

## EXPONENTIAL FAMILIES RELATED TO CHERNOFF-TYPE INEQUALITIES

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ABSTRACT. In this paper, the characterization results related to Chernoff-type inequalities are applied for exponential-type (continuous and discrete) families. Upper variance bound is obtained here with a slightly different technique used in Alharbi and Shanbhag [1] and Mohtashami Borzadaran and Shanbhag [8]. Some results are shown with assuming measures such as non-atomic measure, atomic measure, Lebesgue measure and counting measure as special cases of Lebesgue-Stieltjes measure. Characterization results on power series distributions via Chernoff-type inequalities are corollaries to our results.

### 1. Introduction

Characterization results related to variance bound established by Chernoff [5] and Cacoullos and Papathanasiou [2, 3, 4]. Goldstein and Reinert [6] and Papadatos and Papathanasiou [9] found variance bounds via Stein identity. Alharbi and Shanbhag [1] and Mohtashami Borzadaran and Shanbhag [8] published an extended version of Chernoff inequality. Hudson [7] and Prakasa Rao [11] characterized the exponential families. Also, Papathanasiou [7] characterized the power series distribution via the Chernoff-type inequality. We characterize the generalized exponential-type family by using a version of the Chernoff inequality. The following theorem and corollaries lead us to versions of the exponential families:

**THEOREM 1.1.** *With  $w(x) > 0$  for almost all  $[\nu_{F^*}]x \in \mathfrak{R}$ ,  $F$  is a distribution function that absolutely continuous with respect to  $\nu_{F^*}$  and*

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there exists a point  $x_0$  such that  $t(x) > \theta$  for almost all  $[\nu_{F^*}]x \in \mathfrak{R}$ , lying in  $(x_0, \infty)$  and  $t(x) < \theta$  for almost all  $[\nu_{F^*}]x \in \mathfrak{R}$ , lying in  $(-\infty, x_0)$  with  $E(t(X)) = \theta$  and  $t(x_0) \geq \theta$ , where  $X$  is a random variable with a distribution function  $F$ . Then  $F$  satisfies (10) if and only if for some  $c \in (0, \infty)$ ,

$$(1) \quad dF(x) = \begin{cases} c \left\{ \frac{1}{w(x)} \left\{ \prod_{x_r \in D_x^{(1)}} (1 - m^{(1)}(\{x_r\})) \right\} e^{-H_c^{(1)}(x)} \right\} d\nu_{F^*}(x) & \text{if } x \geq x_0 \\ c \left\{ \frac{1}{w^*(x)} \left\{ \prod_{x_r \in D_x^{(2)}} (1 - m^{(2)}(\{x_r\})) \right\} e^{-H_c^{(2)}(x)} \right\} d\nu_{F^*}(x) & \text{if } x < x_0, \end{cases}$$

where

$$m^{(1)}(\bullet) = \int_{[x_0, \infty) \cap \bullet} (t(y) - \theta)(w(y))^{-1} d\nu_{F^*}(y), \quad H_c^{(1)}(x) = m_c^{(1)}((-\infty, x]),$$

and

$$m^{(2)}(\bullet) = \int_{(-\infty, x_0) \cap \bullet} (\theta - t(y))(w^*(y))^{-1} d\nu_{F^*}(y), \quad H_c^{(2)}(x) = m_c^{(2)}([x, \infty))$$

with  $0 < w^*(x) = w(x) + (\theta - t(x))\nu_{F^*}(\{x\})$ ,  $m_c^{(1)}$  and  $m_c^{(2)}$  as continuous parts of  $m^{(1)}$  and  $m^{(2)}$  respectively, and  $D_x^{(1)}$  and  $D_x^{(2)}$  as the sets of discontinuity points of  $m^{(1)}$  that lie in  $(-\infty, x)$  and of  $m^{(2)}$  that lie in  $(x, \infty)$ , respectively.

**COROLLARY 1.2.** *If we have the assumptions in Theorem 1.1 met with  $F^*$  continuous, then the conclusion of the theorem holds with the following in place of (2):*

$$(2) \quad dF_\theta(x) = \frac{c(\theta)}{w(x)} \exp\left\{-\int_{(x_0, x]} \frac{t(y) - \theta}{w(y)} d\nu_{F^*}(y)\right\} d\nu_{F^*}(x),$$

where  $x_0$  is as the statement of Theorem 1.1.

**COROLLARY 1.3.** *If we have the assumptions in Theorem 1.1 met with  $F^*$  continuous, then for any distribution  $F_\theta$  (that is, absolutely continuous with respect to  $\nu_{F^*}$ ) the conclusion of the theorem holds on taking, in place of (2), that  $F_\theta$  is concentrated on  $\mathcal{D}$  and it satisfies the*

following:

$$\begin{aligned}
 (3) \quad dF_\theta(x) &= \frac{c(\theta)}{w(x)} \exp\left\{-\int_{(x_0,x]} \frac{t(y)}{w(y)} d\nu_{F^*}(y)\right\} \exp\left\{\theta \int_{(x_0,x]} \frac{1}{w(y)} d\nu_{F^*}(y)\right\} d\nu_{F^*}(x) \\
 &= c(\theta)k_1(x) \exp\left\{\theta \int_{(x_0,x]} \frac{1}{w(y)} d\nu_{F^*}(y)\right\} d\nu_{F^*}(x), \quad x \in \mathcal{D},
 \end{aligned}$$

where

$$(4) \quad k_1(x) = \frac{1}{w(x)} \exp\left\{-\int_{(x_0,x]} \frac{t(y)}{w(y)} d\nu_{F^*}(y)\right\}.$$

REMARK 1.4. In Corollary 1.3,  $F^*(x) = x$ ,  $x \in \mathfrak{R}$ , implies that we have

$$\begin{aligned}
 dF_\theta(x) &= \frac{c(\theta)}{w(x)} \exp\left\{-\int_{(x_0,x]} \frac{t(y)}{w(y)} dy\right\} \exp\left\{\theta \int_{(x_0,x]} \frac{1}{w(y)} dy\right\} dx \\
 (5) \quad &= c(\theta)k_1(x) \exp\left\{\theta \int_{(x_0,x]} \frac{1}{w(y)} dy\right\} dx,
 \end{aligned}$$

in place of (4) and

$$(6) \quad k_1(x) = \frac{1}{w(x)} \exp\left\{-\int_{(x_0,x]} \frac{t(y)}{w(y)} dy\right\},$$

in place of (4).

Let us now define an exponential family that is a special case of (4). Let  $X$  be a random variable with a distribution function  $F_\theta$  that is absolutely continuous with respect to  $\nu_{F^*}$  with density  $f_\theta(x)$  of the form:

$$(7) \quad f_\theta(x) = c(\theta)k(x)e^{\theta\mu^*(x)}, \quad x \in \mathfrak{R}, \quad \theta \in \mathfrak{R},$$

where  $0 < k(x) = \exp\{-\int_{(x_0,x]} t(y)d\nu_{F^*}(y)\}$ , and  $\mu^*(x) = F^*(x) - F^*(a)$  and  $E_\theta[t(X)] = \theta$ . If  $w(x) \equiv 1$ , then we have the family (7) in place of (4). In (7) if we take,  $F^*(x) = x$ ,  $x \in \mathfrak{R}$ , then (7) simplifies to

$$(8) \quad f_\theta(x) \propto k(x)e^{\theta x}, \quad x \in \mathfrak{R}, \quad \theta \in \mathfrak{R}.$$

## 2. Exponential families via Chernoff-type inequalities

We characterize the generalized exponential-type family introduced in (4) by using a version of the Chernoff inequality using Alharbi and Shanbhag [1] and Mohtashami Borzadaran and Shanbhag [8].

LEMMA 2.1. *Let  $F^*$  be a non-constant Lebesgue-Stieltjes measure function on  $\mathfrak{R}$  and  $\nu_{F^*}$  be the measure on the Borel  $\sigma$ -field of  $\mathfrak{R}$  determined by it. Let  $X$  with a distribution function  $F$  be a random variable such that  $E\{t(X)\} = \theta$ , where  $t(\cdot)$  is increasing and let  $g'$  be as defined before. Then, for almost all  $[F]$   $a \in \mathfrak{R}$ ,*

$$\begin{aligned} & \int_{\mathfrak{R}} (g'(y))^2 \left\{ \int_{[y, \infty)} [t(x) - \theta] dF(x) \right\} d\nu_{F^*}(y) \\ &= \int_{\mathfrak{R}} [t(x) - \theta] \left( \int_{(a, x]} (g'(y))^2 d\nu_{F^*}(y) \right) dF(x), \end{aligned}$$

provided the left hand side of the identity is finite.

*Proof.* For details of the proof of Lemma see Alharbi and Shanbhag [1] and Mohtashami Borzadaran and Shanbhag [8].  $\square$

THEOREM 2.2. *Let  $F^*$  be a non-constant Lebesgue-Stieltjes measure function on  $\mathfrak{R}$  and  $\nu_{F^*}$  be the measure on the Borel  $\sigma$ -field of  $\mathfrak{R}$  determined by it. Let  $X$  be a random variable and  $t$  be a function that is absolutely continuous with respect to  $\nu_{F^*}$ , such that  $E\{t(X)\} = \theta$  and  $E(\{t(X)\}^2) < \infty$ . Further, let  $w$  be a Borel measurable function such that  $w(X) > 0$  a.s. and  $Var\{t(X)\} = E\{t'(X)w(X)\}$ , where  $t'$  is the Radon-Nikodym derivative of  $t$  with respect to  $\nu_{F^*}$ . Assume that  $t' > 0$  and let  $\tau$  be the class of real-valued absolutely continuous functions  $g$  with Radon-Nikodym derivative  $g'$  with respect to the measure  $\nu_{F^*}$  satisfying  $E(\{g(X)\}^2) < \infty$  and  $0 < E\{w(X) \frac{[g'(X)]^2}{t'(X)}\} < \infty$ . Then*

$$(9) \quad \sup_{g \in \tau} \frac{Var[g(X)]}{E\{w(X) \frac{[g'(X)]^2}{t'(X)}\}} = 1$$

if and only if the distribution function of  $X$  satisfies

$$(10) \quad \left( \int_{[x, \infty)} (t(y) - \theta) dF(y) \right) d\nu_{F^*}(x) = w(x) dF(x), \quad x \in \mathfrak{R}.$$

*Proof.* Note that if (9) holds with  $g = t$ , we have

$$\frac{\text{Var}[g(X)]}{E\{w(X) \frac{[g'(X)]^2}{t'(X)}\}} = 1,$$

and hence to have the “ if ” part of the theorem, it is sufficient if we show that

$$(11) \quad \int_{\mathfrak{R}} w(y) \frac{[g'(y)]^2}{t'(y)} dF(y) \geq \text{Var}\{g(X)\}, \quad g \in \tau.$$

By using Lemma 2.1, and assuming that  $X$  and  $X^*$  are independent random variables with the distribution function  $F$ , we have for almost all  $[F]$   $a \in \mathfrak{R}$ ,

$$\begin{aligned} \text{Var}\{g(X)\} &= \frac{1}{2} E\left\{ \left( \int_{(X^*, X]} g'(y) d\nu_{F^*}(y) \right)^2 \right\} \\ &= \frac{1}{2} E\left\{ \left( \int_{(X^*, X]} \frac{\sqrt{t'(y)} g'(y)}{\sqrt{t'(y)}} d\nu_{F^*}(y) \right)^2 \right\} \\ &\leq \frac{1}{2} E\left\{ \left( \int_{(X^*, X]} t'(y) d\nu_{F^*}(y) \right) \left( \int_{(X^*, X]} \frac{(g'(y))^2}{t'(y)} d\nu_{F^*}(y) \right) \right\} \\ (12) \quad &= \frac{1}{2} E\left\{ (t(X) - t(X^*)) \left( \int_{(X^*, X]} \frac{(g'(y))^2}{t'(y)} d\nu_{F^*}(y) \right) \right\} \\ &= E\left\{ (t(X) - \theta) \left( \int_{(a, X]} \frac{(g'(y))^2}{t'(y)} d\nu_{F^*}(y) \right) \right\} \\ &= \int_{\mathfrak{R}} (t(x) - \theta) \left( \int_{(a, x]} \frac{(g'(y))^2}{t'(y)} d\nu_{F^*}(y) \right) dF(x) \\ &= \int_{\mathfrak{R}} \frac{(g'(y))^2}{t'(y)} \left\{ \int_{[y, \infty)} (t(x) - \theta) dF(x) \right\} d\nu_{F^*}(y) \\ &= \int_{\mathfrak{R}} \frac{(g'(y))^2}{t'(y)} w(y) dF(y) \\ &= E\left\{ w(X) \frac{[g'(X)]^2}{t'(X)} \right\}. \end{aligned}$$

(Note that here and in what follows, we take  $\int_{(a,b]} = -\int_{(b,a]}$  if  $a > b$ .) To prove the “ only if ” part, we use an approach somewhat different to that used in Alharbi and Shanbhag [1]. For every real  $\gamma$  and  $u$ , let  $g_u$  be such that  $g'_u(x) = \gamma \cos(ux) + t'(x)$  be the Radon-Nikodym derivative of

it with respect to  $\nu_{F^*}$ . Then (11) gives that

$$\begin{aligned}
 \text{Var}\{g_u(X)\} &= \text{Var}\{g_u(X) - g_u(a)\} \\
 &= \text{Var}\left\{\int_{(a, X]} (t'(x) + \gamma \cos(ux)) d\nu_{F^*}(x)\right\} \\
 (13) \quad &= \text{Var}\{t(X)\} + \gamma^2 \text{Var}\left\{\int_{(a, X]} (\cos(ux)) d\nu_{F^*}(x)\right\} \\
 &\quad + 2\gamma \text{Cov}\left\{t(X), \int_{(a, X]} \cos(ux) d\nu_{F^*}(x)\right\} \\
 &\leq E\left\{\frac{w(X)}{t'(X)} [t'(X) + \gamma \cos(uX)]^2\right\}.
 \end{aligned}$$

It follows from (14), in view of the assumption

$$E(t'(X)w(X)) = \text{Var}(t(X)),$$

that

$$\begin{aligned}
 (14) \quad &\gamma^2 \left\{ \text{Var}\left\{\int_{(a, X]} \cos(ux) d\nu_{F^*}(x)\right\} - E\left\{\frac{w(X)}{t'(X)} \cos^2(uX)\right\} \right\} \\
 &+ 2\gamma \left[ \text{Cov}\left\{t(X), \int_{(a, X]} \cos(ux) d\nu_{F^*}(x)\right\} - E(w(X) \cos(uX)) \right] \leq 0.
 \end{aligned}$$

In view of (15), we get that

$$E\left\{(t(X) - \theta) \int_{(a, X]} \cos(ut) d\nu_{F^*}(t)\right\} = E(w(X) \cos(uX)).$$

This relation implies by Fubini's theorem that

$$\int_{\mathfrak{R}} \cos(ux) w(x) dF(x) = \int_{\mathfrak{R}} \cos(uy) \left\{ \int_{[y, \infty)} (t(x) - \theta) dF(x) \right\} d\nu_{F^*}(y).$$

The relation above also holds if  $\cos(ux)$  is replaced by  $\sin(ux)$  and hence when  $\cos(ux)$  is replaced by  $\exp\{iux\}$ ; then from the uniqueness theorem of the Fourier transforms the required result follows.  $\square$

REMARK 2.3. We can establish the "if" part of Theorem 2.2 via a slightly different way the same as mentioned in previous chapter.

The following theorems are essential versions of theorems related to covariance identities.

**THEOREM 2.4.** *Let  $F^*$  be a non-constant Lebesgue-Stieltjes measure function on  $\mathfrak{R}$  and  $\nu_{F^*}$  be the measure on the Borel  $\sigma$ -field of  $\mathfrak{R}$  determined by it, and let  $t$  and  $Z$  be Borel measurable functions. Let  $X$  be a random variable with a distribution function  $F_\theta$  such that  $t(X)$  is integrable with  $\theta = E_\theta[t(X)]$  and  $E_\theta(|Z(X)|I_{\{X \in (a,b)\}}) < \infty$  for every  $-\infty < a < b < \infty$  and satisfying the condition that  $\liminf_{x \rightarrow \infty} (t(x) - \theta) > 0$  if the right extremity of  $F_\theta$  equals  $\infty$ , and the condition that  $\liminf_{x \rightarrow -\infty} (\theta - t(x)) > 0$  if the left extremity of  $F_\theta$  equals  $-\infty$ . Further let  $\tau$  be the class of real-valued absolutely continuous functions  $g$  with Radon-Nikodym derivative  $g'$  with respect to the measure  $\nu_{F^*}$  (i.e. such that  $g(b) - g(a) = \int_{(a,b]} g'(x) d\nu_{F^*}(x)$  for all  $a$  and  $b$  with  $a < b$ ). Then, we have the condition*

$$(15) \quad \text{Cov}_\theta\{g(X), t(X)\} = E_\theta\{Z(X)g'(X)\},$$

met for all  $g \in \tau$  with  $E_\theta(|Z(X)g'(X)|) < \infty$ , if and only if

$$(16) \quad Z(x)dF_\theta(x) = \left\{ \int_{[x,\infty)} [t(z) - \theta]dF_\theta(z) \right\} d\nu_{F^*}(x), \quad x \in \mathfrak{R}.$$

**THEOREM 2.5.** *Let  $X$ ,  $g$ ,  $\tau$ ,  $Z$  and  $t$  be defined as in Theorem 2.4, but additionally with  $t(\cdot)$  absolutely continuous with respect to  $\nu_{F^*}$  and  $t(X)$  as nondegenerate square integrable satisfying*

$$(17) \quad \text{Var}_\theta\{t(X)\} = E_\theta(Z(X)t'(X))$$

(with two sides of the identity well defined and finite). Furthermore, let  $\tau^*$  be the set of  $g \in \tau$  for which  $g(X)$  is square integrable and  $E_\theta\{Z(X)g'(X)\}$  is defined and nonzero. Then

$$(18) \quad \inf_{g \in \tau^*} \frac{\text{Var}_\theta[g(X)]\text{Var}_\theta[t(X)]}{E_\theta^2\{Z(X)g'(X)\}} = 1$$

if and only if (16) holds.

Let  $\nu_{F^*}$  be a non-atomic measure, then the “only if” parts of Theorems 2.2, 2.4 and 2.5 characterize generalized continuous exponential family with the form (4). We have the following theorem related to characterization of generalized continuous exponential family for the case that  $F^*$  is continuous:

COROLLARY 2.6. Let  $X$ ,  $g$ ,  $\tau$ ,  $w$ , and  $t$  be defined as in Theorem 2.2. Also, let

$$\text{Var}_\theta(g(X)) \leq E_\theta\left\{w(X) \frac{[g'(X)]^2}{t'(X)}\right\}, \text{ for all } g \in \tau,$$

and equality holds when  $g$  is linear in  $t$ . Then the distribution of the random variable  $X$  is given by (4).

COROLLARY 2.7. Let  $X$ ,  $g$ ,  $\tau$ ,  $Z$ , and  $t$  be defined as in Theorem 2.4 but additionally with  $t$  absolutely continuous with respect to non-atomic measure  $\nu_{F^*}$ ,  $\tau^*$  as a subset of  $g \in \tau$  for which  $g(X)$  is square integrable with  $E_\theta(Z(X)g'(X)) \neq 0$  and  $t^2(X)$  integrable and  $\text{Var}_\theta\{t(X)\} = E_\theta(Z(X)t'(X))$ . Then

$$(19) \quad \text{Var}_\theta(g(X)) \geq \frac{E_\theta^2\{Z(X)g'(X)\}}{E_\theta(Z(X)t'(X))}, \text{ for all } g \in \tau^*$$

if and only if for all  $g \in \tau^*$ , the distribution of the random variable  $X$  is given by (4). Equality holds when  $g$  is linear in  $t$ .

- Corollary 2.7 with  $Z(x) \equiv 1$ , Corollary 2.6 with  $w(x) \equiv 1$ , Theorem 2.4 with  $Z(x) \equiv 1$  and Theorem 2.2 with  $w(x) \equiv 1$ , yield characterizations of the exponential family of the form (7).
- Let  $X$ ,  $g$ ,  $\tau$ ,  $Z$ , and  $t$  be defined as in Corollary 2.7 but  $F^*(x) = x$ ,  $x \in \mathfrak{R}$  and  $Z(x) \equiv 1$ . Then

$$\text{Var}_\theta(g(X)) \geq \frac{E_\theta^2\{g'(X)\}}{E_\theta(t'(X))}, \text{ for all } g \in \tau^*,$$

in place of (19), if and only if for all  $g \in \tau^*$ , the distribution of the random variable  $X$  is given by (8). Equality holds when  $g$  is linear in  $t$ .

- Let  $X$ ,  $g$ ,  $\tau$ ,  $w$ , and  $t$  be defined as in Theorem 2.2 but  $F^*(x) = x$ ,  $x \in \mathfrak{R}$  and  $w(x) \equiv 1$ . Also, let

$$\text{Var}_\theta\{g(X)\} \leq E_\theta\left\{\frac{[g'(X)]^2}{t'(X)}\right\}, \text{ for all } g \in \tau,$$

and equality holds when  $g$  is linear in  $t$ . Then the distribution of the random variable  $X$  is given by (8).



TABLE 1. Characterization Based on  $t(x)$ ,  $w(x)$  and Upper Bound of  $Var_{\theta}(g(X))$  in Continuous Case

| Upper bound for $Var_{\theta}(g(X))$ <sup>a</sup> | $w(x)$  | $t(x)$               | Range of the random variable $X$ | Name of Distributions |
|---|---------|----------------------|----------------------------------|-----------------------|
| $E_{\theta}\{[g'(X)]^2\}$                         | 1       | $x$                  | $x \in \mathfrak{R}$             | Normal                |
| $E_{\theta}\{\frac{X^2c^2}{2}[g'(X)]^2\}$         | $x$     | $2c^{-2} \ln x$      | $x \in (0, \infty)$              | Lognormal             |
| $E_{\theta}\{\frac{X}{c}[g'(X)]^2\}$              | $x$     | $cx - 1$             | $x \in (0, \infty), c > 0$       | Gamma                 |
| $E_{\theta}\{\frac{X^2(X-1)}{1+c}[g'(X)]^2\}$     | $x - 1$ | $\frac{1-c(x+1)}{x}$ | $x \in (0, 1), c > 0$            | Beta                  |
| $E_{\theta}\{\frac{1}{4c}[g'(X)]^2\}$             | $x$     | $2cx^2 - 1$          | $x \in (0, \infty), c > 0$       | Generalized Rayleigh  |
| $E_{\theta}\{e^{-X}[g'(X)]^2\}$                   | 1       | $e^x$                | $x \in (0, \infty), c > 0$       | Standard Log-gamma    |

<sup>a</sup>In this table  $c$  is a positive constant.

Based on the Corollary 2.6 when  $F^*(x) = x, x \in \mathfrak{R}$ , we have the following characterizations via  $t(x)$  and upper bounds of the variance of  $g$  as seen by Table 1.

REMARK 2.8. We can find based on Corollary 2.7 via  $t$  and lower bound of the variance of  $g$ , characterizations for some distributions analogous to those in Table 1.

### 3. Discrete families via Chernoff-type inequalities

Let  $\nu_{F^*}$  be concentrated on a countable set  $\mathcal{C}$ , such that  $\nu_{F^*}(\{x\}) = \beta > 0$ , then we have a discrete version of the Theorems 2.2, 2.4 and 2.5 as follows:

**COROLLARY 3.1.** *Let  $F^*$  be a non-constant Lebesgue-Stieltjes measure such that  $\nu_{F^*}$  is concentrated on a countable set  $\mathcal{C}$  with each point  $x$  of  $\mathcal{C}$  having  $\nu_{F^*}(\{x\}) = \beta$ ,  $\beta > 0$ . Also, let  $X$  be a random variable and  $t$  a strictly increasing function with  $E_\theta\{t(X)\} = \theta$ , and  $E_\theta(\{t(X)\}^2) < \infty$ . Further, let  $w$  be a function such that  $w(x) > 0$  for each  $x \in \mathcal{C}$ , and*

$$\text{Var}_\theta\{t(X)\} = E_\theta\left\{\frac{t(X) - t(X-)}{\beta}w(X)\right\}.$$

Further, let  $\tau$  be the class of real-valued functions  $g$  satisfying

$$E_\theta(\{g(X)\}^2) < \infty$$

and

$$0 < E_\theta\left\{w(X)\frac{[g(X) - g(X-)]^2}{t(X) - t(X-)}\right\} < \infty.$$

Then

$$(20) \quad \sup_{g \in \tau} \frac{\beta \text{Var}_\theta[g(X)]}{E_\theta\left\{w(X)\frac{[g(X) - g(X-)]^2}{t(X) - t(X-)}\right\}} = 1,$$

if and only if the probability density function of  $X$  satisfies in the following:

$$(21) \quad w(x)f(x) = \beta \sum_{\{y \in \mathcal{C}, y \geq x\}} (t(y) - \theta)f(y), \quad x \in \mathcal{C}.$$

*Proof.* The result follows as a corollary to Theorem 2.2 on taking  $F^*$  such that  $\nu_{F^*}$  is concentrated on  $\mathcal{C}$  such that  $\nu_{F^*}(\{x\}) = \beta$ ,  $x \in \mathcal{C}$ ; note that we have now  $g'(x) = \frac{g(x) - g(x-)}{\beta}$ ,  $x \in \mathcal{C}$  and  $t'(x) = \frac{t(x) - t(x-)}{\beta}$ ,  $x \in \mathcal{C}$ .  $\square$

Let  $X$  be a random variable with values in  $\mathcal{B}^*$ ; we call a discrete exponential family a shifted scaled discrete exponential family if  $F^*(x) = \beta \lfloor \frac{x-\alpha}{\beta} \rfloor$ ,  $x \in \mathcal{B}^*$ . Also, we define for each  $\beta > 0$ ,  $\Delta_\beta g(x)$  as

$$(22) \quad \Delta_\beta g(x) = \frac{g(x + \beta) - g(x)}{\beta}, \quad x \in \mathcal{B}^*,$$

where  $g(\cdot)$  is a real-valued function. Under the stated assumptions, we have the following specialized theorem leading us to a characterization of the shifted scaled discrete exponential family.

**THEOREM 3.2.** *Let  $g$  be a real-valued function defined on  $\mathcal{B}^*$  with  $E_\theta\{\frac{[\Delta_\beta g(X)]^2}{\Delta_\beta t^*(X)}\} < \infty$  for all  $g$  and  $X$  be distributed as (8) and  $\Delta_\beta t^*(x) > 0$  for all  $x \in \mathcal{B}^*$ . Then we have the following inequality:*

$$(23) \quad V[g(X)] \leq \beta^2 \theta E\left\{\frac{[\Delta_\beta g(X)]^2}{\Delta_\beta t^*(X)}\right\}.$$

*Proof.* The result follows on noting that in (8) the function  $k$  is such that  $w(x + \beta)k(x + \beta) = \beta k(x)$ ,  $x \in \mathcal{B}^*$  and hence, we have under the validity of (8),

$$\begin{aligned} \beta^2 \theta E\left\{\frac{[\Delta_\beta g(X)]^2}{\Delta_\beta t^*(X)}\right\} &= \beta \theta E\left\{\frac{[\Delta_\beta g(X)]^2}{\Delta_\beta t(X)}\right\} \\ &= \theta \sum_{\{x: x \in \mathcal{B}^*\}} \left\{\frac{[g(x + \beta) - g(x)]^2}{t(x + \beta) - t(x)} c(\theta) k(x) \theta^{\frac{x}{\beta}}\right\} \\ (24) \quad &= \sum_{\{x: x \in \mathcal{B}^*\}} \left\{\frac{[g(x) - g(x - \beta)]^2}{t(x) - t(x - \beta)} c(\theta) k(x - \beta) \theta^{\frac{x}{\beta}}\right\} \\ &= \frac{1}{\beta} \sum_{\{x: x \in \mathcal{B}^*\}} \left\{w(x) \frac{[g(x) - g(x - \beta)]^2}{t(x) - t(x - \beta)} c(\theta) k(x) \theta^{\frac{x}{\beta}}\right\} \\ &= \frac{1}{\beta} E\left\{w(X) \frac{[g(X) - g(X - \beta)]^2}{t(X) - t(X - \beta)}\right\}. \end{aligned}$$

From the above equality, we get the inequality (23). □

**THEOREM 3.3.** *Let the inequality (23) be satisfied for all real-valued functions  $g$  defined on  $\mathcal{B}^*$  for some function  $t^*$  with  $\Delta_\beta t^*(x) > 0$  for all  $x \in \mathcal{B}^*$ . Further suppose  $E_\theta[t^*(X)] = \theta$  and the equality holds in (23), when  $g$  is linear in  $t$ . Then the distribution of  $X$  belongs to a family of the form (8).*

*Proof.* On taking  $w(x + \beta)k(x + \beta) = \beta k(x)$ ,  $x \in \mathcal{B}^*$ , lead us to a characterization of a family of the form (8). □

REMARK 3.4. We can prove Theorem 3.2 and Theorem 3.3 directly based on the definition of  $\Delta_\beta$ .

- If  $\mathcal{B}^* = \{n\beta + \alpha : n \in Z\}$ , then in Theorem 3.2, the inequality (23) is valid for the case of the probability density function of the random variable  $X$  belonging to a shifted scaled bilateral power series family; also, Theorem 3.3 characterizes a shifted scaled bilateral power series family.
- If  $\mathcal{B}^* = \{n\beta + \alpha : n \in N\}$ , then in Theorem 3.2, the inequality (23) is valid for the case of the probability density function of the random variable  $X$  belonging to a shifted scaled power series family; also, Theorem 3.3 characterizes a shifted scaled power series family.
- If  $\mathcal{B}^* = \{n\beta + \alpha : n \in N_{n_0} = \{0, 1, 2, \dots, n_0\}\}$ , then in Theorem 3.2, the inequality (23) is valid for the case of the probability density function of the random variable  $X$  belonging to a shifted scaled binomial family; also, Theorem 3.3 characterizes a shifted scaled binomial family.
- For  $\beta = 1$ , and  $\alpha = 0$ , based on Theorem 3.3, we can obtain characterization of the bilateral polynomial power series, discrete quartic, scaled discrete normal, Poisson, binomial, negative binomial, Heine, Euler, Pseudo-Euler and Polya Eggenberger) distributions respectively. In this case, we obtain the first theorem in Papathanasiou [10, Theorem 2.1, p. 165] as a corollary of the above results.

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