The Strong Laws of Large Numbers for Weighted Averages of Dependent Random Variables

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Abstract

We derive the strong laws of large numbers for weighted averages of partial sums of random variables which are either associated or negatively associated. Our theorems extend and generalize strong law of large numbers for weighted sums of associated and negatively associated random variables of Matula(1996; Probab. Math. Statist. 16) and some results in Birkel(1989; Statist. Probab. Lett. 7) and Matula (1992; Statist. Probab. Lett. 15).

Keywords : associated, negatively associated, weighted average, strong law of large number.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on some probability space (Ω, A, P). A finite family $\{X_1, \cdots, X_n\}$ of random variables is called positively associated if $Cov(f(X_1, \cdots, X_n), g(X_1, \cdots, X_n)) \geq 0$ any real coordinatewise nondecreasing functions f, g on R^n such that this covariance exists. It is called negatively associated if for any disjoint subsets $A, B \subset \{1, 2, \cdots, n\}$ and any real coordinatewise nondecreasing functions f on R^A and g on R^B, $Cov(f(X_k, k \in A), g(X_k, k \in B)) \leq 0$. An infinite family of random variables is associated(negatively associated) if every finite subfamily is associated(negatively associated). These concepts of dependence were introduced by Esary, Proschan and Walkup(1967) and Joag-Dev and Proschan(1983). Basic properties of associated and negatively associated random variables may be found in many papers(see [4], [5], [10] and [11]). Strong laws of large numbers for positively and negatively associated random variables were studied by

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Birkel(1989) and Matula(1992), respectively. Almost sure convergences for weighted sums of positively and negatively associated random variables were derived by Matula(1996) and the strong law of large numbers for weighted averages under dependence assumptions was studied by Chandra and Ghosal(1996). Matula(1998) also proved almost sure central limit theorem for associated random variables. Liu, Gan, and Chen(1999) obtained the Hajeck-Renyi inequality and discussed the Marcinkiewicz strong law of large numbers for negatively associated random variables and Christofides(2000) derived a maximal inequality for demimartingales and obtained a maximal inequality and a strong law of large numbers for associaed random variables as special cases.

In this paper we derive strong laws of large numbers for weighted averages of positively and negatively associated sequences. Our theorems extend and generalize some results in Birkel(1988) and Matula(1992, 1996).

2. Results

Lemma 1(Matula(1996)). Assume that $X_1, \cdots, X_n$ are associated zero mean random variables with finite second moments. Then, for every $\varepsilon > 0$

$$P[\max(|S_1|, \cdots, |S_n|) \geq \varepsilon] \leq 8\varepsilon^{-2}ES_n^2,\tag{1}$$

where $S_n = \sum_{i=1}^{n} X_i$.

Theorem 1. Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with finite second moments. Assume that

$$\sum_{j=1}^{\infty} \sum_{i=1}^{j} (f(j))^{-2} Cov(X_i, X_j) < \infty,\tag{2}$$

where $\{f(n)\}$ is a positive sequence increasing to infinity. Then, as $n \to \infty$

$$f(n)^{-1}(S_n - ES_n) \to 0, \qquad \text{almost surely},\tag{3}$$

where $S_n = \sum_{i=1}^{n} X_i$.

Proof Without loss of generality we assume that $EX_n = 0, n \geq 1$. Since $f(n) \to \infty$, for each $n \geq 1$ we may define n_k such that

$$2^k \leq f(n_k) < 2^{k+1}.$$

It is easy to see that if $n_k \geq j$, then $2^{-(k+1)} < f(j)^{-1}$. Let $\varepsilon > 0$ be given, then we have by Chebyshev's inequality

$$\sum_{k=1}^{\infty} P[f(n_k)^{-1}|S_{n_k}| \geq \varepsilon]$$

$$\leq \varepsilon^{-2} \sum_{k=1}^{\infty} f(n_k)^{-2} Var(S_{n_k})$$

$$\leq 2\varepsilon^{-2} \sum_{k=1}^{\infty} \frac{1}{4^k} \sum_{j=1}^{n_k} \sum_{i=1}^{j} Cov(X_i, X_j)$$

$$\leq 2\varepsilon^{-2} \sum_{j=1}^{\infty} (\sum_{k:n_k \geq j} 4^{-k}) \sum_{i=1}^{j} Cov(X_i, X_j) \tag{4}$$

$$\leq \frac{8}{3} \varepsilon^{-2} \sum_{j=1}^{\infty} f(j)^{-2} \sum_{i=1}^{j} Cov(X_i, X_j) < \infty.$$

The Borel-Cantelli lemma implies that $f(n_k)^{-1} S_{n_k} \to 0$ almost surely as $k \to \infty$. Thus it suffices to prove that

$$(f(n_k)^{-1} \max_{n_k < i \leq n_{k+1}} |S_i - S_{n_k}|) \to 0 \quad \text{almost surely as} \quad k \to \infty \tag{5}$$

where $S_{n_k} = \sum_{i=1}^{n_k} X_i$. By applying Lemma 1 and consideration in (4) we get

$$\sum_{k=1}^{\infty} P[f(n_k)^{-1} \max_{n_k < i \leq n_{k+1}} |S_i - S_{n_k}| \geq \varepsilon]$$

$$\leq 8 \varepsilon^{-2} \sum_{k=1}^{\infty} f(n_k)^{-2} E(S_{n_{k+1}} - S_{n_k})^2$$

$$\leq 32 \varepsilon^{-2} \sum_{k=1}^{\infty} f(n_{k+1})^{-2} Var(S_{n_{k+1}}) < \infty,$$

since the X_j are nonnegatively correlated. Again applying the Borel-Cantelli lemma, we obtain (5) which completes the proof of Theorem 1.

Theorem 2. Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with finite second moments and $\{a_n, n \geq 1\}$ a sequence of positive numbers such that

$$\frac{a_n}{b_n} \to 0 \quad \text{and} \quad b_n \to \infty \quad \text{as} \quad n \to \infty, \tag{6}$$

where $b_n = \sum_{i=1}^{n} a_i$. Assume that for $0 < p < 2$

$$\sum_{j=1}^{\infty} \sum_{i=1}^{j} \frac{a_i^{\frac{1}{p}} a_j^{\frac{1}{p}} Cov(X_i, X_j)}{b_j^{\frac{2}{p}}} < \infty. \tag{7}$$

Then, as $n \to \infty$ $b_n^{-\frac{1}{p}} \sum_{k=1}^{n} a_k^{\frac{1}{p}} (X_k - EX_k) \to 0$ almost surely.

Proof. Let $f(n) = b_n^{\frac{1}{p}}$ and $Z_k = a_k^{\frac{1}{p}} X_k$. Since $\{f(n)\}$ is a positive sequence increasing to

infinity and Z_k's are associated the desired result is obtained by Theorem 1.

Remark. Theorem 2 indicates that Theorem 1 generalizes Theorem 1 of Matula(1996).

By setting $p = 1$ Theorem 2 yields the following corollary ;

Corollary 1(Theorem 1 of Matula(1996)). Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with finite second moments and $\{a_n, n \geq 1\}$ a sequence of positive numbers satisfying (6). Assume that

$$\sum_{j=1}^{\infty} \sum_{i=1}^{i} a_i a_j \, Cov(X_i, X_j) / b_j^2 < \infty. \tag{8}$$

Then, as $n \to \infty$ $(S_n^* - ES_n^*)/b_n \to 0$ almost surely, where $S_n^* = \sum_{i=1}^{n} a_i X_i$.

Corollary 2(Theorem 2 of Birkel(1989)). Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with finite second moments. If

$$\sum_{j=1}^{\infty} j^{-2} \sum_{i=1}^{i} Cov(X_i, X_j) < \infty, \tag{9}$$

then $\{X_n, n \geq 1\}$ fulfills the strong law of large number(SLLN), that is $(S_n - ES_n)/n \to 0$ almost surely as $n \to \infty$.

Corollary 3. Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with finite second moments. If

$$\sum_{j=1}^{\infty} \sum_{i=1}^{i} j^{-2} \frac{Cov(X_i, X_j)}{ij} < \infty, \tag{10}$$

then

$$n^{-1} \sum_{k=1}^{n} (X_k - EX_k)/k \to 0 \quad \text{almost surely}.$$

Proof. Let $Z_n = X_n/n$. Clearly Z_n's are associated. It follows from (7) that $\sum_{j=1}^{\infty} \sum_{i=1}^{i} Cov(Z_i, Z_j) / f(j)^2 < \infty$ by setting $f(n) = n$. Thus $n^{-1} \sum_{k=1}^{n} (Z_k - EZ_k) \to 0$ almost surely, by Theorem 1, that is the desired result follows.

Corollary 4. Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with positive means and finite second moments. Assume that

$$\{ES_n\} \quad \text{is a positive sequence increasing to infinity}, \tag{11}$$

$$\sum_{j=1}^{\infty} \sum_{i=1}^{j} \frac{Cov(X_i, X_j)}{(ES_j)^2} < \infty. \tag{12}$$

Then, as $n \to \infty$ $(S_n - ES_n)/ES_n \to 0$ almost surely.

Proof. Put $f(n) = ES_n$. Then the desired result follows according to Theorem 1.

Now let us state analogues of Theorems 1 and 2 in the case of negatively associated random variables. This theorem extends some of the results obtained in Matula(1992).

Theorem 3. Let $\{X_n, n \geq 1\}$ be a sequence of negatively associated random variables with finite second moments. Assume that

$$\sum_{j=1}^{\infty} (f(j))^{-2} Var(X_j) < \infty, \tag{13}$$

where $\{f(n)\}$ is a positive sequence increasing to infinity. Then, as $n \to \infty$

$$(f(n))^{-1}(S_n - ES_n) \to 0 \quad \text{almost surely}.$$

Proof. The proof of Theorem 3 goes the lines of the proof of Theorem 1 and is based on Lemma 4 of Matula(1992) instead of Lemma 1, so we omit details.

Theorem 4. Let $\{X_n, n \geq 1\}$ be a sequence of negatively associated random variables with finite second moments and $\{a_n, n \geq 1\}$ a sequence of positive numbers satisfying (6). Assume that for $0 < p < 2$

$$\sum_{j=1}^{\infty} a_j^{\frac{2}{p}} \frac{Var(X_j)}{b_j^{\frac{2}{p}}} < \infty. \tag{14}$$

Then, as $n \to \infty$ $b_n^{-\frac{1}{p}} \sum_{k=1}^{n} a_k^{\frac{1}{p}}(X_k - EX_k) \to 0$ almost surely.

Proof. The proof of Theorem 4 goes the lines of the proof of Theorem 2 and is based on (14) instead of (7).

Remark. Theorem 4 indicates that Theorem 3 generalizes Theorem 2 of Matula(1996).

Corollary 5(Theorem 2 of Matula(1996)). Let $\{X_n, n \geq 1\}$ be a sequence of negatively associated random variables with finite second moments and $\{a_n, n \geq 1\}$ a sequence of positive number satisfying (6). Assume that

$$\sum_{j=1}^{\infty} \frac{a_j^2}{b_j^2} Var(X_j) < \infty. \tag{15}$$

Then, as $n \to \infty$ $(S_n^* - ES_n^*)/b_n \to 0$ almost surely where $S_n^* = \sum_{i=1}^{n} a_i X_i$.

Corollary 6. Let $\{X_n, n \geq 1\}$ be a sequence of negatively associated random variables with finite second moments. If

$$\sum_{j=1}^{\infty} j^{-2} Var(X_j) < \infty, \tag{16}$$

then $\{X_n, n \geq 1\}$ fulfills the strong law of large number, that is $n^{-1}(S_n - ES_n) \to 0$ almost surely as $n \to \infty$.

Corollary 7. Let $\{X_n, n \geq 1\}$ be a sequence of negatively associated random variables with positive means and finite second moments. Assume that

$$\{ES_n\} \text{ is a positive sequence increasing to infinity} \tag{17}$$

$$\sum_{j=1}^{\infty} (\log(ES_j))^{-2} \frac{Var(X_j)}{(ES_j)^2} < \infty. \tag{18}$$

Then, as $n \to \infty$ $(\log ES_n)^{-1} \sum_{k=1}^{n} (X_k - EX_k)/ES_k \to 0$ almost surely.

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