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On the Conditionally Independent and Positive and Negative Dependence of Bivariate Stochastic Processes[1)]

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Abstract

We introduce a new concept of θ conditionally independent and positive and negative dependence of bivariate stochastic processes and their corresponding hitting times. We have further extended this theory to stronger conditions of dependence similar to those in the literature of positive and negative dependence and developed theorems which relate these conditions. Finally we are given some examples to illustrate these concepts.

Keywords and phrases : hitting times, conditionally independent and positive and negat ive quadrant dependence, conditionally stochastically increasing(decreasing)

1. Introduction

Lehmann[12] introduced the concept of positive(negative) quadrant dependence together with some other dependence concepts. Since then, concepts of this dependence have subsequently been extended to stochastic processes in different directions by many authors. After this a number of aspects of dependence notions have been studied for several decades. For a bibliography of available results see Friday[9]. Recently Ebrahimi [7] defined that $\{\boldsymbol{X(t)}=(X_1(t),X_2(t))\mid t\in\Lambda\}$ is positive(negative) quadrant dependent $(PQD(NQD))$ if

$$P(\bigcap_{i=1}^{2}(T_i(a_i)>x_i))\geq(\leq)\prod_{i=1}^{2}P(T_i(a_i)>x_i)$$

where $T_i(a_i)=\inf\{t\in\Lambda\mid X_i(t)\leq a_i, 0\leq t\leq\infty\}$, $i=1,2$.

Consider a system of two components with random time $T_1(a_1)$and $T_2(a_2)$, operating in an

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environment which is characterized by an abstract(idealized and unobservable) parameter $\theta \in R^+$. Suppose that $I_1, I_2,$ and I_3 partition R^+ such that $I_1 \cup I_2 \cup I_3 = R^+$ and that when $\theta \in I_1$, the operating environment is classified as being "average" or "normal"

whereas when $\theta \in I_2$ or I_3 the operating environment is classified as being "mild" or "harsh," respectively, then, we can seek conditional inequalities for system reliability. That is, we can obtain that

$$P(\bigcap_{i=1}^{2} T_i(a_i) > x_i \mid \theta \in I_1) = \prod_{i=1}^{2} P(T_i(a_i) > x_i \mid \theta \in I_1),$$

$$P(\bigcap_{i=1}^{2} T_i(a_i) > x_i \mid \theta \in I_2) \geq \prod_{i=1}^{2} P(T_i(a_i) > x_i \mid \theta \in I_2), \text{and}$$

$$P(\bigcap_{i=1}^{2} T_i(a_i) > x_i \mid \theta \in I_3) \leq \prod_{i=1}^{2} P(T_i(a_i) > x_i \mid \theta \in I_3).$$

We have extended this theory to stronger conditions of dependence similar to those in the literature of positive and negative dependence and developed theorems which relate these conditions. Furthermore, these results are of value as they help us to understand in what ways the hitting times for dependence structures of hitting times can be inherited from the corresponding processes.

In this paper we introduce a new notion of θ conditionally independent and positive and negative dependent defined over bivariate stochastic processes. Bivariate stochastic processes are not unconditionally dependent or independent but are probably dependent or independent, depending on some other conditioning process. This paper lays the foundation for a new concept in the theory of dependent stochastic processes and the groundwork for incorporating this stochastic nature into dependence theory by defining stochastic dependence, proposing a measure of stochastic dependence and developing theorems based on this concept.

In Section 2, some of notations, properties and definitions are given, in Section 3, we prove some theorems which help us to identify conditionally independent and positive and negative dependence on I_1, I_2 and I_3. Moreover, it is shown that θ conditionally independent and positive and negative dependence on I_1, I_2 and I_3 is closed under limit in distribution, transformations of increasing functions, convolution. Finally we are given some examples to illustrate these concepts in Section 4.

2. Preliminaries

In this section we present definitions, notations and basic facts. Suppose we are given two-dimensional stochastic processes $\{\boldsymbol{X}(t) = (X_1(t), X_2(t)) \mid t \in \Lambda\}$, where the index set Λ is a subset of $R_+ = [0, \infty]$. The state space of $\boldsymbol{X}(t)$ is the cartesian product $E = E_1 \times E_2$, which will be a subset of two-dimensional Euclidean space R^2. For any states

$a_i \in E_i, i=1,2,$ we define the random times as follows

$$T_i(a_i) = \inf\{t \in \Lambda \mid X_i(t) \le a_i, 0 \le t \le \infty\}, \quad i=1,2,$$

that is, $T_i(a_i)$ is the first time that the process $X_i(t)$ reaches or goes below a_i[6]. If we base the dependence between two stochastic processes on the dependence of their hitting times, we then have the following definitions.

Definition 2.1. The stochastic process $\{(X_1(t), X_2(t)) \mid t \in \Lambda\}$ is θ conditionally independent and positive and negative quadrant dependent (CPQD(CNQD)) on I_1, I_2 and I_3 if

(a) $P(T_1(a_1) > t_1,\ T_2(a_2) > t_2 \mid \theta \in I_1) = P(T_1(a_1) > t_1 \mid \theta \in I_1) P(T_2(a_2) > t_2 \mid \theta \in I_1)$,

(b) $P(T_1(a_1) > t_1,\ T_2(a_2) > t_2 \mid \theta \in I_2) \ge P(T_1(a_1) > t_1 \mid \theta \in I_2) P(T_2(a_2) > t_2 \mid \theta \in I_2)$, and

(c) $P(T_1(a_1) > t_1,\ T_2(a_2) > t_2 \mid \theta \in I_3) \le P(T_1(a_1) > t_1 \mid \theta \in I_3) P(T_2(a_2) > t_2 \mid \theta \in I_3)$

for all t_1, t_2, a_1 and a_2.

Definition 2.2. The stochastic process $\{(X_1(t), X_2(t)) \mid t \in \Lambda\}$ is θ conditionally independent and positive and negative associated on I_1, I_2 and I_3 if

(a) $Cov(f(T_1(a_1),\ T_2(a_2)),\ g(T_1(a_1),\ T_2(a_2)) \mid \theta \in I_1) = 0$

(b) $Cov(f(T_1(a_1),\ T_2(a_2)),\ g(T_1(a_1),\ T_2(a_2)) \mid \theta \in I_2) \ge 0$, and

(c) $Cov(f(T_1(a_1),\ T_2(a_2)),\ g(T_1(a_1),\ T_2(a_2)) \mid \theta \in I_3) \le 0$

for all increasing functions f and g for which the covariance exists and a_1 and a_2.

Lemma 2.3. Let f and g be increasing functions of $X_1(t)$ and $X_2(t)$, respectively. Then Definition 2.1 implies

(a) $Cov(f(T_1(a_1),\ T_2(a_2)),\ g(T_1(a_1),\ T_2(a_2)) \mid \theta \in I_1) = 0$

(b) $Cov(f(T_1(a_1),\ T_2(a_2)),\ g(T_1(a_1),\ T_2(a_2)) \mid \theta \in I_2) \ge 0$, and

(c) $Cov(f(T_1(a_1),\ T_2(a_2)),\ g(T_1(a_1),\ T_2(a_2)) \mid \theta \in I_3) \le 0$

Proof. This follows by an extension of a proof of Ebrahimi[6].

Definition 2.4. The stochastic process $\{X_2(t) \mid t \in \Lambda\}$ are θ conditionally right tail dependent(CRTD) on $\{X_1(t) \mid t \in \Lambda\}$ on I_1, I_2 and I_3 if

(a) $P(T_2(a_2) > t_2 \mid T_1(a_1) > t_1),\ \theta \in I_1)$ is constant in t_1 for all t_2, a_1 and a_2,

(b) $P(T_2(a_2) > t_2 \mid T_1(a_1) > t_1),\ \theta \in I_2)$ is increasing in t_1 for all t_2, a_1 and a_2, and

(c) $P(T_2(a_2) > t_2 \mid T_1(a_1) > t_1),\ \theta \in I_3)$ is decreasing in t_1 for all t_2, a_1 and a_2.

Before introducing the main results let us present some basic properties of conditionally independent and positive and negative quadrant dependent stochastic process. It is not difficult to show that :

Property 1. Non-decreasing functions of θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3 are θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Property 2. Any subset of θ conditionally independent and positive and negative quadrant dependent stochastic processes on I_1, I_2 and I_3 are θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Property 3. A set of θ conditionally independent stochastic processes on I_1, I_2 and I_3 are θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Property 4. The union of θ conditionally independent and positive and negative quadrant dependent stochastic processes on I_1, I_2 and I_3 are θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Proof. Let the stochastic processes $\{(X_1(t), X_2(t)) \mid t \in \Lambda\}$ and $\{(Y_1(t), Y_2(t)) \mid t \in \Lambda\}$ be independent, each of which is θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3. Then

$$
\begin{aligned}
& P\Big(\bigcap_{j=1}^{2}(T_j(a_j) > t_j),\ \bigcap_{k=1}^{2}(S_k(b_k) > s_k) \mid \theta \in I_i\Big) \\
&= P\Big(\bigcap_{j=1}^{2}(T_j(a_j) > t_j) \mid \theta \in I_i)\Big) P\Big(\bigcap_{k=1}^{2}(S_k(b_k) > s_k) \mid \theta \in I_i)\Big) \\
&= \prod_{j=1}^{2} P(T_j(a_j) > t_j \mid \theta \in I_1) \prod_{k=1}^{2} P((S_k(b_k) > s_k \mid \theta \in I_1) \\
&\geq \prod_{j=1}^{2} P(T_j(a_j) > t_j \mid \theta \in I_2) \prod_{k=1}^{2} P(S_k(b_k) > s_k \mid \theta \in I_2) \\
&\leq \prod_{j=1}^{2} P(T_j(a_j) > t_j \mid \theta \in I_3) \prod_{j=1}^{2} P(S_k(b_k) > s_k \mid \theta \in I_3).
\end{aligned}
$$

Let $\{\underline{X}(t) \mid t \in \Lambda\}$ and $\{\underline{Y}(t) \mid t \in \Lambda\}$ be two vectors of stochastic processes, of dimension m and n respectively. Then we have the following definition.

Definition 2.5. The stochastic process $\{\underline{X}(t) \mid t \in \Lambda\}$ is θ conditionally stochastically increasing(decreasing) in the stochastic process $\{\underline{Y}(t) \mid t \in \Lambda\}$ ($\underline{X}(t) \uparrow cst.$ in $\underline{Y}(t)$) if $E(f(\underline{T}(\underline{a})) \mid \underline{S}(\underline{b}) = \underline{s}, \theta)$ is increasing(decreasing) in $\{\underline{Y}(t) \mid t \in \Lambda\}$ for all real valued increasing functions f and θ.

The following theorem gives a sufficient condition for θ conditionally independent and positive and negative associated on I_1, I_2 and I_3.

Theorem 2.6. Let (a) $\{(X_1(t), X_2(t)) \mid t \in \Lambda\}$ given $\{\underline{Y}(t) \mid t \in \Lambda\}$ another stochastic processes be θ conditionally independent and positive and negative associated on I_1, I_2 and I_3 and (b) $\{X_1(t) \mid t \in \Lambda\}$ be stochastically increasing in $\{\underline{Y}(t) \mid t \in \Lambda\}$ and $\{X_2(t) \mid t \in \Lambda\}$ be conditionally stochastically increasing in $\{\underline{Y}(t) \mid t \in \Lambda\}$ given $\theta \in I_2$, constant in $\{\underline{Y}(t) \mid t \in \Lambda\}$ given $\theta \in I_1$, and θ conditionally stochastically decreasing in $\{\underline{Y}(t) \mid t \in \Lambda\}$ given $\theta \in I_3$ or (b') $\{X_1(t) \mid t \in \Lambda\}$ be θ conditionally stochastically increasing in $\{\underline{Y}(t) \mid t \in \Lambda\}$ given $\theta \in I_2$, constant in $\{\underline{Y}(t) \mid t \in \Lambda\}$ given $\theta \in I_1$, and θ conditionally stochastically decreasing in $\{\underline{Y}(t) \mid t \in \Lambda\}$ given $\theta \in I_3$ and $\{X_2(t) \mid t \in \Lambda\}$ stochastically increasing in $\{\underline{Y}(t) \mid t \in \Lambda\}$. Then $\{(X_1(t), X_2(t)) \mid t \in \Lambda\}$ is θ condition ally independent and positive and negative associated on I_1, I_2 and I_3.

Proof. $Cov(f(T_1(a_1)), g(T_2(a_2)) \mid \theta \in I_i)$

$$= Cov(E(f(T_1(a_1)) \mid \theta \in I_i, \underline{S(b)}), \ E(g(T_2(a_2)) \mid \theta \in I_i, \underline{S(b)})) + E(Cov(f(T_1(a_1)), g(T_2(a_2)) \mid \theta \in I_i, \underline{S(b)}))$$

The first term on the right is non-negative when $\theta \in I_2$, zero when $\theta \in I_1$ and non-positive when $\theta \in I_3$ by (b) and (b') for increasing f and g. For such f and g the second term is non-negative when $\theta \in I_2$, zero when $\theta \in I_1$, and non-positive when $\theta \in I_3$ using assumption (a). Thus, it follows that $\{(X_1(t), X_2(t)) \mid t \in \Lambda\}$ is θ conditionally independent and positive and negative associated on I_1, I_2 and I_3.

For proving the next theorem we need the following definition.

Definition 2.7. The stochastic process $\{\underline{Y}(t) \mid t \in \Lambda\}$ is θ conditionally stochastically right tail dependent in the stochastic process $\{\underline{X}(t) \mid t \in \Lambda\}$ on I_1, I_2 and I_3 if $E(f(\underline{T(a)}) \mid \underline{S(b)} > \underline{s}, \theta)$ is constant, increasing, and decreasing given $\theta \in I_1$, I_2 and I_3, respectively, for any real valued increasing f.

Theorem 2.8. Let (a) $\{X_i(t) \mid t \in \Lambda\}, i = 1, 2, \cdots, m$ be θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3, (b) $\{Y_l(t) \mid t \in \Lambda\}$, $l = 1, \cdots, n$ be conditionally independent given $\{\underline{X}(t) \mid t \in \Lambda\}$ and θ and (c) $Y_l(t)$ be θ condition ally stochastically right tail dependent in $\{\underline{X}(t) \mid t \in \Lambda\}$ on I_1, I_2 and I_3, for all $l = 1, 2, \cdots, n$. Then (i) $\{(\underline{X}(t), \underline{Y}(t)) \mid t \in \Lambda\}$ is θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3 and (ii) $\{\underline{Y}(t) \mid t \in \Lambda\}$is θ con ditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Proof. (*i*) $P(\bigcap_{k=1}^{m}(T_k(a_k)>t_k),\ \bigcap_{l=1}^{n}(S_l(b_l)>s_l)\mid\theta\in I_i)$

$= P(\bigcap_{l=1}^{n}(S_l(b_l)>s_l)\mid\bigcap_{k=1}^{m}(T_k(a_k)>t_k),\ \theta\in I_i)P(\bigcap_{k=1}^{m}(T_k(a_k)>t_k)\mid\theta\in I_i)$

$= \prod_{l=1}^{n}P(S_l(b_l)>s_l)\mid\bigcap_{k=1}^{m}(T_k(a_k)>t_k),\ \theta\in I_i)P(\bigcap_{k=1}^{m}(T_k(a_k)>t_k)\mid\theta\in I_i)$

using (b),

$= \prod_{l=1}^{n}P(S_l(b_l)>s_l\mid\theta\in I_1)\prod_{k=1}^{m}P(T_k(a_k)>t_k\mid\theta\in I_1)$

$\geq \prod_{l=1}^{n}P(S_l(b_l)>s_l\mid\theta\in I_2)\prod_{k=1}^{m}P(T_k(a_k)>t_k\mid\theta\in I_2)$

$\leq \prod_{l=1}^{n}P(S_l(b_l)>s_l\mid\theta\in I_3)\prod_{k=1}^{m}P(T_k(a_k)>t_k\mid\theta\in I_3)$

using (c) and (a)

(*ii*) By making $t_k\to 0$ $(k=1,2\cdots,m)$ in (*i*), (*ii*) follows.

3. Theoretical Results

Barlow and Proschan[3] and others have considered a number of alternative notions of dependence and studied the relationship among them. Motivated by this we now extend the multivariate stochastic dependence to stronger conditions of dependence.

The next theorem demonstrates the preservation of θ conditionally independent and positive and negative dependent property on I_1, I_2 and I_3 under limits.

We now show that θ conditionally right tail dependent implies θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Theorem 3.1. Let $\{(X_1(t),X_2(t))\mid t\in\Lambda\}$ be θ conditionally right tail dependent on I_1, I_2 and I_3, then $\{(X_1(t),X_2(t))\mid t\in\Lambda\}$ is θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Proof. $P(T_1(a_1)>t_1,\ T_2(a_2)>t_2\mid\theta\in I_i)$

$= P(T_1(a_1)>t_1\mid\theta\in I_i)P(T_2(a_2)>t_2\mid T_1(a_1)>t_1,\ \theta\in I_i)$

$= \prod_{j=1}^{2}P(T_j(a_j)>t_j\mid\theta\in I_1)$

$\geq \prod_{j=1}^{2}P(T_j(a_j)>t_j\mid\theta\in I_2)$

$\leq \prod_{j=1}^{2}P(T_j(a_j)>t_j\mid\theta\in I_3)$

making $t_1 \to 0$.

Theorem 3.2. Let $\{(X_1(t), X_2(t)) | t \in \Lambda\}$ be θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3 and f_1 and f_2 are non-negative incr easing functions, then $\{f_1(X_1(t)) | t \in \Lambda\}$and $\{f_2(X_2(t)) | t \in \Lambda\}$ are θ conditionally indepe ndent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Proof. $P[(\inf\{t | f_1(X_1(t)) \le a_1\}) > t_1, (\inf\{t | f_2(X_2(t)) \le a_2\}) > t_2 | \theta \in I_i]$

$$= P[(\inf\{t | X_1(t) \le f_1^{-1}(a_1)\}) > t_1, (\inf\{t | X_2(t) \le f_2^{-1}(a_2)\}) > t_2 | \theta \in I_i]$$

$$= P[(T_1(f_1^{-1}(a_1)) > t_1, T_2(f_2^{-1}(a_2)) > t_2 | \theta \in I_i]$$

$$= \prod_{i=1}^{2} P(T_i(f_i^{-1}(a_i)) > t_i) | \theta \in I_1)$$

$$= \prod_{i=1}^{2} P[(\inf\{t | f_i(X_i(t)) \le a_i\}) > t_i | \theta \in I_1]$$

$$\ge \prod_{i=1}^{2} P(T_i(f_i^{-1}(a_i)) > t_i) | \theta \in I_2)$$

$$= \prod_{i=1}^{2} P[(\inf\{t | f_i(X_i(t)) \le a_i\}) > t_i | \theta \in I_2]$$

$$\le \prod_{i=1}^{2} P(T_i(f_i^{-1}(a_i)) > t_i) | \theta \in I_3)$$

$$= \prod_{i=1}^{2} P[(\inf\{t | f_i(X_i(t)) \le a_i\}) > t_i | \theta \in I_3].$$

We show that the theorem demonstrates preservation of the conditionally independent and positive and negative quadrant on I_1, I_2 and I_3 among the random times under limits.

Theorem 3.3. Let $\{X_n(t) | t \in \Lambda\}$ be θ conditionally independent and positive and nega tive quadrant dependent on I_1, I_2 and I_3, 2-dimensional stochastic vectors with distribution functions $H_n \to H$ weakly as $n \to \infty$, where H is the distribution function of a stochastic process $\{(X_1(t), X_2(t)) | t \in \Lambda\}$. Then $\{(X_1(t), X_2(t)) | t \in \Lambda\}$ is θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Proof. For any t_1, t_2 writing $\{X_n(t) = (X_{1n}(t), X_{2n}(t)) | t \in \Lambda\}$, $n \ge 1$

$$P(T_{1n}(a_1) > t_1,\ T_{2n}(a_2) > t_2 | \theta \in I_i)$$

$$= \lim_{n \to \infty} P(T_{1n}(a_1) > t_1,\ T_{2n}(a_2) > t_2 | \theta \in I_i)$$

$$= \lim_{n \to \infty} \prod_{j=1}^{2} P(T_{jn}(a_i) > t_j | \theta \in I_1) = \prod_{j=1}^{2} P(T_j(a_j) > t_j | \theta \in I_1)$$

$$\geq \lim_{n\to\infty} \prod_{j=1}^{2} P(T_{jn}(a_j) > t_j \mid \theta \in I_2) = \prod_{j=1}^{2} P(T_j(a_j) > t_j \mid \theta \in I_2)$$

$$\leq \lim_{n\to\infty} \prod_{j=1}^{2} P(T_{jn}(a_j) > t_j \mid \theta \in I_3) = \prod_{j=1}^{2} P(T_j(a_j) > t_j \mid \theta \in I_3).$$

The following theorem provides a characterization of conditionally right tail dependent in the bivariate case.

Theorem 3.4. Let the stochastic process $\{(X_1(t), X_2(t)) \mid t \in \Lambda\}$. Then $X_2(t)$ is θ conditionally right tail dependent on $X_1(t)$ on I_1, I_2 and $I_3 \Leftrightarrow P(T_2(a_2) > t_2 \mid T_1(a_1),$ $\theta \in I_2) \uparrow$ in t_1 for all t_2, $P(T_2(a_2) > t_2 \mid T_1(a_1) > t_1, \theta \in I_3) \downarrow$ in t_1 for all t_2 and $P(T_2(a_2) > t_2 \mid T_1(a_1) > t_1, \theta \in I_1)$ is constant in t_1 for all $t_2 \Leftrightarrow E(f(T_2(a_2)) \mid T_1(a_1) > t_1$ $, \theta \in I_2) \uparrow$, $E(f(T_2(a_2)) \mid T_1(a_1) > t_1, \theta \in I_3) \downarrow$, $E(f(T_2(a_2)) \mid T_1(a_1) > t_1, \theta \in I_1)$ is cons tant in t_1 for all real valued increasing f.

Proof. $\{X_1(t) \mid t \in \Lambda\}$ and $\{X_2(t) \mid t \in \Lambda\}$ θ conditionally right tail dependent on I_1, I_2 and I_3 $\Leftrightarrow P(T_2(a_2) > t_2 \mid T_1(a_1) > t_1, \theta \in I_2) \uparrow$ in t_1 for all t_2, $P(T_2(a_2) > t_2 \mid T_1(a_1) > t_1, \theta \in I_3) \downarrow$ in t_1 for all t_2 and $P(T_2(a_2) > t_2 \mid T_1(a_1) > t_1, \theta \in I_1)$ is constant in t_1 for all t_2 by Definition2.3 $\Leftrightarrow P(T_2(a_2) \geq t_2 \mid T_1(a_1) > t_1, \theta \in I_2) \uparrow$ in $t_1$for all t_2, $P(T_2(a_2) \geq t_2 \mid T_1(a_1) > t_1, \theta \in I_3) \downarrow$ in t_1 for all t_2 and $P(T_2(a_2) \geq t_2 \mid T_1(a_1) > t_1, \theta \in I_1)$ is constant in t_1 forall t_2. Now assume

$$E(f(T_2(a_2)) \mid T_1(a_1) > t_1, \theta \in I_2) \uparrow,$$

$E(f(T_2(a_2)) \mid T_1(a_1) > t_1, \theta \in I_3) \downarrow$, and $E(f(T_2(a_2)) \mid T_1(a_1) > t_1, \theta \in I_1)$ is constant in t_1 for all real valued increasing f. Putting $f(T_2(a_2)) = I_{[T_2(a_2) \geq t_2]}$, it follows that $P(T_2(a_2) \geq t_2 \mid T_1(a_1) > t_1, \theta \in I_2) \uparrow$ in t_1 for all t_2, $P(T_2(a_2) \geq t_2 \mid T_1(a_1) > t_1, \theta \in I_3) \downarrow$ in t_1 for all t_2 and $P(T_2(a_2) \geq t_2 \mid T_1(a_1) > t_1, \theta \in I_1)$ is constant in t_1 for all t_2. Conversely, suppose $E(I_{[T_2(a_2) \geq t_2]} \mid T_1(a_1) > t_1, \theta \in I_2) \uparrow$, $E(I_{[T_2(a_2) \geq t_2]} \mid T_1(a_1) > t_1, \theta \in I_3) \uparrow$, and $E(I_{[T_2(a_2) \geq t_2]} \mid T_1(a_1) > t_1, \theta \in I_1)$ is constant in t_1 for all t_2, then $E(f(T_2(a_2)) \mid T_1(a_1) >$ $t_1, \theta \in I_2) \uparrow$, $E(f(T_2(a_2)) \mid T_1(a_1) > t_1, \theta \in I_3) \downarrow$, and $E(f(T_2(a_2)) \mid T_1(a_1) > t_1, \theta \in I_1)$ is constant in t_1 for all real valued increasing f. Using the monotone convergence theorem Chung[4] the same is true for every non-negative increasing f. Consequently, the result hold for every increasing f.

The following theorem exhibits a conditionally right tail dependent preservation property.

Theorem 3.5. Let (a) $\{X_2(t) \mid t \in \Lambda\}$ be θ conditionally right tail dependent in $\{X_1(t) \mid t \in \Lambda\}$ given $\{\underline{Y}(t) \mid t \in \Lambda\}$ on I_1, I_2 and I_3. Then $\{X_2(t) \mid t \in \Lambda\}$ is θ condi tionally right tail dependent in $\{X_1(t) \mid t \in \Lambda\}$ on I_1, I_2 and I_3.

Proof. Let $t_1 > t_1'$. Then by (a)

$$P(T_2(a_2) \geq t_2 \mid T_1(a_1) \geq t_1,\ \theta \in I_2)$$
$$= E\,P(T_2(a_2) > t_2 \mid T_1(a_1) > t_1,\ \theta \in I_3,\ \underline{S(b)})$$
$$\geq EP(T_2(a_2) > t_2 \mid T_1(a_1) > t_1',\ \theta \in I_3,\ \underline{S(b)})$$
$$= P(T_2(a_2) > t_2 \mid T_1(a_1) > t_1',\ \theta \in I_3) \text{ for all } t_2.$$

Similarly, one handless the cases for $\theta \in I_1$ and I_3.

Next, we show that conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3 is invariant under transformations of increasing functions.

Theorem 3.6. Let (a) $((X_{i1}(t), X_{i2}(t)), i = 1, 2, \cdots, n$ are n conditionally independent 2-variate processes with increasing sample paths, (b) $((X_{i1}(t), X_{i2}(t))$ are θ condi- tionally independent and positive and negative quadrant dependent processes on I_1, I_2 and I_3 for each $i = 1, \cdots, n$, and (c) $f_i : R^n \to R$, $j = 1, 2$ are increasing functions then the processes $Y_j(t)$, $j = 1, 2$, given by $Y_j(t) = f_j((X_{1j}(t), \cdots, X_{nj}(t))$ are θ condition ally independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Proof. The proof will be given for the case $n = 2$ when $\theta \in I_2$. For the general n, the proof is similar. For fixed $t_i \geq 0$, $j = 1, 2$, and introduce the variables $V_j = X_{2j}(t_j)$ and $U_j = \sup_{0 \leq s \leq t_j} f_j(X_{1j}(s), X_{2j}(s))$, $j = 1, 2$, where, for simplicity, t_1, t_2 have been suppressed in V_j and U_j. Consider only hitting times of $Y_j(s) = f_j(X_{1j}(s), X_{2j}(s))$ given by

$$W_j(a_j) = \inf\{s \mid Y_j(s) \geq a_j\},\ j = 1, 2.$$

It suffices to show that

$$P(W_1(a_1) > t_1,\ W_2(a_2) > t_2 \mid \theta \in I_2) \geq P(W_1(a_1) > t_1 \mid \theta \in I_2)\, P(W_2(a_2) > t_2 \mid \theta \in I_2).$$

Note the facts that $U_j = \sup_{0 \leq s \leq t_j} f_j(X_{1j}(s), V_j)$ and that, by hypothesis, V_1 and V_2 are θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3. Now, we obtain

$$P(W_1(a_1) > t_1,\ W_2(a_2) > t_2 \mid \theta \in I_i)$$
$$= P(U_1 < a_1,\ U_2 < a_2 \mid \theta \in I_i)$$
$$= E\,P(U_1 < a_1,\ U_2 < a_2 \mid V_1,\ V_2,\ \theta \in I_i)$$

$$\geq E\prod_{i=1}^{2} P(U_i < a_i \mid V_i,\ \theta\in I_2)$$

$$\geq \prod_{i=1}^{2} P(U_i < a_i \mid V_i,\ \theta\in I_2) = \prod_{i=1}^{2} P(W_i(a_i) > t_i \mid \theta\in I_2).$$

Similarly, one handless the cases for $\theta\in I_1$ and I_3.

Next we show that conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3 is preserved under convolution.

Corollary 3.7. If (a) $\{(X_1(t), X_2(t)) \mid t\in\Lambda\}$ is θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3, (b) $\{(Z_1(t), Z_2(t)) \mid t\in\Lambda\}$ is θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3, (c) and $(X_1(t), X_2(t))$ and $(Z_1(t), Z_2(t))$ are independent and have increasing sample paths, then $X_1(t)+Z_1(t)$ and $X_2(t)+Z_2(t)$ are θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

4. Examples

Example 1. Consider a bivariate process $\{(X_n, Y_n) \mid n\geq 1\}$ such that $(X_1, Y_1), (X_2, Y_2), \cdots$ are independent and X_i and Y_i, $i=1,2,\cdots$ are θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3. Then we have the following theorem.

Theorem 4.1. Consider a bivariate processes $\{(X_n, Y_n) \mid n\geq 1\}$ is θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Proof. Let $P(T_1(x) > n_1,\ T_2(y) = n_2 \mid \theta\in I_i)$

$$= P(X_1>x,\ X_2>x, \cdots, X_{n1}>x, Y_1>y\cdots, Y_{n2}>y \mid I_i)$$

Now consider the following cases.

Case 1. $n_1 = n_2$,

$$P(T_1(x) > n_1,\ T_2(y) > n_2 \mid \theta\in I_i)$$
$$= P(X_1>x,\ Y_1>y \mid \theta\in I_i) P(X_2>x,\ Y_2>y \mid \theta\in I_i) \cdots P(X_{n1}>x,\ Y_{n1}>y \mid \theta\in I_i)$$
$$= P(X_1>x, X_2>x, \cdots X_{n1}>x \mid \theta\in I_i) P(Y_1>y, Y_2>y, \cdots Y_{n1}>y \mid \theta\in I_i)$$

$$= \prod_{i=1}^{2} P(T_i(x) > n_i | \theta \in I_1)$$

$$\geq P(X_1 > x, \cdots, X_{n1} > x | \theta \in I_2) P(Y_1 > y, \cdots, Y_{n1} > y | \theta \in I_2)$$

$$= \prod_{i=1}^{2} P(T_i(x) > n_i | \theta \in I_2)$$

$$\leq P(X_1 > x, \cdots, X_{n1} > x | \theta \in I_3) P(Y_1 > y, \cdots, Y_{n1} > y | \theta \in I_3)$$

$$= \prod_{i=1}^{2} P(T_i(x) > n_i | \theta \in I_3)$$

The proof of the $n_1 < n_2$ and $n_1 \geq n_2$ is similar to Case 1.

Example 2. Consider a simple form of econometrical model relating the investment and capital gain. Let $X_1(t)$ and $X_2(t)$, $t \in \Lambda$, denote respectively the investment and capital gain at time t. The model is

$$X_1(t) = aX_2(t) + Z_1(t) \tag{4.1}$$

where $a > 0$, $Z_1(t)$ is a white noise process independent of $X_1(t)$, and $X_2(t)$ is θ conditionally independent and positive and negative associated dependent on I_1, I_2 and I_3 Then, it is clear that $X_1(t)$ is θ conditionally independent and positive and negative associated dependent on I_1, I_2 and I_3. Now, for any increasing functions f and g,

$Cov(f(T_1(a_1), T_2(a_2)), g(T_1(a_1), (T_2(a_2)) | \theta \in I_i)$

$= Cov(f(T_1(a_1) + S_1(b_1), T_2(a_2)), g(T_2(a_1) + S_1(b_1), T_2(a_2)) | \theta \in I_i)$.

The last term is non-negative when $\theta \in I_2$, zero when $\theta \in I_1$, and non-positive when $\theta \in I_3$ by assumption for increasing f and g. Consequently $(X_1(t), X_2(t))$is θ conditionally independent and positive and negative associated dependent on I_1, I_2 and I_3.

Example 3. Block et al[4] proposed a bivariate geometric autoregressive model of order m, BGAR(m),

$$X(n) = \begin{cases} M(n) & n = 0, 1, 2, \cdots, m-1 \\ \sum_{q=1}^{m} C(n, q)\, G(n-q) + N(n), & n = m, m+1, \cdots, \end{cases} \tag{4.2}$$

where $M(n) = (M_1(n), M_2(n))$ is a sequence of independent bivariate geometric random vectors with mean (p_1^{-1}, p_2^{-1}), $p_1, p_2 > 0$. $C(n, q)$ is a 2×2 random diagonal matrix with $C(n, q) = diag\{J_1(n, q), J_2(n, q)\}$, $q = 1, 2, \cdots, m$.

We assume that for $l = 1, 2$,

$$\sum_{j=1}^{m} p\{J_l(n, 1), \cdots, J_l(n, m)) = e_j'\} = 1 - \alpha_l(n)$$

and that

$$P\{J_l(n, 1), \cdots, J_l(n, m)) = 0'\} = \alpha_l(n).$$

The following theorem gives the result about dependence structure of the bivariate process BGAR(m), $X(n) = (X_1(n), X_2(n))$

Theorem 4.2. Suppose for $j = 0, 1, \cdots, m-1$, the random variables $X_1(j)$ and $X_2(j)$ in (4.2) are θ conditionally independent and positive and negative associated dependent on I_1, I_2 and I_3. Then $X(n)$ is θ conditionally independent and positive and negative associated dependent on I_1, I_2 and I_3.

Proof. This follows by an extension of a proof of Block et al[4].

Example 4. Consider a system with two components which is subjected to shocks. Let $N(t)$ be the number of shocks received by time t and let $Z_1(t) = \sum_{i=1}^{N(t)} X_i$, $Z_2(t) = \sum_{i=1}^{N(t)} Y_i$ be total damages to components 1 and 2 by time t, respectively. Here X_i and Y_i are damages to components 1 and 2 by shock i, respectively. Then we have the following theorem.

The following theorem is very important in recognizing conditionally independent and positive and negative quadrant dependent on compound distribution which arise naturally in stochastic processes.

Theorem 4.3. Let (a) $N(t)$ be a poison process which is independent of X_i's and Y_i's given θ, $i = 1, 2, \cdots$, (b) (X_1, Y_1), $(X_2, Y_2), \cdots$ be independent, (c) X_i and Y_i be θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3, $i = 1, 2, \cdots$, then $(Z_1(t), Z_2(t))$ is θ conditionally independent and positive and negative quadrant dependent on I_1, I_2 and I_3.

Proof. $P(T_1(a_1) \leq t_1,\ T_2(a_2) \leq t_2 \mid \theta \in I_2)$

$$= P\Big(\sum_{i=1}^{N(s)} X_i \geq a_1,\ t_1 \leq s < \infty,\ \sum_{i=1}^{N(s)} Y_i \geq a_i,\ t_2 \leq s < \infty \mid \theta \in I_2\Big)$$

$$= P\Big(\sum_{i=1}^{N(t_1)} X_i \geq a_1,\ \sum_{i=1}^{N(t_2)} Y_i \geq a_2 \mid \theta \in I_2\Big)$$

$$= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} P(N(t_1) = k_1,\ N(t_2) = k_2 \mid \theta \in I_2) P\Big(\sum_{i=1}^{k_1} X_i \geq a_1,\ \sum_{i=1}^{k_2} Y_i \geq a_2 \mid \theta \in I_2\Big)$$

$$\geq \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} P(N(t_1)=k_1, N(t_2)=k_2 \mid \theta \in I_2) P(\sum_{i=1}^{k_1} X_i \geq a_1 \mid \theta \in I_2) P(\sum_{i=1}^{k_2} Y_i \geq a_2 \mid \theta \in I_2)$$

$$\geq \prod_{j=1}^{2} P(T_i(a_i) > t_i \mid \theta \in I_2)$$

Similarly, one handless the case for $\theta \in I_1$ and I_3.

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