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UNITARY INTERPOLATION FOR OPERATORS IN TRIDIAGONAL ALGEBRAS

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ABSTRACT. Given operators X and Y acting on a Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator A such that $AX = Y$. An interpolating operator for the n-operators satisfies the equation $AX_i = Y_i$, for $i = 1, 2, \cdots, n$. In this article, we obtained the following : Let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators acting on $\mathcal{H}$ such that $x_{i\sigma(i)} \neq 0$ for all i. Then the following statements are equivalent.

(1) There exists a unitary operator A in Alg$\mathcal{L}$ such that $AX = Y$ and every E in $\mathcal{L}$ reduces A.

(2) $\sup\left\{\frac{\|\sum_{i=1}^{n} E_i Y f_i\|}{\|\sum_{i=1}^{n} E_i X f_i\|} : n \in N, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} < \infty$ and $\frac{|y_{i\sigma(i)}|}{|x_{i\sigma(i)}|} = 1$ for all $i = 1, 2, \cdots$.

1. Introduction

Let $\mathcal{C}$ be a collection of operators acting on a Hilbert space $\mathcal{H}$ and let x and y be vectors on $\mathcal{H}$. An *interpolation question* for $\mathcal{C}$ asks for which x and y is there a bounded operator $T \in \mathcal{C}$ such that $Tx = y$. A variation, the 'n-vector interpolation problem', asks for an operator T such that $Tx_i = y_i$ for fixed finite collections $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$. The n-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison [3]. In the case that $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance [4]: His result was extended by Hopenwasser [2] to the case that $\mathcal{U}$ is a CSL-algebra.

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When an operator maps one thing to another, we think of the operator as the interpolating operator and the equation representing the mapping as the interpolation equation. The equations $Ax = y$ and $AX = Y$ are indistinguishable if spoken aloud, but we mean the change to capital letters to indicate that we intend to look at fixed operators X and Y, and ask under what conditions there will exist an operator A satisfying the equation $AX = Y$.

In this article, we investigate unitary interpolation problems in tridiagonal algebra : Given operators X and Y acting on a Hilbert space, when does there exist a unitary operator A in tridiagonal algebra such that $AX = Y$?

First, we establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, Alg$\mathcal{L}$ is called a CSL-algebra. The symbol Alg$\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let x and y be two vectors in some Hilbert space. Then $< x, y >$ means the inner product of the vectors x and y. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers.

2. Unitary interpolation for operators in tridiagonal algebra

Let $\mathcal{H}$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \cdots\}$. Let x_1, x_2, $\cdots$, x_n be vectors in $\mathcal{H}$. Then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors x_1, x_2, $\cdots$, x_n. Let $\mathcal{L}$ be a subspace lattice of orthogonal projections generated by the subspaces $[e_{2k-1}], [e_{2k-1}, e_{2k}, e_{2k+1}]$ $(k = 1, 2, \cdots)$. Then the algebra Alg$\mathcal{L}$ is called a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson [1]. These algebras have been found to be useful counterexample to a number of plausible conjectures. Recently, such algebras have been found to be use in physics, in electrical engineering and in general system theory.

Let $\mathcal{A}$ be the algebra consisting of all bounded operators acting on $\mathcal{H}$

of the form

$$\begin{pmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \ddots \end{pmatrix}$$

with respect to the orthonormal basis $\{e_1, e_2, \cdots\}$, where all non-starred entries are zero. It is easy to see that Alg$\mathcal{L}$=$\mathcal{A}$. Let D=$\{A : A$ is diagonal acting on $\mathcal{H}$ $\}$. Then D is a masa(maximal abelian subalgebra) of Alg$\mathcal{L}$ and $\mathcal{D}$=(Alg$\mathcal{L}$)$\cap$ (Alg$\mathcal{L}$)*, where (Alg$\mathcal{L}$)* $= \{A^* : A \in$Alg$\mathcal{L}\}$.

In this paper, we use the convention $\frac{0}{0}$=0, when necessary.

Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded operators acting on $\mathcal{H}$ and let U be an operator in $\mathcal{B}(\mathcal{H})$. U is called a unitary operator if $U^*U = UU^* = I$.

Then the following is well-known.

THEOREM 1. *Let A be a diagonal operator in $\mathcal{B}(\mathcal{H})$ with diagonal $\{a_n\}$. A is a unitary operator if and only if $|a_n| = 1$ for all $n = 1, 2, \cdots$.*

From now, let $\sigma : \mathbb{N} \to \mathbb{N}$ be a mapping in this paper.

THEOREM 2. *Let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators acting on $\mathcal{H}$ such that $x_{i\sigma(i)} \neq 0$ for all i. Then the following statements are equivalent.*

(1) *There exists a unitary operator A in Alg$\mathcal{L}$ such that $AX = Y$ and every E in $\mathcal{L}$ reduces A.*

(2) $\sup\left\{\dfrac{\|\sum_{i=1}^n E_iYf_i\|}{\|\sum_{i=1}^n E_iXf_i\|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} < \infty$ *and* $|y_{i\sigma(i)}||x_{i\sigma(i)}|^{-1} = 1$ *for all* $i = 1, 2, \cdots$.

PROOF. (1) $\Rightarrow$ (2). Since E reduces A and $AX = Y$, $AEX = EY$ for every E in $\mathcal{L}$. So $A(\sum_{i=1}^n E_iXf_i) = \sum_{i=1}^n E_iYf_i$ and hence $\|\sum_{i=1}^n E_iYf_i\| \leq \|A\|\|\sum_{i=1}^n E_iXf_i\|$, $n \in \mathbb{N}$, $E_i \in \mathcal{L}$ and $f_i \in \mathcal{H}$. If $\|\sum_{i=1}^n E_iXf_i\| \neq 0$, then $\dfrac{\|\sum_{i=1}^n E_iYf_i\|}{\|\sum_{i=1}^n E_iXf_i\|} \leq \|A\|$. Hence

$$\sup\left\{\frac{\|\sum_{i=1}^n E_iYf_i\|}{\|\sum_{i=1}^n E_iXf_i\|} : n \in \mathbb{N},\ E_i \in \mathcal{L}, \text{ and } f_i \in \mathcal{H}\right\} < \infty.$$

Since every E in $\mathcal{L}$ reduces A, A is a diagonal operator. Let $A = (a_{ii})$. Since $AX = Y$, $y_{ij} = a_{ii}x_{ij}$ for all i and all j. Since A is a unitary operator, $|y_{i\sigma(i)}||x_{i\sigma(i)}|^{-1} = 1$ for all $i = 1, 2, \cdots$.

Conversely, if

$$\sup\left\{\frac{\|\sum_{i=1}^n E_iYf_i\|}{\|\sum_{i=1}^n E_iXf_i\|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} < \infty,$$

then, without loss of generality, we may assume that

$$\sup\left\{\frac{\|\sum_{i=1}^n E_iYf_i\|}{\|\sum_{i=1}^n E_iXf_i\|} : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\} = 1.$$

Then $\|\sum_{i=1}^n E_iYf_i\| \leq \|\sum_{i=1}^n E_iXf_i\|$, $n \in \mathbb{N}$, $E_i \in \mathcal{L}$ and $f_i \in \mathcal{H}$ $\cdots(*)$. Let

$$\mathcal{M} = \left\{\sum_{i=1}^n E_iXf_i : n \in \mathbb{N}, E_i \in \mathcal{L} \text{ and } f_i \in \mathcal{H}\right\}.$$

Then $\mathcal{M}$ is a linear manifold. Define $A : \mathcal{M} \longrightarrow \mathcal{H}$ by $A(\sum_{i=1}^n E_iXf_i) = \sum_{i=1}^n E_iYf_i$. Then A is well-defined by $(*)$. Extend A to $\overline{\mathcal{M}}$ by continuity. Define $A|_{\overline{\mathcal{M}}^\perp} = 0$. Then $\|A\| \leq 1$ and $AX = Y$.

$$\begin{aligned} AE(\sum_{i=1}^n E_iXf_i) &= A(\sum_{i=1}^n EE_iXf_i) \\ &= \sum_{i=1}^n EE_iYf_i \end{aligned}$$

and

$$\begin{aligned} EA(\sum_{i=1}^n E_iXf_i) &= E(\sum_{i=1}^n E_iYf_i) \\ &= \sum_{i=1}^n EE_iYf_i. \end{aligned}$$

And $EA(g) = E(0) = 0$ and $AE(g) = 0$ for g in $\overline{\mathcal{M}}^\perp$ since $\langle Eg, \sum_{i=1}^n E_i\ Xf_i\ \rangle = \langle g, \sum_{i=1}^n EE_iXf_i\rangle = 0$. Hence every E in $\mathcal{L}$ reduces A. So A is a diagonal operator. Let $A = (a_{ii})$. Since $AX = Y$, $y_{ij} = a_{ii}x_{ij}$ for all i and all j. Since $|y_{i\sigma(i)}||x_{i\sigma(i)}|^{-1} = 1$ for all $i = 1, 2, \cdots$, A is a unitary operator. □

THEOREM 3. *Let* $X_p = (x_{ij}^{(p)})$ *and* $Y_p = (y_{ij}^{(p)})$ *be operators in* $\mathcal{B}(\mathcal{H})$ $(p = 1, 2, \cdots, n)$ *such that* $x_{i\sigma(i)}^{(q)} \neq 0$ *for some* q. *Then the following statements are equivalent.*

(1) *There exists a unitary operator* A *in* $Alg\mathcal{L}$ *such that* $AX_p = Y_p$ $(p = 1, 2, \cdots, n)$ *and every* E *in* $\mathcal{L}$ *reduces* A.

(2) $\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|} : m_i \in \mathbb{N}, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}$ $< \infty$ *and* $|y_{i\sigma(i)}^{(q)}||x_{i\sigma(i)}^{(q)}|^{-1} = 1$ *for all* $i = 1, 2, \cdots$.

PROOF. (1) $\Rightarrow$ (2). If $AX_p = Y_p$, then $EAX_p = AEX_p = EY_p$ for every E in $\mathcal{L}$ $(p = 1, 2, \cdots, n)$. So $AEX_p f = EY_p f$ for every E in $\mathcal{L}$ and every f in $\mathcal{H}$ $(p = 1, 2, \cdots, n)$. Thus $A(\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}) = \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}$, $m_i \in N$, $l \leq n$, $E_{k,i} \in \mathcal{L}$ and $f_{k,i} \in \mathcal{H}$. So

$$\begin{aligned}\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\| &= \|A(\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i})\| \\ &\leq \|A\|\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|.\end{aligned}$$

If $\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\| \neq 0$, then

$$\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{ki}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{ki}\|} \leq \|A\|.$$

Hence

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|} : l \leq n,\ m_i \in \mathbb{N},\ E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}$$
$\leq \|A\|$.

Since every E in $\mathcal{L}$ reduces A, A is diagonal. Let $A = (a_{ii})$. Since $AX_p = Y_p$, $y_{ij}^{(p)} = a_{ii}x_{ij}^{(p)}$ for all p, all i and all j. Since A is a unitary operator, $|y_{i\sigma(i)}^{(q)}||x_{i\sigma(i)}^{(q)}|^{-1} = 1$ for all $i = 1, 2, \cdots$.

(2) $\Rightarrow$ (1). If

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}Y_if_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}X_if_{k,i}\|} : l\leq n, m_i\in\mathbb{N}, E_{k,i}\in\mathcal{L} \text{ and } f_{k,i}\in\mathcal{H}\right\} < \infty,$$

then, without loss of generality, we may assume that

$$\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}Y_if_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}X_if_{k,i}\|} : l\leq n, m_i\in\mathbb{N}, E_{k,i}\in\mathcal{L} \text{ and } f_{k,i}\in\mathcal{H}\right\}$$

$= 1$. So $\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}Y_if_{k,i}\| \leq \|\sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}X_if_{k,i}\| \cdots\cdots (*)$.

Let $\mathcal{M} = \left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}X_if_{k,i} : l\leq n, m_i\in\mathbb{N}, E_{k,i}\in\mathcal{L} \text{ and } f_{k,i}\in\mathcal{H}\right\}$. Then $\mathcal{M}$ is a linear manifold. Define $A : \mathcal{M} \to \mathcal{H}$ by

$$A(\sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}X_if_{k,i}) = \sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}Y_if_{k,i}.$$

Then A is well-defined by $(*)$. Clearly $AX_p = Y_p$. Extend A to $\overline{\mathcal{M}}$ by continuity. Define $A|_{\overline{\mathcal{M}}^\perp} = 0$. Then

$$\begin{aligned} AE(\sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}X_if_{k,i}) &= A(\sum_{k=1}^{m_i}\sum_{i=1}^{l}EE_{k,i}X_if_{k,i}) \\ &= \sum_{k=1}^{m_i}\sum_{i=1}^{l}EE_{k,i}Y_if_{k,i} \end{aligned}$$

and

$$\begin{aligned} EA(\sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}X_if_{k,i}) &= E(\sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}Y_if_{k,i}) \\ &= \sum_{k=1}^{m_i}\sum_{i=1}^{l}EE_{k,i}Y_if_{k,i}. \end{aligned}$$

And for every g in $\overline{\mathcal{M}}^\perp$, $EA(g) = E(0) = 0$ and $AE(g) = 0$ since

$$\langle Eg, \sum_{k=1}^{m_i}\sum_{i=1}^{l}E_{k,i}X_if_{k,i}\rangle = \langle g, \sum_{k=1}^{m_i}\sum_{i=1}^{l}EE_{k,i}X_if_{k,i}\rangle = 0.$$

Hence every E in $\mathcal{L}$ reduces A. So A is diagonal. Let $A = (a_{ii})$. Since $AX_p = Y_p$, $y_{ij}^{(p)} = a_{ii}x_{ij}^{(p)}$ for all i, all j and all $p = 1, 2, \cdots, n$. Since $|y_{i\sigma(i)}^{(q)}||x_{i\sigma(i)}^{(q)}|^{-1} = 1$ for all $i = 1, 2, \cdots$, A is a unitary operator. □

With the similar proof as Theorem 3, we can get the following theorem.

THEOREM 4. *Let $X_p = (x_{ij}^{(p)})$ and $Y_p = (y_{ij}^{(p)})$ be operators in $\mathcal{B}(\mathcal{H})$ $(p = 1, 2, \cdots)$ such that $x_{i\sigma(i)}^{(q)} \neq 0$ for some fixed q and for all i. Then the following statements are equivalent.*

(1) *There exists a unitary operator A in Alg$\mathcal{L}$ such that $AX_p = Y_p$ $(p = 1, 2, \cdots)$ and every E in $\mathcal{L}$ reduces A.*

(2) $\sup\left\{\dfrac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}\|} : m_i, l \in \mathbb{N}, E_{k,i} \in \mathcal{L} \text{ and } f_{k,i} \in \mathcal{H}\right\}$ $< \infty$ *and* $|y_{i\sigma(i)}^{(q)}||x_{i\sigma(i)}^{(q)}|^{-1} = 1$ *for all* $i = 1, 2, \cdots$.

References

[1] F. Gilfeather and D. Larson, *Commutants modulo the compact operators of certain CSL algebras*, Operator Theory: Adv. Appl., Birkhauser, Basel **2** (1981), 105–120.

[2] A. Hopenwasser, *The equation $Tx = y$ in a reflexive operator algebra*, Indiana University Math. J. **29** (1980), 121–126.

[3] R. Kadison, *Irreducible Operator Algebras*, Proc. Nat. Acad. Sci. U.S.A. (1957), 273–276.

[4] E. C. Lance, *Some properties of nest algebras*, Proc. London Math. Soc., (3), **19** (1969), 45–68.

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