Commun. Korean Math. Soc. **17** (2002), No. 3, pp. 505–517

CERTAIN CONFORMALLY INVARIANT CONNECTIONS OF RIZZA MANIFOLDS

EUN-SEO CHOI

ABSTRACT. We introduce certain conformally invariant h-Finsler connection in a Rizza manifold. Using this connection, we find some conformally invariant Finsler tensors. The conformal flatness and the Kaehlerian Finsler manifold with respect to the above connection are investigated.

1. Introduction

Let M be a $2n$-dimensional differentiable manifold admitting an almost complex structure $f^i{}_j(x)$ and Finsler metric $g_{ij}(x,y)$. If the couple $(f^i{}_j(x), g_{ij}(x,y))$ satisfies the Rizza condition:

$$\{g_{ij}(x,y) - g_{pq}(x,y)f^p{}_i(x)f^q{}_j(x)\}y^i = 0, \tag{1.1}$$

then the manifold M is called a *Rizza manifold* ([4], [12], and [13]). In [7], Ichijyō has introduced the notions of a $(G_{ij}(x,y), N^i{}_j(x,y))$-structure and its conformal changes, where $G_{ij}(x,y)$ is a generalized Finsler metric and $N^i{}_j(x,y)$ is the generalized Chern's non-linear connection in a Rizza manifold. In [11], Park has introduced a conformally invariant h-Finsler connection $(M_j{}^i{}_k(x,y), M^i{}_j(x,y))$, where $M^i{}_j(x,y)$ is a non-linear connection constructed from the generalized Chern's non-linear connection $N^i{}_j(x,y)$ and a generalized Finsler metric $G_{ij}(x,y)$ and an almost complex structure $f^i{}_j(x)$. In the same paper, he found some conformally invariant Finsler tensors and investigated the conformal flatness of the $(G_{ij}, M^i{}_j)$-structure.

Received October 16, 2001.
2000 Mathematics Subject Classification: 53B40.
Key words and phrases: Rizza manifold, generalized Finsler metric, generalized Chern's non-linear connection, conformally invariant Finsler connection, conformal flatness, Kaehlerian Finsler manifold.
This research was supported by the Yeungnam University research grants in 2001.

In the present paper the author introduces a special conformally invariant h-Finsler connection $(H_j{}^i{}_k(x,y), H^i{}_j(x,y))$ in a Rizza manifold, where $H^i{}_j(x,y)$ is constructed from the generalized Chern's non-linear connection $N^i{}_j(x,y)$ and hv-torsion $P^i{}_{jk}(x,y)$. We find some conformally invariant Finsler tensors and conformal flatness of a $(G_{ij}, H^i{}_j)$-structure. We also investigate the Kaehlerian Finsler manifold with respect to the $(H_j{}^i{}_k(x,y), H^i{}_j(x,y))$-connection in a Rizza manifold.

2. Preliminaries

In a Rizza manifold, if we put

$$G_{ij}(x,y) = \frac{1}{2}(g_{ij}(x,y) + g_{pq}(x,y)f^p{}_i f^q{}_j), \tag{2.1}$$

then it is seen that $G_{ij}(x,y) = G_{ji}(x,y)$, $G_{ij}(x,y)$ is a positively homogeneous tensor of degree zero for y^i and $G_{ij}(x,y)\xi^i\xi^j$ is positive definite. That is, $G_{ij}(x,y)$ is a *generalized Finsler metric* ([2], [8]). This is not a Finsler metric. With respect to this generalized Finsler metric $G_{ij}(x,y)$, we have

$$G_{pq}(x,y)f^p{}_i(x)f^q{}_j(x) = G_{ij}(x,y), \quad y^r\dot{\partial}_r G_{ij}(x,y) = 0. \tag{2.2}$$

Moreover, due to the Rizza condition (1.1) we can see easily

$$\begin{aligned} &y^m f^r{}_m(x)\dot{\partial}_r G_{ij}(x,y) = 0, \quad \dot{\partial}_i G_{pq}(x,y)y^p y^q = 0, \\ &G_{ij}(x,y) = G_{pq}(x,y)f^p{}_i(x)f^q{}_j(x). \end{aligned} \tag{2.3}$$

In a Rizza manifold M, the *generalized Chern's non-linear connection* is given by ([8]):

$$\begin{aligned} N^i{}_j =& \frac{1}{2}(G^{ih}\partial_j G_{hs} - f^i{}_h G^{hr} f^t{}_j \partial_t G_{rs} + S^i{}_{sj} - G^{ih}G_{ms}S^m{}_{jh} \\ &- G^{ih}\dot{\partial}_r G_{hm} y^m S^r{}_{sj} + G^{ih} f^m{}_h G_{rs} f^r{}_t S^t{}_{mj})y^s, \end{aligned} \tag{2.4}$$

where $S^i{}_{kj} = (\partial_k f^i{}_r)f^r{}_j$.

It is known that if the given almost complex structure $f^i{}_j(x)$ in the Rizza manifold is integrable, then $N^i{}_j(x,y)$ defined by (2.4) coincides with the Chern's non-linear connection ([7]). With respect to a generalized Finsler metric $G_{ij}(x,y)$ and the generalized Chern's non-linear connection $N^i{}_j(x,y)$ respectively defined by (2.1) and (2.4), we introduce a symmetric Finsler connection $F\Gamma = (\Gamma_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$ as follows ([1], [9]):

$$\begin{aligned} \Gamma_j{}^i{}_k(x,y) &= \frac{1}{2}G^{im}(X_jG_{mk} + X_kG_{mj} - X_mG_{jk}), \\ C_j{}^i{}_k(x,y) &= \frac{1}{2}G^{im}(\dot{\partial}_jG_{mk} + \dot{\partial}_kG_{mj} - \dot{\partial}_mG_{jk}), \end{aligned} \tag{2.5}$$

where $X_j = \partial_j - N^m{}_j\dot{\partial}_m$, $\partial_j = \partial/\partial x^j$ and $\dot{\partial}_j = \partial/\partial y^j$. Denoting the h-covariant, v-covariant derivative with respect to $F\Gamma$ by ∇ and $\dot{\nabla}$ respectively, we have directly

$$\nabla_kG_{ij} = 0, \quad \dot{\nabla}_kG_{ij} = 0.$$

The above Finsler connection $F\Gamma = (\Gamma_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$ is said to be the *Finsler connection associated with the* (G,N)*-structure.*

According to Matsumoto [9], we write the h-torsion and hv-torsion of $F\Gamma$ as

$$R^i{}_{jk} = X_kN^i{}_j - j/k, \quad P^i{}_{jk} = \dot{\partial}_kN^i{}_j - \Gamma_j{}^i{}_k, \tag{2.6}$$

and the h-curvature and hv-curvatures of $F\Gamma$ as

$$R_h{}^i{}_{jk} = K_h{}^i{}_{jk} + C_h{}^i{}_mR^m{}_{jk}, \quad P_h{}^i{}_{jk} = \dot{\partial}_k\Gamma_h{}^i{}_j - Q_h{}^i{}_{jk}, \tag{2.7}$$

where j/k denotes the interchange of indices j and k of the preceding terms, and we put

$$\begin{aligned} K_h{}^i{}_{jk} &= X_k\Gamma_h{}^i{}_j + \Gamma_m{}^i{}_k\Gamma_h{}^m{}_j - j/k, \\ Q_h{}^i{}_{jk} &= \nabla_jC_h{}^i{}_k - C_h{}^i{}_mP^m{}_{jk}. \end{aligned} \tag{2.8}$$

3. Conformally invariant h-connections

In a Rizza manifold M, let us consider the conformal changes:

$$G_{ij}(x,y) \longrightarrow \overline{G}_{ij}(x,y) = e^{2\sigma(x)}G_{ij}(x,y), \tag{3.1}$$

where $\sigma(x)$ is any scalar. Then we can see easily

$$\begin{aligned} \partial_k\overline{G}_{ij}(x,y) &= e^{2\sigma(x)}\partial_kG_{ij}(x.y) + 2\sigma_ke^{2\sigma}G_{ij}(x,y), \\ \overline{G}^{ij}(x,y) &= e^{-2\sigma(x)}G^{ij}(x,y), \quad \dot{\partial}_k\overline{G}_{ij}(x,y) = e^{2\sigma(x)}\dot{\partial}_kG_{ij}(x,y). \end{aligned} \tag{3.2}$$

The conformal change of the generalized Chern's non-linear connection $N^i{}_j$ defined (2.4) is given as follows ([7]):

$$\overline{N}^i{}_j = N^i{}_j + y^i\sigma_j - f^i{}_h y^h f^r{}_j \sigma_r, \tag{3.3}$$

where $\sigma_k = \partial_k \sigma(x)$, and we have

$$\overline{\Gamma}_j{}^i{}_k = \Gamma_j{}^i{}_k + \delta^i_j \sigma_k + \delta^i_k \sigma_j - G_{jk} G^{im} \sigma_m, \tag{3.4}$$

$$\overline{P}^i{}_{jk} = P^i{}_{jk} - f^i{}_k f^r{}_j \sigma_r - \delta^i{}_j \sigma_k + G_{jk} G^{ir} \sigma_r. \tag{3.5}$$

If we put $P_k = P^r{}_{rk}$, then we have $\overline{P}_k = P_k - 2n\sigma_k$, from which

$$\sigma_k = (P_k - \overline{P}_k)/2n. \tag{3.6}$$

Substituting (3.6) in (3.3) and (3.4), we have

$$\overline{N}^i{}_j + (y^i \overline{P}_j - f^i{}_h y^h f^r{}_j \overline{P}_r)/(2n) = N^i{}_j + (y^i P_j - f^i{}_h y^h f^r{}_j P_r)/2n,$$

$$\begin{aligned} \overline{\Gamma}_j{}^i{}_k &+ (\delta^i_j \overline{P}_k + \delta^i_k \overline{P}_j - \overline{G}_{jk} \overline{G}^{ir} \overline{P}_r)/(2n) \\ &= \Gamma_j{}^i{}_k + (\delta^i_j P_k + \delta^i_k P_j - G_{jk} G^{ir} P_r)/2n \end{aligned}$$

respectively. If we put

$$H^i{}_j = N^i{}_j + (y^i P_j - f^i{}_h y^h f^r{}_j P_r)/2n, \tag{3.7}$$

and

$$H_j{}^i{}_k = \Gamma_j{}^i{}_k + (\delta^i_j P_k + \delta^i_k P_j - G_{jk} G^{ir} P_r)/2n, \tag{3.8}$$

then $\overline{H}^i{}_j = H^i{}_j$ and $\overline{H}_j{}^i{}_k = H_j{}^i{}_k$, that is, $(H^i{}_j, H_j{}^i{}_k)$ is a conformally invariant h-connection. Thus we have

THEOREM 3.1. *In a Rizza manifold M, we have a conformally invariant h-Finsler connection $(H_j{}^i{}_k, H^i{}_j)$ given by* (3.7) *and* (3.8).

In a Rizza manifold M with a conformally invariant h-Finsler connection $FH = (H_j{}^i{}_k, H^i{}_j, C_j{}^i{}_k)$ associated (G, H)-structure, the h- and

v-covariant derivatives of a Finsler tensor field, for example $K^i{}_j(x,y)$, are given by

$$\begin{aligned} &\overset{h}{\nabla}_k K^i{}_j = \overset{*}{X}_k K^i{}_j + K^r{}_j H_r{}^i{}_k - K^i{}_r H_j{}^r{}_k, \\ &\overset{*}{\nabla}_k K^i{}_j = \dot{\partial}_k K^i{}_j + K^r{}_j C_r{}^i{}_k - K^i{}_r C_j{}^r{}_k, \end{aligned} \tag{3.9}$$

where $\overset{*}{X}_k = \partial_k - H^r{}_k \dot{\partial}_r$.

For the supporting element y^i we have

$$\overset{h}{\nabla}_k y^i = H_0{}^i{}_k - H^i{}_k, \qquad \overset{*}{\nabla}_k y^i = \delta^i_k + C_0{}^i{}_k,$$

where the index 0 denotes the contraction with the supporting element y^i. The Finsler tensor field $\overset{h}{D}$ given by

$$\overset{h}{D}{}^i{}_k = H_0{}^i{}_k - H^i{}_k$$

is called the *deflection tensor field* with respect to connection $FH = (H_j{}^i{}_k, H^i{}_j, C_j{}^i{}_k)$. Thus *the deflection tensor vanishes if and only if* $H^i{}_j = H_0{}^i{}_j$.

The Ricci identities applying to G_{ij} are

$$\begin{aligned} &\overset{h}{\nabla}_l \overset{h}{\nabla}_k G_{ij} - \overset{h}{\nabla}_k \overset{h}{\nabla}_l G_{ij} \\ &\quad = -G_{rj} \overset{h}{R}_i{}^r{}_{kl} - G_{ir} \overset{h}{R}_j{}^r{}_{kl} - (\overset{h}{\nabla}_r G_{ij}) \overset{h}{T}_k{}^r{}_l - (\overset{*}{\nabla}_r G_{ij}) \overset{h}{R}_k{}^r{}_l, \\ &\overset{*}{\nabla}_l \overset{h}{\nabla}_k G_{ij} - \overset{h}{\nabla}_k \overset{*}{\nabla}_l G_{ij} \\ &\quad = -G_{rj} \overset{h}{P}_i{}^r{}_{kl} - G_{ir} \overset{h}{P}_j{}^r{}_{kl} - (\overset{h}{\nabla}_r G_{ij}) C_k{}^r{}_l - (\overset{*}{\nabla}_r G_{ij}) \overset{h}{P}_k{}^r{}_l, \end{aligned} \tag{3.10}$$

where torsion tensors $\overset{h}{T}_j{}^i{}_k$, $\overset{h}{R}_k{}^i{}_l$, $\overset{h}{P}_k{}^i{}_l$ and curvature tensors $\overset{h}{R}_j{}^i{}_{kl}$, $\overset{h}{P}_j{}^i{}_{kl}$ are given by

$$\begin{aligned} &\overset{h}{T}_j{}^i{}_k = H_j{}^i{}_k - j/k, \quad \overset{h}{R}_k{}^i{}_l = \overset{*}{X}_l H^i{}_k - k/l, \\ &\overset{h}{P}_k{}^i{}_l = \dot{\partial}_l H^i{}_k - H_l{}^i{}_k, \\ &\overset{h}{R}_j{}^i{}_{kl} = \overset{h}{K}_j{}^i{}_{kl} + C_j{}^i{}_r \overset{h}{R}_k{}^r{}_l, \quad \overset{h}{P}_j{}^i{}_{kl} = \dot{\partial}_l H_j{}^i{}_k - \overset{h}{Q}_j{}^i{}_{kl}, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \overset{h}{K}_j{}^i{}_{kl} &= (\overset{*}{X}_l H_j{}^i{}_k + H_j{}^r{}_k H_r{}^i{}_l) - k/l, \\ \overset{h}{Q}_j{}^i{}_{kl} &= \overset{h}{\nabla}_k C_j{}^i{}_l + C_j{}^i{}_r \overset{h}{P}_k{}^r{}_l. \end{aligned} \tag{3.12}$$

Since $\overset{h}{T}_j{}^i{}_k = 0$ and $\overset{*}{\nabla}_k G_{ij} = \dot{\nabla}_k G_{ij} = 0$, (3.10) is written as

$$\begin{aligned} \overset{h}{\nabla}_l \overset{h}{\nabla}_k G_{ij} - \overset{h}{\nabla}_k \overset{h}{\nabla}_l G_{ij} &= -\overset{h}{R}_{ijkl} - \overset{h}{R}_{jikl}, \\ \overset{*}{\nabla}_l \overset{h}{\nabla}_k G_{ij} &= -\overset{h}{P}_{ijkl} - \overset{h}{P}_{jikl} - (\overset{h}{\nabla}_r G_{ij}) C_k{}^r{}_l, \end{aligned} \tag{3.13}$$

where $\overset{h}{R}_{ijkl} = G_{rj} \overset{h}{R}_i{}^r{}_{kl}$, $\overset{h}{P}_{ijkl} = G_{jr} \overset{h}{P}_i{}^r{}_{kl}$.

On the other hand, substituting (3.7) and (3.8) in

$$\overset{h}{\nabla}_k G_{ij} = \overset{*}{X}_k G_{ij} - G_{rj} H_i{}^r{}_k - G_{ir} H_j{}^r{}_k$$

and using (2.3) and $\nabla_k G_{ij} = 0$, we have

$$\overset{h}{\nabla}_k G_{ij} = P_k G_{ij}/n, \quad \overset{h}{\nabla}_k G^{ij} = -P_k G^{ij}/n. \tag{3.14}$$

Therefore we get

$$\overset{h}{\nabla}_l \overset{h}{\nabla}_k G_{ij} - \overset{h}{\nabla}_k \overset{h}{\nabla}_l G_{ij} = (\overset{h}{\nabla}_l P_k - \overset{h}{\nabla}_k P_l) G_{ij}/n. \tag{3.15}$$

From (3.13), (3.14), (3.15) and $H_j{}^i{}_k = H_k{}^i{}_j$ we have

$$\begin{aligned} n(\overset{h}{R}_{ijkl} + \overset{h}{R}_{jikl}) &= (\overset{h}{X}_k P_l - \overset{h}{X}_l P_k) G_{ij}, \\ n(\overset{h}{P}_{ijkl} + \overset{h}{P}_{jikl}) &= -(\dot{\partial}_l P_k) G_{ij}. \end{aligned} \tag{3.16}$$

Transvecting (3.16) by G^{ij} respectively, we have

$$\overset{h}{R}_{ijkl} G^{ij} = \overset{h}{X}_k P_l - \overset{h}{X}_l P_k, \quad \overset{h}{P}_{ijkl} G^{ij} = -\dot{\partial}_l P_k. \tag{3.17}$$

This shows that if $\overset{h}{R}_{ijkl} = \overset{h}{P}_{ijkl} = 0$, then the (G, N)-structure in a Rizza manifold is conformally flat. In fact, from $\overset{h}{P}_{ijkl} = 0$ and

$\overset{h}{P}_{ijkl}G^{i}j = -\dot{\partial}_l P_k$, we have $P_k = P_k(x)$, and thus, from $\overset{h}{R}_{ijkl}G^{ij} = \overset{h}{X}_k P_l - \overset{h}{X}_l P_k$, we have

$$\frac{\partial P_k}{\partial x^l} = \frac{\partial P_l}{\partial x^k}.$$

Therefore there exists a local function $\sigma(x)$ on $U(x)$ such that

$$\frac{\partial \sigma(x)}{\partial x^k} = \frac{1}{2n} P_k.$$

By this local function $\sigma(x)$, the local metric $\overline{G}_{ij} = e^{-2\sigma(x)} G_{ij}$ is flat metric on $U(x)$. In fact, from the equations (3.3) and (3.7) imply the equation $\overline{N}^i_j = 0$, and moreover, from the equations (3.4) and (3.8) imply the equation $\overline{\Gamma}_j{}^i{}_k = 0$. Hence the metrical condition $\nabla_k G_{ij} = 0$ implies $G_{ij} = G_{ij}(y)$ on $U(x)$.

The following definition is well-known: If for a any point p of M with (G,N)-structure, there exists a neighborhood (U, x^i) containing p such that G_{ij} is a locally Minkowski metric on $U(x)$, the (G,N)-structure on M is said to be a *conformally flat.* Thus we have

THEOREM 3.2. *The (G,N)-structure on a Rizza manifold (M,G,f) is conformally flat if and only if*

$$\overset{h}{R}_{ijkl} = \overset{h}{P}_{ijkl} = 0$$

are satisfied.

Next, we put

$$\overset{*}{\Gamma}_j{}^i{}_k = \frac{1}{2} G^{ir} (\overset{*}{X}_j G_{rk} + \overset{*}{X}_k G_{jr} - \overset{*}{X}_r G_{jk}). \tag{3.18}$$

From $(2.2)_2$ and $(2.3)_1$, we get $\overset{*}{X}_k G_{ij} = X_k G_{ij}$ and hence $\overset{*}{\Gamma}_j{}^i{}_k = \Gamma_j{}^i{}_k$.

The triple $\overset{*}{F}\Gamma = (\overset{*}{\Gamma}_j{}^i{}_k, H^i{}_j, C_j{}^i{}_k)$ is, of course, a sort of Finsler connection ([9]). With respect to connection $\overset{*}{F}\Gamma$, (2.6), (2.7) and (2.8) are expressed by

$$\begin{aligned}
&\overset{*}{R}_j{}^i{}_k = \overset{*}{X}_k H^i{}_j - j/k, \quad \overset{*}{P}_j{}^i{}_k = \dot{\partial}_k H^i{}_j - \overset{*}{\Gamma}_j{}^i{}_k,\\
&\overset{*}{R}_h{}^i{}_{jk} = \overset{*}{K}_h{}^i{}_{jk} + C_h{}^i{}_r \overset{*}{R}_j{}^r{}_k, \quad \overset{*}{P}_h{}^i{}_{jk} = \dot{\partial}_k \overset{*}{\Gamma}_h{}^i{}_j - \overset{*}{Q}_h{}^i{}_{jk},\\
&\overset{*}{K}_h{}^i{}_{jk} = (\overset{*}{X}_k \overset{*}{\Gamma}_h{}^i{}_j + \overset{*}{\Gamma}_r{}^i{}_k \overset{*}{\Gamma}_h{}^r{}_j) - j/k,\\
&\overset{*}{Q}_h{}^i{}_{jk} = \overset{*}{\nabla}_j C_h{}^i{}_k - C_h{}^i{}_r \overset{*}{P}_j{}^r{}_k.
\end{aligned} \tag{3.19}$$

4. The flatness of (G, H)-structure

In a Rizza manifold with $\overset{*}{F}\Gamma$, we shall define the notion of flatness similar to a Finsler space with a generalized Finsler metric G_{ij} and a non-linear connection $N^i{}_j$ ([8]).

DEFINITION 4.1 Let M be a Rizza manifold with $\overset{*}{F}\Gamma$-connection admitting a (G, H)-structure. If, for any point p of M, there exists a coordinate neighborhood (U, x^i) containing p such that $\overset{*}{X}_k G_{ij} = 0$ holds on U, then the (G, H)-structure is said to be *weakly flat.*

The definition of the flatness of (G, H)-structure is equivalent to the one of the (G, N)-structure. However, here we shall find some quantities of the flatness in a Rizza manifold with $\overset{*}{F}\Gamma$-connection.

In a Rizza manifold M, if a (G, H)-structure is weakly flat, then from Definition 4.1, M is covered by a system of local coordinate neighborhoods $\{(U, x^i)\}$ such that $\overset{*}{X}_k G_{ij} = 0$ holds good in each U. Therefore we have $\overset{*}{\Gamma}_j{}^i{}_k = 0$ in each U from (3.18). By virtue of $(3.19)_5$, we get $\overset{*}{K}_h{}^i{}_{jk} = 0$. And from $(3.19)_4$, we have

$$\overset{*}{P}_{hijk} = -\overset{*}{Q}_{hijk}, \tag{4.1}$$

where $\overset{*}{P}_{hijk} = G_{ir}\overset{*}{P}_h{}^r{}_{jk}$, $\overset{*}{Q}_{hijk} = G_{ir}\overset{*}{Q}_h{}^r{}_{jk}$. Applying the Ricci identities to $\overset{*}{\nabla}_k G_{ij} = 0$ and $\dot{\nabla}_k G_{ij} = 0$, we have

$$\overset{*}{R}_{hijk} = -\overset{*}{R}_{ihjk}, \quad \overset{*}{P}_{hijk} = -\overset{*}{P}_{ihjk}, \tag{4.2}$$

where $\overset{*}{R}_{hijk} = G_{ir}\overset{*}{R}_h{}^r{}_{jk}$. Hence, from (4.1) and (4.2), we have

$$\overset{*}{Q}_{ihjk} + \overset{*}{Q}_{hijk} = 0. \tag{4.3}$$

Conversely, we suppose that $\overset{*}{K}_h{}^i{}_{jk} = 0$ and (4.3) holds good. By the second Bianchi identity for the connection $\overset{*}{F}\Gamma$, we have

$$\overset{*}{\nabla}_j C_{khi} - \overset{*}{\nabla}_k C_{jhi} + C_{jhr}\overset{*}{P}^r{}_{ki} - C_{khr}\overset{*}{P}^r{}_{ji} - \overset{*}{P}_{jhki} + \overset{*}{P}_{khji} = 0,$$

that is,

$$\overset{*}{Q}_{khji} - \overset{*}{Q}_{jhki} - \overset{*}{P}_{jhki} + \overset{*}{P}_{khji} = 0. \tag{4.4}$$

Applying the Christoffel process with respect to k, h and j to (4.4) and using (4.3), we get

$$2\overset{*}{P}_{jhki} = \overset{*}{Q}_{khji} + \overset{*}{Q}_{hjki} + \overset{*}{Q}_{hkji} - \overset{*}{Q}_{jhki} - \overset{*}{Q}_{kjhi} - \overset{*}{Q}_{jkhi}.$$

From (4.3), the above equation is reduced to $\overset{*}{P}_{jhki} = -\overset{*}{Q}_{jhki}$. Therefore, from $(3.19)_4$, we have $\dot{\partial}_k \overset{*}{\Gamma}{}_h{}^i{}_j = 0$. Thus $\overset{*}{\Gamma}{}_j{}^i{}_k$ is a function of the position only on M.

On the other hand, $\overset{*}{K}_h{}^i{}_{jk} = 0$ tells us that M is covered by a system of local coordinate neighborhoods such that $\overset{*}{\Gamma}{}_j{}^i{}_k = 0$ hold on each U. Hence $\overset{*}{\nabla}_k G_{ij} = 0$ leads us to $\overset{*}{X}_k G_{ij} = 0$ on U. Therefore the given (G, H)-structure is weakly flat. Thus we have

THEOREM 4.1. *A (G, H)-structure in a Rizza manifold with $\overset{*}{F}\Gamma$-connection is weakly flat if and only if*

$$\overset{*}{K}_h{}^i{}_{jk} = 0, \quad \overset{*}{Q}_{ihjk} + \overset{*}{Q}_{hijk} = 0 \tag{4.5}$$

are satisfied.

DEFINITION 4.2. In a Rizza manifold M, with $\overset{*}{F}\Gamma$-connection if for every point p of M, there exists a coordinate neighborhood (U, x^i) containing p such that $\partial_k G_{ij} = 0$ and $H^r{}_k \dot{\partial}_r G_{ij} = 0$ hold on U, then the (G, H)-structure is said to be *flat*.

THEOREM 4.2. *A (G, H)-structure in Rizza manifold is flat if and only if*

$$\overset{*}{K}_h{}^i{}_{jk} = 0, \quad \overset{*}{Q}_{ihjk} + \overset{*}{Q}_{hijk} = 0, \quad \overset{*}{P}{}^r{}_{k0}(C_{ijr} + C_{jir}) = 0 \tag{4.6}$$

are satisfied, where $P^r{}_{k0} = P^r{}_{kj} y^j$, $C_{ijr} = G_{jm} C_i{}^m{}_r$.

PROOF. Let a (G, H)-structure be flat. By Definition 4.1, the (G, H) -structure is weakly flat. By virtue of Theorem 4.1, the former two equation of (4.6) is satisfied. The Rizza manifold M is covered by a system of local coordinate neighborhoods $\{(U, x^i)\}$ such that $\partial_k G_{ij} = 0$ and $H^r{}_k \dot{\partial}_r G_{ij} = 0$ in each U. On the other hand, from $\dot{\nabla}_k G_{ij} = 0$ we have that

$$\dot{\partial}_m G_{jk} = C_{mjk} + C_{mkj}. \tag{4.7}$$

From $(3.19)_2$, we have $\overset{*}{P}{}^i{}_{jr} y^r = H^i{}_j - \overset{*}{\Gamma}{}_j{}^i{}_r y^r$. Since $\overset{*}{\Gamma}{}_j{}^i{}_k = 0$ in U, we have $\overset{*}{P}{}^i{}_{jr} y^r = H^i{}_j$. Therefore, from the definition of the flatness and (4.7), we get $(C_{mjk} + C_{mkj}) \overset{*}{P}{}^m{}_{ir} y^r = 0$.

Conversely, we suppose that (4.6) holds good. By virtue of Theorem 4.1, we see that the (G, H)-structure is weakly flat. Hence, with respect to the assigned coordinate neighborhood U of the weak flatness, $\overset{*}{X}_k G_{ij} = 0$, from which $\overset{*}{\Gamma}{}_j{}^i{}_k = 0$. Thus, from $(3.19)_2$ $\overset{*}{P}{}^i{}_{jr} y^r = H^i{}_j$ holds in each U. Therefore, from this equation and (4.7) and third equation of (4.6), we have $H^m{}_k \dot{\partial}_m G_{ij} = 0$ in each U. Since $\overset{*}{X}_k G_{ij} = 0$ is shown, $\partial_k G_{ij} = 0$ is also true in each U. Hence the given (G, H)-structure is flat. □

5. A Kaehlerian Finsler (G, H)-structure

In a Rizza manifold, as a non-linear connection, we shall adapt a non-linear connection $H^i{}_j$ which is given by (3.7). In this case, a (G, H)-structure satisfying $\overset{\circ}{\nabla}_k f^i{}_j = 0$ is said a *Kaehlerian Finsler* (G, H)-*structure*, where $\overset{\circ}{\nabla}_k$ denotes the h-covariant derivative with respect to the $\overset{*}{F}\Gamma$-connection.

THEOREM 5.1. *If the Kaehlerian Finsler (G, H)-structure is flat, the relations*

$$\overset{*}{R}{}^i{}_{jk} = \overset{*}{P}{}^i{}_{jk} = 0, \quad \overset{*}{P}{}_h{}^i{}_{jk} = \overset{*}{Q}{}_h{}^i{}_{jk} = 0, \quad \overset{*}{R}{}_h{}^i{}_{jk} = \overset{*}{K}{}_h{}^i{}_{jk} = 0$$

hold true.

PROOF. Since the given (G, H)-structure is flat, from Definition 4.2, we see that the manifold is covered by a system of local coordinate neighborhoods $\{(U, x^i)\}$ such that, in each (U, x^i), $G_{ij} = G_{ij}(y)$ and $H^r{}_k \partial_r G_{ij} = 0$. Therefore $\overset{*}{X}_k G_{ij} = 0$. Thus we get $\overset{*}{\Gamma}{}_j{}^i{}_k = 0$. By assumption, in the above coordinate neighborhood U, $\overset{\circ}{\nabla}_k f^i{}_j = 0$ is also true. Hence it follows $\partial_k f^i{}_j = 0$, from which $S_k{}^i{}_j = (\partial_k f^i{}_r) f^r{}_j = 0$. Therefore the generalized Chern's non-linear connection $N^i{}_j$ given by (2.4) vanishes in U. From $N^i{}_j = 0$ and the latter of (2.6), we have $P_i = 0$. Therefore $H^i{}_j = 0$ from (3.7). Hence, from $H^i{}_j = 0$, $\overset{*}{\Gamma}{}_j{}^i{}_k = 0$ and $\overset{*}{\nabla}_h C_j{}^i{}_k = 0$, the proof is evidently completed. □

THEOREM 5.2. *Let M be a $2n$-dimensional $(n > 2)$ Rizza manifold with the Kaehlerian (G,H)-structure. If M is locally conformal to a flat (G,H)-structure, then $\overset{*}{K}{}_h{}^i{}_{jk} = 0$ holds true.*

PROOF. Let the $(\overline{G}, \overline{H})$ be flat and locally conformal to a Rizza manifold M with the Kaehlerian Finsler (G,H)-structure. Then it follows that $\partial_k \overline{G}_{ij} = 0$ and $\overline{H}^m{}_k \dot{\partial}_m \overline{G}_{ij} = 0$ locally, and naturally $\overline{X}_k \overline{G}_{ij} = 0$ from which $\overline{\overset{*}{\Gamma}}{}_j{}^i{}_k = 0$. Also $\partial_k(e^{2\sigma} G_{ij}) = 0$ leads to

$$\partial_k \overline{G}_{ij} = 2\sigma_k G_{ij} \tag{5.1}$$

and

$$\partial_k \overline{G}^{ij} = -2\sigma_k G^{ij}. \tag{5.2}$$

From (3.7) and $\overline{H}^m{}_k = H^m{}_k$, we have

$$H^m{}_k \dot{\partial}_m G_{ij} = 0. \tag{5.3}$$

From this equation and

$$0 = H^m{}_k \dot{\partial}_m (G^{ir} G_{rj}) = H^m{}_k \{\dot{\partial}_m (G^{ir}) G_{rj} + G^{ir} \dot{\partial}_m G_{rj}\},$$

we have

$$H^m{}_k \dot{\partial}_m G^{ij} = 0. \tag{5.4}$$

On the other hand, by the Rizzi identity for $\overset{\circ}{\nabla}_k f^i{}_j = 0$, we have

$$\begin{aligned} 0 &= \overset{\circ}{\nabla}_k \overset{\circ}{\nabla}_h f^i{}_j - \overset{\circ}{\nabla}_h \overset{\circ}{\nabla}_k f^i{}_j \\ &= f^m{}_j \overset{*}{R}{}_m{}^i{}_{hk} - f^i{}_m \overset{*}{R}{}_j{}^m{}_{hk} - C_m{}^i{}_r f^m{}_j \overset{*}{R}{}^r{}_{hk} + C_j{}^m{}_r f^i{}_m \overset{*}{R}{}^r{}_{hk} \\ &= f^m{}_j (\overset{*}{R}{}_m{}^i{}_{hk} - C_m{}^i{}_r \overset{*}{R}{}^r{}_{hk}) - f^i{}_m (\overset{*}{R}{}_j{}^m{}_{hk} - C_j{}^m{}_r \overset{*}{R}{}^r{}_{hk}) \\ &= f^m{}_j \overset{*}{K}{}_m{}^i{}_{hk} - f^i{}_m \overset{*}{K}{}_j{}^m{}_{hk} \end{aligned}$$

from (3.19), that is

$$f^i{}_m \overset{*}{K}_j{}^m{}_{hk} = f^m{}_j \overset{*}{K}_m{}^i{}_{hk}. \tag{5.5}$$

Since $\overline{\overset{*}{\Gamma}}_j{}^i{}_k = 0$, (3.4) shows us that

$$\overset{*}{\Gamma}_j{}^i{}_k = -\delta^i_j\sigma_k - \delta^i_k\sigma_j + G_{jk}G^{im}\sigma_m. \tag{5.6}$$

Using (5.1), (5.2), (5.3), (5.4) and (5.6), we get

$$\begin{aligned}\overset{*}{K}_h{}^i{}_{jk} = {}& \delta^i_k(\partial_j\sigma_h + \sigma_j\sigma_h - G_{hj}\sigma^m\sigma_m) - \delta^i_j(\partial_k\sigma_h + \sigma_k\sigma_h - G_{hk}\sigma^m\sigma_m) \\ & + G_{hj}G^{im}(\partial_k\sigma_m + \sigma_k\sigma_m) - G_{hk}G^{im}(\partial_j\sigma_m + \sigma_j\sigma_m).\end{aligned}$$

Putting $\sigma_{jh} = \partial_j\sigma_h + \sigma_j\sigma_h - G_{hj}\sigma^m\sigma_m/2$, we have $\sigma_{jh} = \sigma_{hj}$. Therefore we find

$$\overset{*}{K}_h{}^i{}_{jk} = \delta^i_k\sigma_{jh} - \delta^i_j\sigma_{kh} + G_{hj}G^{im}\sigma_{km} - G_{hk}G^{im}\sigma_{jm}. \tag{5.7}$$

Substituting (5.7) in (5.5), we have

$$\begin{aligned} f^i{}_m(\delta^m_k\sigma_{jh} - \delta^m_j\sigma_{kj} + G_{jh}\sigma^m{}_k - G_{jk}\sigma^m{}_h \\ = f^m{}_j(\delta^i_k\sigma_{hm} - \delta^i_h\sigma_{km} + G_{hm}\sigma^i{}_k - G_{km}\sigma^i{}_h),\end{aligned} \tag{5.8}$$

where $\sigma^m{}_r = G^{im}\sigma_{ir}$. Contracting with i and k in (5.8), we have

$$\begin{aligned} -f^m{}_h\sigma_{mj} + G_{jh}f^r{}_m\sigma^m{}_r - f^r{}_mG_{jr}\sigma^m{}_h \\ = 2nf^m{}_j\sigma_{hm} - f^m{}_j\sigma_{hm} + f_{hj}\sigma^m{}_m - f^m{}_j\sigma_{mh}.\end{aligned} \tag{5.9}$$

Transvecting (5.9) by G^{jh}, we get $f^r{}_m\sigma^m{}_r = 0$ and $f^{rm}\sigma_{mr} = 0$, where $f^{rm} = G^{im}f^r{}_i$. From $f_{ij} = -f_{ji}$ and $f^r{}_mG_{jr}\sigma^m{}_h = f_{jh}\sigma^m{}_h$, (5.9) leads us to

$$-f^m{}_h\sigma_{mj} = (2n-3)\sigma_{hm}f^m{}_j + \sigma^m{}_mf_{hj}.$$

Transvecting this equation by $f^h{}_k$, we obtain

$$\sigma_{kj} = (2n-3)\sigma_{pq}f^p{}_kf^q{}_j + \sigma^m{}_mG_{jk}. \tag{5.10}$$

Moreover, transvecting (5.10) by G^{kj}, we get $4(n-1)\sigma^m{}_m = 0$. Since we assume $n > 2$, we have $\sigma^m{}_m = 0$. Therefore (5.10) is written as $\sigma_{kj} = (2n-3)^2\sigma_{kj}$. Thus we have $\sigma_{kj} = 0$. Consequently from (5.7) we obtain $\overset{*}{K}_h{}^i{}_{jk} = 0$. □

ACKNOWLEDGEMENTS. The author thanks the referee for his excellent suggestions and comments.

References

[1] P. L. Antonelli, R. S. Ingarden, and M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kluwer Acad. Dordecht, 1993.

[2] M. Hashiguchi, *On generalized Finsler spaces*, An. Stiint. Univ. "AI. I. Cuza" Iasi, Sect. 1 a Mat. N. S. **30** (1984), no. 1, 69–73.

[3] ———, *Some remarks on linear Finsler connections*, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) **21** (1988), 25–31.

[4] Y. Ichijyō, *Finsler metrics on almost complex manifolds*, Riv. Mat. Univ., Parma **14** (1988), no. 4, 1-28.

[5] ———, *Conformally flat Finsler structures*, J. Math. Tokushima Univ. **25** (1991), 13–25.

[6] ———, *Kaehlerian Finsler manifolds*, J. Math. Tokushima Univ. **28** (1994), 19–27.

[7] ———, *Kaeherian manifolds of Chern type*, Reseach Bull. Tokushima Bunri Univ. **57** (1999), 9–16.

[8] ———, *On flatness of generalized Finsler manifolds and Kaehlerian Finsler manifolds of Chern type*, Reseach Bull. Tokushima Bunri Univ. **59** (2000), 11–18.

[9] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler spaces*, Kaiseisha Press, Ōtsu, Saikawa, 1986.

[10] H. S. Park, *On nearly Kaehlerian Finsler manifolds*, Tensor N. S. **52** (1993), 243–248.

[11] ———, *Conformal changes of Rizza manifolds*, The 35th Symposium on Finsler Geometry, Kagoshima, Japan (2000), 36–39.

[12] G. B. Rizza, *Strutture di Fibsler sulle varita compressea*, Atti Acad. Naz. Lincei (Rendiconti) **33** (1962), 271–275.

[13] ———, *Strutture di Fibsler di tipo quasi Hermitiano*, Riv. Mat. Univ. Parma **4** (1963), no. 2, 83–106.

School of Mathematics and Statistics (Mathematics major), and Institute of Natural Science
Yeungnam University
Kyungpook 712-749, Korea
E-mail: eschoi@ynucc.yeungnam.ac.kr