\{2(p/3)^{1/2} z_0 - (k_{3n}/k_{4n})\}. \tag{6}$$

The value $g'(t_0)$ is positive (negative) if k_{4n} is positive (negative). Hence, if k_{4n} is positive, (6) is the solution t_0 with $\widetilde{R}_n''(t_0) > 0$. If k_{4n} is negative, the solution t_0 with $\widetilde{R}_n''(t_0) > 0$ does not exist, and so the saddlepoint approximations by Easton and Ronchetti (1986) and Na (1998) cannot be used.

If $p < 0$, $g(t)$ has a local maximum and a local minimum. And the cubic equation $g(t) = 0$ is equivalent to $4z^3 - 3z = C$.

Case 4 : $p < 0$ and $C > 1$

Both the local maximum and minimum values of $g(t)$ are negative (positive) if k_{4n} is positive (negative). Using the identity

$$4 \cosh^3 u - 3 \cosh u = \cosh 3u = \cosh(3u - 2l\pi i) \tag{7}$$

for any integer l, the solutions of the cubic equation $4z^3 - 3z = C$ are

$$z = \cosh\left(\frac{1}{3} \cosh^{-1} C\right) \cos \frac{2l\pi}{3} + i \sinh\left(\frac{1}{3} \cosh^{-1} C\right) \sin \frac{2l\pi}{3}.$$

The real root is

$$z_0 = \cosh\left(\frac{1}{3}\cosh^{-1}C\right) = \frac{(C+\sqrt{C^2-1})^{1/3} + (C+\sqrt{C^2-1})^{-1/3}}{2}.$$

Hence, the cubic equation $g(t)=0$ has two conjugate imaginary roots and a real root

$$t_0 = n^{-1}\{2(-p/3)^{1/2}z_0 - (k_{3n}/k_{4n})\}. \tag{8}$$

The value $g'(t_0)$ is positive (negative) if k_{4n} is positive (negative). Hence, if k_{4n} is positive, (8) is the solution t_0 with $\widetilde{R_n}''(t_0) > 0$. If k_{4n} is negative, the solution t_0 with $\widetilde{R_n}''(t_0) > 0$ does not exist, and so the saddlepoint approximations by Easton and Ronchetti (1986) and Na (1998) cannot be used.

Case 5 : $p<0$ and $C<-1$

Both the local maximum and minimum values of $g(t)$ are positive (negative) if k_{4n} is positive (negative). And the cubic equation $g(t)=0$ is equivalent to $4(-z)^3 - 3(-z) = -C$. Using the identity (7), the solutions are

$$z = -\cosh\left(\frac{1}{3}\cosh^{-1}(-C)\right)\cos\frac{2l\pi}{3} - i\sinh\left(\frac{1}{3}\cosh^{-1}(-C)\right)\sin\frac{2l\pi}{3}$$

The real root is

$$z_0 = -\cosh\left(\frac{1}{3}\cosh^{-1}(-C)\right) = -\frac{(-C+\sqrt{C^2-1})^{1/3} + (-C+\sqrt{C^2-1})^{-1/3}}{2}.$$

Hence, the cubic equation $g(t)=0$ has two conjugate imaginary roots and a real root

$$t_0 = n^{-1}\{2(-p/3)^{1/2}z_0 - (k_{3n}/k_{4n})\}. \tag{9}$$

The value $g'(t_0)$ is positive (negative) if k_{4n} is positive (negative). Hence, if k_{4n} is positive, (9) is the solution t_0 with $\widetilde{R_n}''(t_0) > 0$. If k_{4n} is negative, the solution t_0 with $\widetilde{R_n}''(t_0) > 0$ does not exist, and so the saddlepoint approximations by Easton and Ronchetti (1986) and Na (1998) cannot be used.

Case 6 : $p<0$ and $|C|<1$

The cubic equation $g(t)=0$ has three distinct real roots. Using the identity

$$4\cos^3 u - 3\cos u = \cos 3u,$$

the three distinct real roots of the cubic equation $4z^3 - 3z = C$ are

$$z_1 = -\frac{1}{2}\cos\left(\frac{1}{3}\cos^{-1}C\right) - \frac{\sqrt{3}}{2}\sin\left(\frac{1}{3}\cos^{-1}C\right)$$

$$z_2 = -\frac{1}{2}\cos\left(\frac{1}{3}\cos^{-1}C\right) + \frac{\sqrt{3}}{2}\sin\left(\frac{1}{3}\cos^{-1}C\right)$$

and

$$z_3 = \cos\left(\frac{1}{3}\cos^{-1}C\right).$$

Since $0 < \cos^{-1}C < \pi$, it follows that

$$0 < \frac{1}{3}\cos^{-1}C < \frac{\pi}{3} < \frac{2\pi}{3} - \frac{1}{3}\cos^{-1}C < \frac{2\pi}{3} < \frac{2\pi}{3} + \frac{1}{3}\cos^{-1}C < \pi,$$

and hence

$$1 > z_3 > \frac{1}{2} > z_2 > -\frac{1}{2} > z_1 > -1.$$

Let $t_j = n^{-1}\{2(-p/3)^{1/2}z_j - (k_{3n}/k_{4n})\}$, $j=1,2,3$. Then $t_1 < t_2 < t_3$ are the three distinct real roots of the cubic equation $g(t)=0$.

If k_{4n} is positive, $g'(t_1)$ and $g'(t_3)$ are positive, and $g'(t_2)$ is negative. The smaller the absolute value of t is, the more accurate the approximation of K_n by $\widetilde{K}_n$ is. For this reason, if $|t_1| < |t_3|$, t_1 should be chosen as the t_0 for the general saddlepoint approximations, and if $|t_3| < |t_1|$, t_3 should be chosen as the t_0 for the general saddlepoint approximations.

If k_{4n} is negative, $g'(t_1)$ and $g'(t_3)$ are negative, and $g'(t_2)$ is positive. So t_2 is the solution t_0 with $\widetilde{R}_n''(t_0) > 0$.

Case 7 : $p < 0$ and $C = 1$

In this case, $g(t)$ is reduced to

$$g(t) = \frac{k_{4n}}{3!}2\left(\frac{-p}{3}\right)^{3/2}(z-1)(2z+1)^2.$$

Hence, the cubic equation $g(t)=0$ has a double root and a single real root

$$t_0 = n^{-1}\{2(-p/3)^{1/2} - (k_{3n}/k_{4n})\}. \qquad (10)$$

The value $g'(t_0)$ is positive (negative) if k_{4n} is positive (negative). Hence, if k_{4n} is positive, (10) is the solution t_0 with $\widetilde{R}_n''(t_0) > 0$. If k_{4n} is negative, the solution t_0 with $\widetilde{R}_n''(t_0) > 0$ does not exist, and so the saddlepoint approximations by Easton and Ronchetti (1986) and Na (1998) cannot be used.

Case 8 : $p < 0$ and $C = -1$

In this case, $g(t)$ is reduced to

$$g(t) = \frac{k_{4n}}{3!}2\left(\frac{-p}{3}\right)^{3/2}(z+1)(2z-1)^2.$$

Hence, the cubic equation $g(t)=0$ has a double root and a single real root

$$t_0 = n^{-1}\{-2(-p/3)^{1/2} - (k_{3n}/k_{4n})\}. \qquad (11)$$

The value $g'(t_0)$ is positive (negative) if k_{4n} is positive (negative). Hence, if k_{4n} is positive, (11) is the solution t_0 with $\widetilde{R}_n''(t_0)>0$. If k_{4n} is negative, the solution t_0 with $\widetilde{R}_n''(t_0)>0$ does not exist, and so the saddlepoint approximations by Easton and Ronchetti (1986) and Na (1998) cannot be used.

To summarize, if k_{4n} is positive, the solution t_0 with $\widetilde{R}_n''(t_0)>0$ always exists except the case $p=q=0$. On the other hand, if k_{4n} is negative, the solution t_0 with $\widetilde{R}_n''(t_0)>0$ exists only when $p<0$ and $|C|<1$. Note that the sign of k_{4n} is equal to that of k_{4n}/k_{2n}^2, the kurtosis of the distribution of V_n.

4. Conclusion

Cubic equations in general saddlepoint approximations were discussed. Exact roots were found for various cases and so numerical methods are not necessary for the solution of cubic equations. The selection of the appropriate root for the general saddlepoint approximations was also discussed. The defective cases in which the saddlepoint approximations by Easton and Ronchetti (1986) and Na (1998) cannot be used were found. In general K_n is unknown, and so the approximation of K_n by $\widetilde{K}_n$ proposed by Easton and Ronchetti (1986) is useful although there are some defective cases. Accordingly, the saddlepoint approximations by Easton and Ronchetti (1986) and Na (1998) are available except some defective cases. For the defective cases, other methods of estimating or approximating K_n should be used.

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