J. Korean Math. Soc. **39** (2002), No. 4, pp. 543–558

STOCHASTIC FRAGMENTATION AND SOME SUFFICIENT CONDITIONS FOR SHATTERING TRANSITION

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ABSTRACT. We investigate the fragmentation process developed by Kolmogorov and Filippov, which has been studied extensively by many physicists (independently for some time). One of the most interesting phenomena is the shattering (or disintegration of mass) transition which is considered a counterpart of the well known gelation phenomenon in the coagulation process. Though no masses are subtracted from the system during the break-up process, the total mass decreases in finite time. The occurrence of shattering transition is explained as due to the decomposition of the mass into an infinite number of particles of zero mass. It is known only that shattering phenomena occur for some special types of break-up rates.

In this paper, by considering the n-particle system of stochastic fragmentation processes, we find general conditions of the rates which guarantee the occurrence of the shattering transition.

1. Introduction

The fragmentation process as well as the coagulation (or coalescence) process has received much attention recently [2, 4, 5, 7, 17]. Besides mathematical interests, these processes have many applications in physics and other natural sciences. In particular, the fragmentation process is an important subject for polymer degradation, break-up of many objects such as rocks, liquid droplets, glasses, etc [5, 6, 8, 14, 18]. Suppose there are n particles in a box. We call a cluster consisting of k particles a

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Received August 18, 2001.

2000 Mathematics Subject Classification: Primary 60K35; Secondary 82C22.

Key words and phrases: fragmentation, shattering transition, stochastic fragmentation, stochastic dominance.

This research was supported in part by KOSEF through the Statistical Research Center for Complex Systems at Seoul National University.

k-cluster, and a k-cluster splits into i and $k-i$ cluster $(1 \leq i \leq k/2)$ after waiting exponential amount of time with parameter given a priori. After suitable normalization, Aizenman and Bak derived a governing equation which is equivalent to

$$\frac{d}{dt}c(x,t) = -a(x)c(x,t) + \int_x^\infty b(x|y)a(y)c(y,t)dy, \tag{1}$$

where $c(x,t)$ is the distribution of particles of mass x at time t, $a(x)$ is the break up rate of x-cluster into smaller clusters, and $b(z|x)$ is the rate of creation of z-clusters conditional on the fragmentation of x-cluster [1].

The initial study of this fragmentation process dates back to Kolmogorov for mathematical interest [12]. Filippov soon generalized Kolmogorov's result by finding the limit distribution of the process for the case that, in the above notation, $a(x) = x^\alpha$ with some special types of $b(z|x)$ (Kolmogorov considered only the case $a(x) = x$). On the other hand, physicists have studied fragmentation dynamics, independently from mathematicians by investigating equation (1) for some time. As a result, some exact solutions and interesting phenomena have been proposed by many authors [6, 8, 14, 16, 18].

One of the most interesting phenomena is the shattering (or disintegration of mass) transition which is considered a counterpart of the well known gelation phenomenon in the coagulation process. Different from finding large size clusters, which is common stochastic particle system [12], shattering is related to find small size clusters. Indeed, this was first indicated by Filippov [9]. Though no masses are added to or subtracted from the system during the break-up process, for some kernels, the total mass $\int xc(x,t)dx$ decreases in finite time. This is explained (a little ambiguously) as due to the decomposition of the mass into an infinite number of particles of zero mass.

This counterintuitive phenomenon is due to the fact that the equation (1) is a normalized limit of stochastic process of finite particles [1]. Assume initially there is a single n-cluster. Suppose the speed of the fragmentation is so fast that there could exist $t_0 < \infty$, $\delta > 0$ and $\phi(n)$ such that t_0 does not depend on n, $\phi(n)/n \to 0$ (i.e., $\phi(n) = o(n)$), and at time t_0, the total number of particles which form clusters of size less then or equal to $\phi(n)$ exceeds δn. (This condition is analogous to the definition of the stochastic gelation in Section 1 of [11] and will be used to define the term stochastic shattering. See Definition 1 in Section 2.) Notice that the normalization to get the equation (1) is given by letting each particle has mass $1/n$ so that the total mass of the initial single n-cluster is 1. After letting n tend to infinity, the n-cluster becomes a cluster with mass 1 in the limiting continuous model. Similarly, $[\alpha n]$-cluster becomes a cluster with mass α and we rewrite the cluster α-cluster in continuous model. Any cluster whose size is less than or equal to $\phi(n)$ in stochastic model becomes massless and disappears in continuous model since $\phi(n)/n \to 0$. As a result, in the above stochastic model, since the δn amount of particles (or mass δ) will disappear in the continuous model by time t_0, mass dropping phenomenon occurs. See Theorem 5 in [10] for the analogous result of gelation.

The occurrence of shattering is known only for the case $a(x) \sim 1/x^\alpha$, $\alpha > 0$ with a special form of $b(z|x)$, mostly when the explicit solution is known or after the assumption that the solution has some special types [6, 9, 16]. It is the motivation of our paper to find more general conditions which guarantee the occurrence of the shattering transition. Note that it is a matter of estimating the speed of convergence of the Markov process to its limit distribution. However, though tremendous results are known about the speed of Markov processes, since the configuration space of this fragmentation process is the partitions of n, (see Section 2) and because of the complexity of the space it is difficult to apply the well known theories.

Our idea is to estimate the speed of the process using a comparison with other simple processes. This procedure is carried out through the estimation of jump rates of special types. These types of stochastic dominance are justified by the coupling arguments [10, 11, 15]. Indeed, if we are interested in the speed of two processes, say Y_t^1, Y_t^2 and if the jump rates of Y_t^1 dominate those of Y_t^2 for all steps, then the speed of Y_t^1 stochastically dominates that of Y_t^2. We will use a slight modification of this idea in Section 3. One advantage of this method is that $a(x)$ and $b(z|x)$ do not need to have any form.

In this paper we mainly assume that a cluster breaks up into only two small clusters, i.e., binary fragmentation. Multiple fragmentation is just a simple generalization of this model, and our method can be applied without a big change (see the Remark after the proof of Theorem 1 in Section 3). Though only the case that $a(x)$ is a decreasing function of x was examined by previous authors, we also consider the case that $a(x)$ is increasing.

2. The stochastic fragmentation process and main theorems

In this section, we construct a sequence of finite state Markov chains associated with the rate constants $F(i,j),\ i,j \geq 1$. In the nth Markov chain, there are n particles which form clusters. These clusters fragment at rates determined by $F(i,j)$ to make smaller clusters (any $i+j$-cluster breaks up into i-cluster and j-cluster with rate $F(i,j)$). After suitable scaling, the Markov chains can be thought of as discrete, stochastic approximations to solutions of the fragmentation equation (1) [1].

NOTATION.

(a) Let $\mathbb{N} = \{0,1,2,\cdots\},\ \mathbb{N}^+ = \{1,2,3,\cdots\}$.
(b) Let $E_n = \{\eta : \eta \in \mathbb{N}^{\mathbb{N}^+},\ \sum_{k=1}^{\infty} k\eta(k) = n\}$.
(c) $[\cdot]$ represents the largest integer function.
(d) Let $\{e_i\}_{i=1}^{\infty}$ be the basis of $\mathbb{R}^{\mathbb{N}^+}$, i.e., $e_i = (0,0,\cdots,0,1,0,0,\cdots)$, where 1 is located in the ith coordinate.

REMARK. Note that any $\eta \in E_n$ can be expressed by $(\eta(1),\ \eta(2),\ \cdots,\ \eta(n))$, since all $\eta(i) = 0$ if $i > n$, or using the basis defined on the Notation (d), $\eta = \sum_{i=1}^{n} \eta(i)e_i$. Here, e_i means that there is an i-cluster. The space E_n is indeed the partition of n, and the process X_t^n which will be defined on this space can be considered as a random partition process. The partition of n is known to be hard to analyze, which causes a great complexity for investigating the dynamics of coagulation or fragmentation.

Let $\{F(i,j)\}_{i,j=1}^{\infty}$ be a nonnegative sequence such that $F(i,j) = 0$ if $j < i$. For $i \leq j$, let $\Delta_{ij}^n = (e_i + e_j - e_{i+j})$. Let X_t^n be the Markov chain on E_n with generator

$$(2) \qquad L_1^n f(\eta) = \sum_{i+j \leq n} (f(\eta + \Delta_{i,j}^n) - f(\eta))F(i,j)\eta(i+j)$$

for any bounded function defined on E_n.

We may describe the dynamics as follows.
The process waits at state η for an exponentially distributed amount of time with parameter

$$(3) \qquad \lambda^n(\eta) \doteq \sum_{i+j \leq n} F(i,j)\eta(i+j),$$

then jumps to state $\eta + \Delta^n_{ij}$ (or $i+j$ cluster fragments to form i and j cluster) with probability

$$\frac{F(i,j)\eta(i+j)}{\lambda^n(\eta)}. \tag{4}$$

Since, for each n, the state space consists of finitely many points, i.e., $|E_n| < \infty$, there is a unique well defined pure jump process, say X^n_t on E_n for each n. Moreover, with probability 1, each sample path is right continuous and has left limit. We will call this sequence of processes $\{X^n_t\}_{n=1}^{\infty}$ the system of the stochastic fragmentation process, and we will denote it simply by X^n_t. In general, we assume that the initial configuration $X^n_0 = e_n \in E_n$, i.e., initially, there is a single n cluster.

In this system of processes, we can define the stochastic shattering phenomenon using the idea that in finite time at least δn, for some $\delta > 0$, amount of mass is located in the $o(n)$ order of clusters. More precisely:

DEFINITION 1. For given fragmentation kernels $F(i,j)$, we say stochastic shattering occurs if there exists a function $\phi(n)$ such that $\phi(n) = o(n)$, and exist $t_0 < \infty$, $\delta > 0$ satisfying

$$\liminf_{n\to\infty} P\{\sum_{i=1}^{[\phi(n)]} iX^n_{t_0}(i) \geq \delta n\} > 0.$$

Let $\tilde{E}_n$ be the space obtained from E_n by normalizing the mass of each particle and the coordinates by n, e.g., the kth coordinate becomes k/n-coordinate and the total mass becomes 1. That is,

$$\tilde{E}_n = \{\frac{1}{n}\sum_{i=1}^{n} \eta(i) e_{\frac{i}{n}} : \eta \in E_n\},$$

where $\{e_{\frac{i}{n}}\}$ is a new basis and $e_{\frac{i}{n}}$ means that there is an $\frac{i}{n}$-cluster. Now, let us define the system of scaled fragmentation process Y^n_t on $\tilde{E}_n$ with generator

$$L^n_2 f(\eta) = \sum_{i+j\leq n} (f(\eta + \Delta^n_{i,j}) - f(\eta))\frac{F(i,j)}{F(n)}\eta(i+j) \tag{5}$$

for any bounded function defined on $\tilde{E}_n$, where $\eta \in \tilde{E}_n$, $\Delta^n_{i,j} = \frac{1}{n}(e_{i/n} + e_{j/n} - e_{(i+j)/n})$ and $F(n) = \sum_{1\le i\le n-1} F(i, n-i)$.

The differences between X^n_t and Y^n_t are that k-cluster becomes k/n-cluster and the jump rate $F(k)$ becomes $F(k)/F(n)$ so that n-cluster is normalized to 1-cluster and the fragmentation rate of this 1-cluster is normalized to 1.

DEFINITION 2. For given fragmentation kernels $F(i,j)$, we say shattering occurs if there exists a function $\phi(n)$ such that $\phi(n) = o(n)$, and exist $t_0 < \infty$, $\delta > 0$ satisfying

$$\liminf_{n\to\infty} P\{\sum_{i=1}^{[\phi(n)]} \frac{i}{n} Y^n_{t_0}(\frac{i}{n}) \ge \delta\} > 0.$$

REMARK. Suppose $F(i,j)$ are chosen so that $\sum_{i+j\le k} F(i,j)/F(n) \to a(x)$, and

$$\sum_{i=1}^{j} F(i, k-i)/F(k) \to \int_0^z b(t|x)dt,$$

where $x = \lim_{n,k\to\infty} k/n$ and $z = \lim_{n,j\to\infty} j/n$. Then Y^n_t is a discrete, stochastic approximation of (1). Therefore, the mass located on the $o(n)$ cluster becomes zero mass in the limit.

From now on, to make the notation simple, we will omit the largest integer symbol $[\cdot]$ if there is no difference in calculating the asymptotics. That is, if there is a number which is not an integer form but should be, then notice that $[\cdot]$ is omitted. For example, in the following Condition 1, ϵn means $[\epsilon n]$ and in the following Lemma 1, $2^{l+1}/\epsilon$ is in fact $[2^{l+1}/\epsilon]$.

Throughout this paper we assume that $F(i,j)$ satisfy the following weak scaling property.

CONDITION 1. There exist $\epsilon, 0 < \epsilon < 1/2$ and $\gamma > 0$ such that

$$\frac{\sum_{i=\epsilon k}^{k/2} F(i, k-i)}{F(k)} \ge \gamma \tag{6}$$

for all k, where $F(k) = \sum_{1\le i\le k-1} F(i, k-i)$.

REMARK. This assumption implies that if k-cluster breaks up then with positive probability it becomes two clusters of size bigger than or equal to ϵk. Therefore, it corresponds to the condition of $\int_\epsilon^x b(z|x)dz \geq \gamma$ in (1). This condition excludes the dust evaporation phenomena. For example, if $F(i, k-i) = f(k)\delta_i^1$, where $\delta_a^b = 1$ if $a = b$ and 0 if $a \neq b$ (i.e., Becker-Döring type fragmentation [3]), it does not satisfy the Condition 1. In this case, any cluster emits only a single particle which is invisible in the system ($o(n)$ order). Consequently, for some $F(k)$ (e.g., $F(k) = k^\alpha, \alpha > 1$), a huge cluster, though we can not detect any fragmentation, reduces its mass as time passes. Obviously, there is no deterministic counterpart of this phenomenon.

As explained in Section 1, we consider the two cases that

$$F(k) = \sum_{1 \leq i \leq k-1} F(i, k-i)$$

is increasing and/or is decreasing. The former implies that the bigger cluster has a smaller fragmentation rate, and the latter implies the reverse.

THEOREM 1. *For $\epsilon > 0$, given in Condition 1, let $a_{il} = 2^l/\epsilon^i$.*

(a) *Suppose $F(k)$ is decreasing and there exists $M < \infty$ such that*

$$\sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \frac{a_{il}}{F(a_{il}n)} \leq M \tag{7}$$

for all n, then stochastic shattering occurs.

(b) *Suppose $F(k)$ is decreasing and there exists $M < \infty$ such that*

$$\sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \frac{a_{il}F(n)}{F(a_{il}n)} \leq M \tag{8}$$

for all n, then shattering occurs.

In the deterministic analogue, n-cluster of the stochastic model is normalized to 1-cluster. Obviously, $a(x) = 1/x^\alpha$ in Section 2 matches $F(k) = 1/k^\alpha$ in Y_t^n. In this case we have

COROLLARY 1. *If $F(k) = 1/k^\alpha, \alpha > 1$, then shattering occurs.*

REMARK. For special form of $b(x|y)$ (for example, $b(x|y) = (v + 2)x^v/y^{v+1}, v > -2$), it is known that the phase transition occurs at $\alpha = 0$, i.e., shattering occurs if $\alpha > 0$ but not if $\alpha < 0$ [16]. Our result is weaker than this, however, we relax the condition of $b(x|y)$.

THEOREM 2. *Let $a_{il} = 2^{l+1}/\epsilon^i$. Suppose $F(k)$ is increasing and there exist $\psi_1(n)$, $\psi_2(n)$ such that $\psi_1(n)/\log n \to \infty$ and $\psi_2(n)/\log n \to \infty$ as $n \to \infty$ and*

$$\sum_{i=0}^{\psi_1(n)} \sum_{l=0}^{\psi_2(n)} \frac{a_{il}}{F(a_{il}n)} \leq M < \infty \tag{9}$$

for all n. Then stochastic shattering occurs.

COROLLARY 2. *If $F(k) = k^\alpha, \alpha \geq 1$, then stochastic shattering occurs.*

3. Proofs

First assume $\gamma = 1$ in (6) and consider the case that $F(k)$ is decreasing. Let $n_l = (1/2)^l n$ and let B^l be the set of all configurations whose maximum cluster size is less than or equal to n_l, i.e.,

$$B^l = \{\eta \in E^n : \sum_{i=1}^{n_l} i\eta(i) = n\}$$

for any $l \geq 0$ satisfying $n_l \geq 1$. Note that n_l is decreasing and $B^{l+1} \subset B^l$.

LEMMA 1. *For any $\eta \in B^l$, if there are $2^{l+1}/\epsilon$ jumps including clusters of size bigger than n_{l+1}, then $\eta \in B^{l+1}$.*

Proof. For any $\eta \in B^l$, consider η' which is obtained from η through a fragmentation of a cluster of size bigger than n_{l+1}, i.e., $\eta' = \eta + \Delta_{ij}^n$ for some i, j $(i \leq j)$ such that $i + j > n_{l+1}$. Let $m = n_{l+1} + 1$, then from (6), since $\gamma = 1$, we have $\epsilon n_{l+1} < i \leq \frac{1}{2}n_l = n_{l+1}$ and

$$\begin{aligned} \sum_{k=m}^{n_l} k\eta'(k) &\leq \sum_{k=m}^{n_l} k\eta(k) - i \\ &\leq \sum_{k=m}^{n_l} k\eta(k) - \epsilon n_{l+1}. \end{aligned}$$

Since $\sum_{k=m}^{n_l} k\eta(k) \leq n$, if $\tilde{\eta} \in E^n$ is obtained from η by $2^{l+1}/\epsilon$ jumps including only clusters of size bigger than n_{l+1}, then

$$\sum_{k=m}^{n_l} k\tilde{\eta}(k) \leq n - \epsilon n_{l+1} 2^{l+1}/\epsilon = 0,$$

i.e., there is no cluster of size bigger than n_{l+1}. Since $\eta \in B^l$, $\sum_{k=m}^{n}$ $k\tilde{\eta}(k) = 0$ and therefore $\tilde{\eta} \in B^{l+1}$. □

Now let us estimate the sum of the rates of fragmentation including only clusters of size between n_{l+1} and n_l. Note that each cluster breaks up independently of each other. We begin by letting

$$K^l = \{k \in N : n_{l+1} < k \leq n_l\},$$

and let

$$\lambda^l(\eta) = \sum_{k \in K^l} \eta(k) F(k),$$

i.e., the sum of jump rates of η including only clusters of size in K^l.

LEMMA 2. *For any $\eta \in B^l \setminus B^{l+1}$, $\lambda^l(\eta) \geq F(n_l)$.*

Proof. It is clear from the fact that $F(k)$ is decreasing and there exists $k_0 \in K^l$ such that $\eta(k_0) \geq 1$. □

For any $\eta \in B^l$, let $\tau_B^l(\eta)$ be the first time that X_t^n is in B^{l+1} given $X_0^n = \eta$, i.e.,

$$\tau_B^l(\eta) = \inf\{t > 0 : X_t^n \in B^{l+1} | X_0^n = \eta\}.$$

LEMMA 3. *For any $\eta \in B^l \setminus B^{l+1}$, we have*

$$E\tau_B^l(\eta) \leq \frac{2^l}{\epsilon F(n_l)}. \tag{10}$$

Proof. The proof will be done by using the comparison method introduced in Section 1. Let $\mu = F(n_l)$ and let Y_t be a Poisson process with jump intensity μ. Assume $X_0^n = \eta$ and define $\{T_k\}$, the jump time of X_t^n including only clusters of size in K^l, successively by

$$T_0 = 0,$$
$$T_k = \begin{cases} \inf\{t > 0 : X^n_{(T_{k-1}+t)^-}(m) \neq X^n_{(T_{k-1}+t)}(m) \text{ for some } m \in K^l\} \\ \qquad \text{if } X^n_{\tilde{T}_{k-1}} \in B^l \setminus B^{l+1} \\ 0 \quad \text{if } X^n_{\tilde{T}_{k-1}} \in B^{l+1}, \end{cases}$$

where $\tilde{T}_k = T_1 + T_2 + \cdots + T_k$ and $X^n_{t^-} = \lim_{s\to t^-} X^n_s$ (note that with probability 1, the limit exists) and the infimum of an empty set is ∞ by convention. Note that if $X^n_{\tilde{T}_k} \in B^{l+1}$ then $T_j = 0$ for all $j \geq k+1$. Clearly, we have

$$\tau^l_B(\eta) = T_1 + T_2 + \cdots + T_{2^l/\epsilon}$$

from Lemma 1. Indeed, we know that before making $2^k/\epsilon$ jumps $X^n_t \in B^{l+1}$, i.e., $T_j = 0$ for all $j > 2^l/\epsilon$.

Now, define $\{T'_k\}$, the jump time of Y^n_t successively by

$$T'_0 = 0,$$
$$T'_k = \inf\{t > 0 : Y_{(T_{k-1}+t)^-} \neq Y_{(T_{k-1}+t)}\}.$$

Then T'_k's are iid and exponential random variables with parameter μ. Since if $X^n_t \in B^l \setminus B^{l+1}$, then by Lemma 2 the total jump rate including only clusters of size in K^l is greater than or equal to μ, T_k is stochastically dominated by T'_k for $k = 1, 2, \cdots$. Therefore, the expectation of $\tau^l_B(\eta)$ can be estimated by

$$\begin{aligned} E\tau^l_B(\eta) &= E(T_1 + T_2 + \cdots + T_{2^l/\epsilon}) \\ &\leq E(T'_1 + T'_2 + \cdots + T'_{2^l/\epsilon}) \\ &\leq 2^l/\epsilon E(T'_1) \\ &= \frac{2^l}{\epsilon F(n_l)}, \end{aligned} \tag{11}$$

and we are done. □

Recall ϵ given in (6), and let A^l be the set of all configurations whose maximum cluster size is less than or equal to $\epsilon^l n$, i.e.,

$$A^l = \{\eta \in E_n : \sum_{i=1}^{\epsilon^l n} i\eta(i) = n\}.$$

Note that $A^{l+1} \subset A^l$ for any l satisfying $\epsilon^l n \geq 1$.

For any $\eta \in A^0$, let

$$\tau_A^0(\eta) = \inf\{t > 0 : X_t^n \in A^1 | X_0^n = \eta\},$$

and let l_0 be the largest integer satisfying $n_{l_0} \geq \epsilon n$, i.e., $(1/2)^{l_0} \geq \epsilon$. Recall $n_l = (1/2)^l n$.

LEMMA 4. *For any* $\eta \in A^0$,

$$E\tau_A^0(\eta) \leq \frac{1}{\epsilon} \sum_{l=0}^{l_0} \frac{2^l}{F(n_l)}. \tag{12}$$

Proof. Since $n_{l_0+1} < \epsilon n$, $B^{l_0+1} \subset A^1$. Therefore

$$\begin{aligned}\tau_A^0(\eta) &= \inf\{t > 0 : X_t^n \in A^1 | X_0^n = \eta\} \\ &\leq \inf\{t > 0 : X_t^n \in B^{l_0+1} | X_0^n = \eta\} \\ &\leq \tau_B^0(\eta_0) + \tau_B^1(\eta_1) + \cdots + \tau_B^{l_0}(\eta_{l_0})\end{aligned}$$

for some $\eta_l \in B^l$, $l = 0, 1, \cdots, l_0$. From Lemma 3,

$$E\tau_A^0(\eta) \leq \sum_{l=0}^{l_0} E\tau_B^l(\eta_l) \leq \frac{1}{\epsilon} \sum_{l=0}^{l_0} \frac{2^l}{\epsilon F(n_l)},$$

as desired. □

Now let us consider the general case. For any $\eta \in A^i \setminus A^{i+1}$, let

$$\tau_A^i(\eta) = \inf\{t > 0 : X_t^n \in A^{i+1} | X_0^n = \eta\},$$

i.e., the first time that X_t^n has maximum cluster size less than or equal to $\epsilon^{i+1} n$, given initial maximum cluster size between $\epsilon^{i+1} n + 1$ and $\epsilon^i n$.

LEMMA 5.

$$E\tau_A^i(\eta) \leq \frac{1}{\epsilon^{i+1}} \sum_{l=0}^{l_0} \frac{2^l}{F(\epsilon^i n_l)}. \tag{13}$$

Proof. The worst case is that $X_0^n = 1/\epsilon^i e_{\epsilon^i n}$, i.e., there are $1/\epsilon^i$ amount of $\epsilon^i n$ clusters. In fact, using the largest integer function, we can express X_0^n rigorously by

$$X_0^n = [\frac{n}{[\epsilon^i n]}] e_{[\epsilon^i n]} + e_{n-[\epsilon^i n][\frac{n}{[\epsilon^i n]}]}.$$

But, since the final term is ignorable, there is no difference in the estimation. Now for each cluster of size $\epsilon^i n$, since it fragments independently of others, we see easily

$$\tau_A^i(\eta) \leq \frac{1}{\epsilon^i}\tau^i,$$

where τ^i is the first time $X_t^n \in A^{i+1}$, assuming that X_0^n is given by a single cluster of size $\epsilon^i n$ and all others are just single particles, i.e.,

$$\tau^i = \inf\{t > 0 : X_t^n \in A^{i+1} | X_0^n = e_{\epsilon^i n} + (1 - \epsilon^i) n e_1\}.$$

By the similar way of Lemmas 1 and 3, we make blocks $\tilde{B}^l$, such that

$$\tilde{B}^l = \{\eta \in E_n : \sum_{k=1}^{\epsilon^i n_l} k\eta(k) = n\},$$

and let

$$\tau_{\tilde{B}}^l = \inf\{t > 0 : X_t^n \in \tilde{B}^{l+1} | X_0^n = e_{\epsilon^i n_l} + (n - \epsilon^i n_l) e_1\},$$

i.e., the first time that X_t^n is in $\tilde{B}^{l+1}$ given X_0^n has one $\epsilon^i n_l$-cluster and all others are just single particles. Note that in this case $X_0^n \in \tilde{B}^{l+1}$. Then (10) in Lemma 3 is generalized to

$$E\tau_{\tilde{B}}^l \leq \frac{2^l}{\epsilon F(\epsilon^i n_l)}.$$

By the same way of Lemma 4, with the same l_0,

$$E\tau^i \leq \sum_{l=0}^{l_0} E\tau_{\tilde{B}}^l \leq \frac{1}{\epsilon}\sum_{l=0}^{l_0} \frac{2^l}{F(\epsilon^i n_l)}. \tag{14}$$

Therefore,

$$E\tau_A^i(\eta) \leq \frac{1}{\epsilon^i}\tau^i \leq \frac{1}{\epsilon^{i+1}}\sum_{l=0}^{l_0} \frac{2^l}{F(\epsilon^i n_l)}, \tag{15}$$

and we are done. □

Let $\phi(n)$ be a function such that $\phi(n) = o(n)$ and monotone increasing and $\lim_{n\to\infty} \phi(n) = \infty$. Let

$$A = \{\eta \in E^n : \sum_{i=1}^{\phi(n)} i\eta(i) = n\},$$

i.e., the maximum cluster size is less than or equal to $\phi(n)$, and let T^n be the first time that X_t^n is in A, initially given a single n-cluster, i.e.,

$$T^n = \inf\{t > 0 : X_t^n \in A | X_0^n = e_n\}.$$

Our aim is to show that with positive probability, $T^n \le M$ for some constant M which does not depend on n. This obviously can be interpreted, for large n, the initially given n-cluster fragments into small clusters of order less than n in finite time, i.e., shattering occurs.

LEMMA 6. *Let i_0 be the first number such that $\epsilon^{i_0} n \le \phi(n)$, i.e., $i_0 \le \log_\epsilon \frac{\phi(n)}{n}$, then*

$$ET^n \le \sum_{i=0}^{i_0} \sum_{l=0}^{l_0} \frac{2^l}{\epsilon^{i+1} F(\epsilon^i n_l)}. \tag{16}$$

Proof. It is an easy consequence of Lemma 5;

$$ET^n \le \sum_{i=0}^{i_0} E\tau_A^i \le \sum_{i=0}^{i_0} \sum_{l=0}^{l_0} \frac{2^l}{\epsilon^{i+1} F(\epsilon^i n_l)}, \tag{17}$$

and we are done. □

Proof of Theorem 1. For any $0 < \gamma \le 1$, since (6) implies

$$\sum_{i=\epsilon k}^{k/2} F(i, k-i) F(k) \ge \gamma F(k) \tag{18}$$

for large k, the only difference is that γ is multiplied to the jump rates in the previous lemmas. We can ignore any fragmentation which makes clusters of size less than or equal to ϵk, since such fragmentation makes the speed of the process faster. Therefore,

$$
\begin{aligned}
ET^n &\le \sum_{i=0}^{i_0}\sum_{l=0}^{l_0} \frac{2^l}{\epsilon^{i+1}\gamma F(\epsilon^i n_l)} \\
&\le \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \frac{2^l}{\epsilon^{i+1}\gamma F(\epsilon^i n_l)} \qquad (19)\\
&\le \frac{M}{\epsilon\gamma}
\end{aligned}
$$

for all n. Now let $t_0 = 2M/\epsilon\gamma$, then

$$P\{\sum_{i=1}^{[\phi(n)]} iX_{t_0}^n(i) = n\} = P\{X_{t_0}^n \in A\} = P\{T^n \le t_0\} \ge \frac{1}{2}$$

since $E(T^n) \ge t_0/2$. Therefore, stochastic shattering occurs.

Now for part (b), the only difference is that the term $F(n)$ is multiplied in (16). □

REMARK. As we see in the proof, the fragmentation do not have to be binary. The only important condition is (6) and we can apply the same method for multiple fragmentation to approximate $b(z|x)$.

Proof of Corollary 1. Note that $F(k) = 1/k^\alpha$, $\alpha > 1$, is monotone decreasing, and we have

$$
\begin{aligned}
\sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \frac{a_{il}F(n)}{F(a_{il}n)} &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty} \frac{2^l}{\epsilon^{i+1}\gamma(2^l/\epsilon^i)^\alpha} \qquad (20)\\
&= \frac{1}{\epsilon\gamma}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}(\epsilon^{\alpha-1})^i(\frac{1}{2^{\alpha-1}})^l \\
&< \infty.
\end{aligned}
$$

Therefore $F(k)$ meets the condition (b) of Theorem 1. □

Proof of Theorem 2. In the above lemmas, the only difference is the estimation of jump rates on each block. For example, in Lemma 2, $\lambda^l(\eta) \ge F(n_{l+1})$ and in Lemma 3

$$E\tau_B^l(\eta) \le \frac{2^l}{\epsilon F(n_{l+1})} = \frac{2^{l+1}}{2\epsilon F(n_{l+1})}.$$

Consequently, since l_0 and i_0 are of the order of $\log n$, (19) corresponds to

$$\sum_{i=0}^{\psi_1(n)} \sum_{l=0}^{\psi_2(n)} \frac{a_{il}}{2\epsilon\gamma F(a_{il}n)} \leq M'$$

for some $M' < \infty$. □

Proof of Corollary 2. Note that $F(k) = k^\alpha$, $\alpha \geq 1$, is monotone increasing, and we have

$$(21) \quad \sum_{i=0}^{\psi_1(n)} \sum_{l=0}^{\psi_2(n)} \frac{a_{il}}{F(a_{il}n)} \leq (\frac{1}{n})^\alpha \sum_{i=0}^{\psi_1(n)} \sum_{l=0}^{\psi_2(n)} \frac{1}{a_{il}^{\alpha-1}}$$
$$\leq (\frac{1}{n})^\alpha \sum_{i=0}^{\psi_1(n)} (\epsilon^{\alpha-1})^i \sum_{l=0}^{\psi_2(n)} (\frac{1}{2^{\alpha-1}})^{l+1}$$
$$\leq M'$$

for some M' if $\alpha \geq 1$. □

ACKNOWLEDGEMENT. I would like to thank the referee for valuable comments

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