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STRUCTURE OF THE FLAT COVERS OF ARTINIAN MODULES

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ABSTRACT. The aim of the paper is to obtain information about the flat covers and minimal flat resolutions of Artinian modules over a Noetherian ring. Let R be a commutative Noetherian ring and let A be an Artinian R-module. We prove that the flat cover of A is of the form $\prod_{p\in \mathrm{Att}_R(A)} T_p$, where T_p is the completion of a free R_p-module. Also, we construct a minimal flat resolution for R/xR-module $0 :_A x$ from a given minimal flat resolution of A, when x is a non-unit and non-zero divisor of R such that $A = xA$. This result leads to a description of the structure of a minimal flat resolution for $\mathrm{H}^n_{\underline{m}}(R)$, nth local cohomology module of R with respect to the ideal $\underline{m}$, over a local Cohen-Macaulay ring $(R, \underline{m})$ of dimension n.

1. Preliminaries

Throughout this paper, R denotes a commutative ring with non-zero identity and M an R-module.

Let us recall briefly some notions defined in [3, 4, 11]. By terminology of Enochs in [3], a linear map $\phi : F \longrightarrow M$ with F flat is called a flat precover of M if $\mathrm{Hom}_R(G, F) \longrightarrow \mathrm{Hom}_R(G, M) \longrightarrow 0$ is exact for all flat modules G. If furthermore any linear map $f : F \longrightarrow F$ such that $\phi f = \phi$ is an automorphism of F, then $\phi : F \longrightarrow M$ is called a flat cover of M. It was proved that the flat cover of M exists and is unique up to isomorphism (see [1, 3]). Also, if M has a flat precover F, then M has a flat cover which is a direct summand of F.

DEFINITION 1.1. A *minimal flat resolution* of M is an exact sequence

$$(1) \qquad \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_0 \xrightarrow{d_0} M \longrightarrow 0$$

such that for each i, F_i is a flat cover of $\mathrm{Im}(d_i)$.

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An R-module C is called cotorsion if $\mathrm{Ext}^1_R(F, M) = 0$ for all flat modules F.

For the remaining part of this paper, we shall assume that R is Noetherian.

Note that for any injective R-module E, by Matlis' theorem [9], E is a direct sum of indecomposable injective modules such that each of them is in the form $E(R/p)$ for some prime ideal p. In general we do not have a similar result for any flat R-module. But if F is flat and cotorsion R-module, then it was proved in [4] that F is uniquely a product $F = \prod T_p$. Here T_p is the completion of a free R_p-module with respect to the pR_p-adic topology. Also note that a flat cover of a cotorsion module is flat and cotorsion, and the kernel of a flat cover $F \longrightarrow M$ is cotorsion (see [4]). Therefore, in (1.1), for all $i \geq 1$, F_i is flat and cotorsion, and then it is a product of such T_p. For $i = 0$, F_0 is not cotorsion in general. But we take its cotorsion envelope (or equivalently pure injective envelope). The pure injective envelope $\mathrm{PE}(F_0)$ of F_0 is flat and cotorsion [5, p. 352]. Hence, $\mathrm{PE}(F_0)$ is a product of T_p.

Definition 1.2. Assume M has a minimal flat resolution as (1). For $i \geq 1$ and prime ideal p of R, $\pi_i(p, M)$ is defined to be the *cardinality of the base* of a free R_p-module whose completion is T_p in the product $F_i = \prod T_q$. For $i = 0$, $\pi_0(p, M)$ is defined similarly by using the pure injective envelope $\mathrm{PE}(F_0)$ instead of F_0 itself.

We note that the $\pi_i(p, M)$ are homologically independent and well-defined. We call the $\pi_i(p, M)$ the dual Bass numbers.

2. The main results

Let us recall basic facts concerning a secondary module respectively secondary representation of a module, for the details see [8]. An R-module $S \neq 0$ is called secondary if for each $a \in R$ multiplication by a on S is surjective or nilpotent. When this is the case, $r(\mathrm{Ann}_R(S)) = p$ is a prime ideal of R, in these circumstances, we say that S is a p-secondary R-module. Let X be an R-module. A secondary representation of X is an expression for X as a sum of finitely many secondary submodules of X. Such a secondary representation $X = S_1 + \cdots + S_n$ with S_i, p_i-secondary and $1 \leq i \leq n$ of X is said to be minimal precisely when $p_1, \cdots, p_n$ are n different prime ideals of R and for all $j = 1, \cdots, n$ we have $S_j \not\subseteq \sum_{i=1, i\neq j}^{n} S_i$. We say that X is a representable R-module when it has a secondary representation. The set $\{p_1, \cdots, p_n\}$ depends only on X and called the set of attached prime ideals $\mathrm{Att_R(X)}$.

The following result is essential in the proof of (2.2) and we quote it for the convenience of the reader.

LEMMA 2.1. [8, 2.8] *Let A be a representable R-module and let $\underline{a}$ be an ideal of R. Then $A = \underline{a}A$ if and only if there exists $x \in \underline{a}$ such that $A = xA$.*

Proof. The 'if' part is obvious. Hence we shall prove the 'only if' half. If $A \neq xA$ for any $x \in \underline{a}$ then by using [8, 2.6], $\underline{a} \subseteq \cup_{p\in \mathrm{Att}_R(A)} p$. Since $\mathrm{Att}_R(A)$ is a finite set, there exists p such that $\underline{a} \subseteq p$, $p \in \mathrm{Att}_R(A)$. Now, in view of [8, 2.5], there is a proper submodule B of A such that $p = \mathrm{Ann}_R(A/B)$. Hence $\underline{a}A \subseteq pA \subseteq B \subsetneq A$ contrary to the assumption. □

THEOREM 2.2. *Let A be an Artinian R-module and let $F = \prod T_q$ be the flat cover of A. Then T_p appears in the product of F if and only if $p \in \mathrm{Att}_R(A)$.*

Proof. By using [12, p. 63] A is cotorsion. Hence, as mentioned before, F is flat and cotorsion. Thus, in view of [4, p. 183], $F = \prod T_q$, where T_q is the completion of a free R_q-module with respect to the qR_q-adic topology. Assume that T_p appears in the product of F. It follows from [7, Theorem 2.2] that $k(p) \otimes_{R_p} \mathrm{Hom}_R(R_p, A) \neq 0$, where $k(p)$ denotes the residue field of R_p. So that $pR_p\mathrm{Hom}_R(R_p, A) \neq \mathrm{Hom}_R(R_p, A)$. On the other hand, by using [10, Theorem 3.2], we have that $\mathrm{Hom}_R(R_p, A)$ is a representable R_p-module and

$$\mathrm{Att}_{R_p}(\mathrm{Hom}_R(R_p, A)) = \{qR_p : q \subseteq p, q \in \mathrm{Att}_R(A)\}.$$

Now (2.1) shows that there is not any element $x \in pR_p$ such that $x\mathrm{Hom}_R(R_p, A) = \mathrm{Hom}_R(R_p, A)$. Hence $pR_p \subseteq \cup_{Q\in \mathrm{Att}_{R_p}(\mathrm{Hom}_R(R_p,A))} Q$. Thus $pR_p \in \mathrm{Att}_{R_p}(\mathrm{Hom}_R(R_p, A))$; since pR_p is a maximal ideal of R_p. Therefore, $p \in \mathrm{Att}_R(A)$. Conversely, if $p \in \mathrm{Att}_R(A)$ then $pR_p \in \mathrm{Att}_{R_p}(\mathrm{Hom}_R(R_p, A))$. Hence $k(p) \otimes_{R_p} \mathrm{Hom}_R(R_p, A) \neq 0$. Thus $\pi_0(p, A) \neq 0$. So that T_p is a direct summand of F. The result follows. □

Let $(R, \underline{m})$ be a local ring, let $E(R/\underline{m})$ be the injective envelope of $R/\underline{m}$, and let $F = \prod T_q$ be the flat cover of $E(R/\underline{m})$. In [12, Remark 4.4.9], Xu has proved that, if p is a minimal prime ideal of R then T_p

appears in the product of F. Also, at that time, this problem "for a prime ideal p, when T_p appears in the product of F" was a question for him. The following corollary give an answer to the above question.

COROLLARY 2.3. *Let $(R, \underline{m})$ be a local ring and let $F = \prod T_q$ be a flat cover of $E(R/\underline{m})$. Then T_p appears in the product of F if and only if $p \in \mathrm{Ass}_R(R)$.*

Proof. We know that $E(R/\underline{m})$ is an Artinian R-module with $\mathrm{Att}_R(E(R/\underline{m})) = \mathrm{Ass}_R(R)$. Thus by using (2.2) the result follows. □

In the remaining part of this section, we are primarily concerned with minimal flat resolutions of Artinian R-modules.

THEOREM 2.4. *Let A be an Artinian R-module and let x be a non-unit and non-zero divisor of R such that $A = xA$. If p is a prime ideal of R and $x \in p$ then for all $i \geq 0$,*

$$\mathrm{Tor}_i^{\overline{R}_{\overline{p}}}(k(\overline{p}), \mathrm{Hom}_{\overline{R}}(\overline{R}_{\overline{p}}, 0 :_A x)) \cong \mathrm{Tor}_{i+1}^{R_p}(k(p), \mathrm{Hom}_R(R_p, A))$$

where $\overline{R} = R/xR$, $\overline{p} = p/(x)$ and $k(\overline{p})$ denotes the residue field of $\overline{R}_{\overline{p}}$.

Proof. Let

$$F^{\bullet} \quad \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_0 \xrightarrow{d_0} A \longrightarrow 0$$

be a minimal flat resolution for A. In [7, Theorem 2.7] it was proved that

$$\cdots \longrightarrow \mathrm{Hom}_R(R_p, F_i) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(R_p, F_0) \xrightarrow{\overline{d_0}} \mathrm{Hom}_R(R_p, A) \longrightarrow 0$$

is a minimal flat resolution of $\mathrm{Hom}_R(R_p, A)$ as an R_p-module. Suppose that K=ker $\overline{d}_0$. The commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & K & \longrightarrow & \mathrm{Hom}_R(R_p, F_0) & \longrightarrow & \mathrm{Hom}_R(R_p, A) & \longrightarrow 0 \\ & \downarrow \frac{x}{1} & & \downarrow \frac{x}{1} & & \downarrow \frac{x}{1} & \\ 0 \longrightarrow & K & \longrightarrow & \mathrm{Hom}_R(R_p, F_0) & \longrightarrow & \mathrm{Hom}_R(R_p, A) & \longrightarrow 0 \end{array}$$

in which the rows are exact, induces an exact sequence

$$0 :_{\mathrm{Hom}_R(R_p, F_0)} \frac{x}{1} \longrightarrow 0 :_{\mathrm{Hom}_R(R_p, A)} \frac{x}{1} \longrightarrow K/\frac{x}{1}K$$
$$\longrightarrow \mathrm{Hom}_R(R_p, F_0)/\frac{x}{1}\mathrm{Hom}_R(R_p, F_0).$$

But $0 :_{\mathrm{Hom}_R(R_p, F_0)} \frac{x}{1} = 0$; since $\mathrm{Hom}_R\ (R_p,\ F_0)$ is a flat R_p-module and $x/1$ is a non-zero divisor of R_p. We show that $\mathrm{Hom}_R\ (R_p,\ F_0)\ /\frac{x}{1}$

$\mathrm{Hom}_R(R_p, F_0) = 0$ (or equivalently $\overline{R}_{\overline{p}} \otimes_{R_p} \mathrm{Hom}_R(R_p, F_0) = 0$). By using (2.2) we have $F_0 = \prod_{q \in \mathrm{Att}_R(A)} T_q$, so that

$$\begin{aligned}\overline{R}_{\overline{p}} \otimes_{R_p} \mathrm{Hom}_R(R_p, F_0) &\cong R_p/xR_p \otimes_{R_p} \mathrm{Hom}_R(R_p, \prod_{q\in \mathrm{Att}_R(A)} T_q) \\ &\cong \prod_{q\in \mathrm{Att}_R(A)} (R_p/xR_p \otimes_{R_p} \mathrm{Hom}_R(R_p, T_q)) \\ &\cong \prod_{q\in \mathrm{Att}_R(A)} \mathrm{Hom}_R(R_p, T_q)/x\mathrm{Hom}_R(R_p, T_q) = 0\end{aligned}$$

Hence, $0 :_{\mathrm{Hom}_R(R_p,A)} \frac{x}{1} \cong K/\frac{x}{1}K$ as R_p- and $\overline{R}_{\overline{p}}$-modules. Now it is easy to see that

$$(2) \qquad \cdots \longrightarrow \overline{R}_{\overline{p}} \otimes_{R_p} \mathrm{Hom}_R(R_p, F_2) \longrightarrow \overline{R}_{\overline{p}} \otimes_{R_p} \mathrm{Hom}_R(R_p, F_1) \longrightarrow 0$$

is a flat resolution for $0 :_{\mathrm{Hom}_R(R_p,A)} \frac{x}{1}$ as an $\overline{R}_{\overline{p}}$-module. On the other hand, we have

$$\begin{aligned}0 :_{\mathrm{Hom}_R(R_p,A)} \frac{x}{1} &\cong \mathrm{Hom}_{R_p}(\overline{R}_{\overline{p}}, \mathrm{Hom}_R(R_p, A)) \\ &\cong \mathrm{Hom}_{R_p}(\overline{R} \otimes_R R_p, \mathrm{Hom}_R(R_p, A)) \\ &\cong \mathrm{Hom}_R(\overline{R}, \mathrm{Hom}_R(R_p, A)) \\ &\cong \mathrm{Hom}_R(R_p, 0 :_A x) \cong \mathrm{Hom}_R(R_p, \mathrm{Hom}_{\overline{R}}(\overline{R}, 0 :_A x)) \\ &\cong \mathrm{Hom}_{\overline{R}}(\overline{R}_{\overline{p}}, 0 :_A x).\end{aligned}$$

So that (2) is a flat resolution of $\mathrm{Hom}_{\overline{R}}(\overline{R}_{\overline{p}}, 0 :_A x)$ as an $\overline{R}_{\overline{p}}$-module. If $\mathrm{H}_i(G^\bullet)$ shows the ith homology module of an arbitrary complex $G^\bullet$ then

$$\begin{aligned}\mathrm{Tor}_i^{\overline{R}_{\overline{p}}}(k(\overline{p}), \mathrm{Hom}_{\overline{R}}(\overline{R}_{\overline{p}}, 0 :_A x)) &\cong \mathrm{H}_i(k(\overline{p}) \otimes_{\overline{R}_{\overline{p}}} (\overline{R}_{\overline{p}} \otimes_{R_p} \mathrm{Hom}_R(R_p, F^\bullet))) \\ &\cong \mathrm{H}_i((k(\overline{p}) \otimes_{\overline{R}_{\overline{p}}} \overline{R}_{\overline{p}}) \otimes_{R_p} \mathrm{Hom}_R(R_p, F^\bullet)) \\ &\cong \mathrm{H}_{i+1}(k(p) \otimes_{R_p} \mathrm{Hom}_R(R_p, F^\bullet)) \\ &\cong \mathrm{Tor}_{i+1}^{R_p}(k(p), \mathrm{Hom}_R(R_p, A)).\end{aligned}$$

The result follows. □

COROLLARY 2.5. *Let A be an Artinian R-module and let x be a non-unit and non-zero divisor of R such that $A = xA$. If p is a prime ideal of R and $x \in p$ then for all $i \geq 0$,*

$$\pi_i(p/(x), 0 :_A x) = \pi_{i+1}(p, A).$$

Proof. Assume p is a prime ideal of R such that $x \in p$. In view of [7, Theorem 2.2] and (2.4) the result follows, that is

$$\begin{aligned}\pi_i(p/(x), 0 :_A x) &= \dim_{k(\overline{p})} \operatorname{Tor}_i^{\overline{R}_{\overline{p}}}(k(\overline{p}), \operatorname{Hom}_{\overline{R}}(\overline{R}_{\overline{p}}, 0 :_A x)) \\ &= \dim_{k(p)} \operatorname{Tor}_{i+1}^{R_p}(k(p), \operatorname{Hom}_R(R_p, A)) \\ &= \pi_{i+1}(p, A). \qquad \square\end{aligned}$$

In the following we obtain a minimal flat resolution for $0 :_A x$ as an R/xR-module by using a minimal flat resolution of A, where x and A is as in (2.5).

THEOREM 2.6. *Let A be an Artinian R-module and let x be a non-unit and non-zero divisor of R such that $A = xA$. Let*

$$\cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_0 \xrightarrow{d_0} A \longrightarrow 0$$

be a minimal flat resolution for A. Let $K = \ker d_0$. Then $R/xR \otimes_R K \cong 0 :_A x$ as R-and R/xR-modules, also the induced complex of R/xR-modules and R/xR-homomorphisms

$$(3) \qquad \cdots \to F_i \otimes_R R/xR \to \cdots \to F_1 \otimes_R R/xR \to K \otimes_R R/xR \to 0$$

is a flat resolution for R/xR-module $K \otimes_R R/xR$. Furthermore, if

$$\cdots \longrightarrow G_i \xrightarrow{g_i} G_{i-1} \longrightarrow \cdots \longrightarrow G_0 \xrightarrow{g_0} 0 :_A x \longrightarrow 0$$

is a minimal flat resolution of $0 :_A x$ as an R/xR-module then $G_i \cong F_{i+1} \otimes_R R/xR$, for all $i \geq 0$.

Proof. The commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & K & \longrightarrow & F_0 & \longrightarrow & A & \longrightarrow 0 \\ & \downarrow^x & & \downarrow^x & & \downarrow^x & \\ 0 \longrightarrow & K & \longrightarrow & F_0 & \longrightarrow & A & \longrightarrow 0 \end{array}$$

in which the rows are exact, induces an exact sequence

$$0 :_K x \longrightarrow 0 :_{F_0} x \longrightarrow 0 :_A x \longrightarrow K/xK \longrightarrow F_0/xF_0 \longrightarrow A/xA.$$

Note that x is a non-zero divisor of R and F_0 is a flat module hence $0 :_{F_0} x = 0$. As mentioned in (2.2), $F_0 = \prod T_p$ with $p \in \operatorname{Att}_R(A)$. Then we have

$$\begin{aligned} F_0 \otimes_R R/xR &= (\prod_{p \in \mathrm{Att}_R(A)} T_p) \otimes_R R/xR \\ &\cong \prod_{p \in \mathrm{Att}_R(A)} (T_p \otimes_R R/xR) \\ &\cong \prod_{x \notin p} T_p/xT_p = 0. \end{aligned}$$

Thus $F_0/xF_0 = 0$, so that $0 :_A x \cong K/xK$ as R- and R/xR- modules. The exact sequence $F_2 \longrightarrow F_1 \longrightarrow K \longrightarrow 0$ shows that (3) is exact at $K \otimes_R R/xR$ and $F_1 \otimes_R R/xR$. If $n > 1$, the homology module of the complex

$$F_{i+1} \otimes_R R/xR \longrightarrow F_i \otimes_R R/xR \longrightarrow F_{i-1} \otimes_R R/xR$$

is isomorphic to $\mathrm{Tor}_i^R(A, R/xR)$, which is zero since the R-module R/xR has projective dimension ≤ 1. Thus (3) is exact. Also, $F_i \otimes_R R/xR$ is a flat R/xR-module for all $i \geq 1$. Hence, (3) is a flat resolution for $K \otimes_R R/xR$. Let

$$\cdots \longrightarrow G_i \xrightarrow{g_i} G_{i-1} \longrightarrow \cdots \longrightarrow G_0 \xrightarrow{g_0} 0 :_A x \longrightarrow 0$$

be a minimal flat resolution of $0 :_A x$ as an R/xR-module. So that $G_i = \prod U_{p/(x)}$, where $U_{p/(x)}$ is the completion of a free $(R/xR)_{p/(x)}$-module with respect to the $p/(x)(R/xR)_{p/(x)}$-adic topology, since $0 :_A x$ is an Artinian R/xR-module. On the other hand, for all $i \geq 0$

$$F_{i+1} \otimes_R R/xR = (\prod T_p) \otimes_R R/xR = \prod T_p/xT_p = \prod T_{p/(x)}.$$

Now (2.5) shows that G_i and $F_{i+1} \otimes_R R/xR$ have the same factors when they are written as products as above. More precisely, if $T_p \otimes_R R/xR \subset F_{i+1} \otimes_R R/xR$ then $T_p \subset F_{i+1}$. Hence, in view of (2.5), G_i has a direct summand as $U_{p/(x)}$, where $U_{p/(x)}$ is the completion of a free $(R/xR)_{p/(x)}$-module with a base having the same cardinality as that of the base of the free R_p-module whose completion is T_p. Thus $U_{p/(x)} \cong T_p \otimes_R R/xR$. A same argument shows that if $U_{q/(x)} \subset G_i$ then there is $T_q \subset F_{i+1}$ such that $U_{q/(x)} \cong T_q \otimes_R R/xR$, for some prime ideal q of R with $x \in q$. Now assume $f_i : G_i \longrightarrow F_{i+1} \otimes_R R/xR$ is the induced isomorphism. Then

$$\cdots \longrightarrow F_i \otimes_R R/xR \xrightarrow{e_i} \cdots \longrightarrow F_1 \otimes_R R/xR \xrightarrow{e_0} 0 :_A x \longrightarrow 0$$

is a minimal flat resolution of $0 :_A x$, where $e_0 = g_0 f_0^{-1}$ and $e_i = f_{i-2} g_{i-1} f_{i-1}^{-1}$, for all $i \geq 0$. This completes the proof of theorem. □

The next easy corollary is in fact an important "change of rings" result on flat dimension (which we write as f.dim).

COROLLARY 2.7. *If A is an Artinian R-module and x is a non-unit and non-zero divisor of R such that $A = xA$ then*

$$f.\dim_R A \geq f.\dim_{R/xR}(0 :_A x) + 1.$$

For the remaining part of this section, we shall assume that $(R, \underline{m})$ is a local ring with maximal ideal $\underline{m}$ and dimension d.

Let $\mathrm{H}^d_{\underline{m}}(R)$ denotes dth local cohomology module of R with respect to the ideal $\underline{m}$. In [4] it was proved that if R is Cohen-Macaulay then $f.\dim_R \mathrm{H}^d_{\underline{m}}(R) = d$. The next theorem presents a minimal flat resolution for $\mathrm{H}^d_{\underline{m}}(R)$ when R satisfies (S_n). For $n \in \mathbb{N}$ we say that R satisfies (S_n) if $\min\{\mathrm{ht}p, n\} \leq \mathrm{depth} R_p$ for all $p \in Spec(R)$.

THEOREM 2.8. *If $(R, \underline{m})$ satisfies (S_n) then $\pi_i(p, \mathrm{H}^d_{\underline{m}}(R)) \neq 0$ implies that $\min\{htp, n\} \leq i$, for all $i \geq 0$.*

Proof. Let

$$\cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow \mathrm{H}^d_{\underline{m}}(R) \longrightarrow 0$$

be a minimal flat resolution for $\mathrm{H}^d_{\underline{m}}(R)$. In view of [2, Theorem 7.1.3], $\mathrm{H}^d_{\underline{m}}(R)$ is an Artinian R-module so it is cotorsion. Hence, as mentioned before, $F_i = \prod T_q$, for all $i \geq 0$. We will show that if T_p appears in the product of F_i then $\min\{htp, n\} \leq i$. If $T_p \subset F_0$ then (2.2) shows that $p \in \mathrm{Att}_R(\mathrm{H}^d_{\underline{m}}(R))$. Hence, by using [2, Theorem 7.3.2], we have $p \in \mathrm{Ass}_R(R)$ and finally $pR_p \in \mathrm{Ass}_{R_p}(R_p)$. Thus $\min\{\mathrm{ht}p, n\} \leq \mathrm{depth} R_p = 0$. So assume inductively that $k \geq 0$ and the result has been proved (for all choices of R and A satisfying the hypothesis) when $i = k$; let p be a prime ideal of R and let T_p appear in the product of F_{k+1}. If $p \subseteq Z(R)$ then there is $q \in \mathrm{Ass}_R(R)$ such that $p \subseteq q$. Thus $\min\{\mathrm{ht}p, n\} \leq \min\{\mathrm{ht}q, n\} \leq \mathrm{depth} R_q = 0 \leq k + 1$. Now assume $p \not\subseteq Z(R)$. Hence there is a non-zero divisor $x \in p$ such that $\mathrm{H}^d_{\underline{m}}(R) = x\mathrm{H}^d_{\underline{m}}(R)$; since

$$\bigcup_{q \in \mathrm{Att}_R(\mathrm{H}^d_{\underline{m}}(R))} q \subseteq \bigcup_{q \in \mathrm{Ass}_R(R)} q = Z(R).$$

By using (2.5) and (2.6) we know that $T_p/xT_p = U_{p/(x)}$ appears in the product of the kth term in a minimal flat resolution of $0 :_{\mathrm{H}^d_{\underline{m}}(R)} x$. We show that all conditions are preserved by R/xR and $0 :_{\mathrm{H}^d_{\underline{m}}(R)} x$. It is easy to see that R/xR satisfies (S_{n-1}). It is enough to show that $\mathrm{Att}_{R/xR}(0 :_{\mathrm{H}^d_{\underline{m}}(R)} x) \subseteq \mathrm{Ass}_{R/xR}(R/xR)$. The exact sequence $0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/xR \longrightarrow 0$ induces the following exact sequence

$$\mathrm{H}^{d-1}_{\underline{m}}(R/xR) \longrightarrow \mathrm{H}^d_{\underline{m}}(R) \xrightarrow{x} \mathrm{H}^d_{\underline{m}}(R) \longrightarrow 0.$$

Also, the following sequence is exact

$$\mathrm{H}^{d-1}_{\underline{m}}(R/xR) \longrightarrow 0 :_{\mathrm{H}^d_{\underline{m}}(R)} x \longrightarrow 0.$$

Hence,

$$\begin{aligned} \mathrm{Att}_{R/xR}(0 :_{\mathrm{H}^d_{\underline{m}}(R)} x) &\subseteq \mathrm{Att}_{R/xR}(\mathrm{H}^{d-1}_{\underline{m}}(R/xR)) \\ &= \mathrm{Att}_{R/xR}(\mathrm{H}^{d-1}_{\underline{m}/(x)}(R/xR)) \\ &\subseteq \mathrm{Ass}_{R/xR}(R/xR). \end{aligned}$$

Hence, by the inductive hypothesis $\min\{\, \mathrm{ht}p/(x), n-1\} \leq k$. So that $\min\{\, \mathrm{ht}p, n\} \leq k+1$. This completes the inductive step. The theorem follows by induction. □

COROLLARY 2.9. *If $(R, \underline{m})$ is Cohen-Macaulay ring then*

$$\pi_i(p, \mathrm{H}^d_{\underline{m}}(R)) \neq 0$$

implies that $\mathrm{ht}p \leq i$, *for all* $i \geq 0$.

Proof. By the preceding theorem and Lemma 3.1 in [6] we obtain a minimal flat resolution for $\mathrm{H}^d_{\underline{m}}(R)$ as follow

$$0 \longrightarrow \prod_{\mathrm{ht}p \leq d} T_p \longrightarrow \prod_{\mathrm{ht}p \leq d-1} T_p \longrightarrow \cdots \longrightarrow \prod_{\mathrm{ht}p \leq 0} T_p \longrightarrow \mathrm{H}^d_{\underline{m}}(R) \longrightarrow 0. \ \square$$

The following corollary can be considered as an special case of [11, Theorem 3.1].

COROLLARY 2.10. *Let $(R, \underline{m})$ be a Gorenstein ring. If $\pi_i(p, E(R/\underline{m})) \neq 0$ then* $\mathrm{ht}p \leq i$ *for all* $i \geq 0$.

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