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THE QUOTIENT GROUP GL(A)/exp(A) OF SOME BANACH ALGEBRAS

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ABSTRACT. Let A be $C(\mathbb{D})$, $A(\mathbb{D})$, $P(\mathbb{T})$, $C(\mathbb{T})$, or $H^\infty(U)$. We evaluate the quotient group $\mathrm{GL}(A)$ by $\exp(A)$.

1. Introduction

Let A be a Banach algebra with identity 1 over the complex field $\mathbb{C}$. The carrier space of A is denoted by $\Delta(A)$. Let a be an element of A. The spectrum of a is denoted by $\sigma_A(a)$. Given $a \in A$, $\exp(a)$ is defined by

$$\exp(a) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} a^n.$$

Since the complex variable $\exp(z)$ is an entire function, the $\exp(a)$ can also be defined by the following symbolic calculus:

$$\exp(a) = \frac{1}{2\pi i} \int_\Gamma \exp(z)(z-a)^{-1} dz,$$

where Γ is any contour surrounding $\sigma_A(a)$. We denote by $\exp(A)$ the range of the exponential in A, i.e., $\exp(A) = \{\exp(a) : a \in A\}$.

The set $\mathrm{GL}(A)$ of all invertible elements in A forms a multiplicative group, which is called the *general linear group* of A. With the topology induced from the norm topology of A, the mapping $a \to a^{-1}$ is a homeomorphism of $\mathrm{GL}(A)$ onto itself, and hence $\mathrm{GL}(A)$ is a topological

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group. Since GL(A) is an open subset of the Banach algebra A, GL(A) is a disjoint union of open connected subsets, the component of A. The component containing the identity 1 is called the *principal component* of GL(A). The principal component of GL(A) is a normal subgroup of GL(A). If A is commutative then the principal component of GL(A) is $\exp(A)$. It is well known that the quotient group $\mathrm{GL}(A)/\exp(A)$ is torsion free. But if A is not commutative $\exp(A)$ is not always a group [7]. If A is a commutative Banach algebra then Arens and Royden proved the following representation of the quotient group GL(A) by $\exp(A)$.

$$\mathrm{GL}(A)/\exp(A) = \pi^1(\Delta(A)),$$

where $\pi^1(\Delta(A))$ is the first cohomotopy group of $\Delta(A)$ [6]. And it is known that $\pi^1(\Delta(A))$ is naturally isomorphic to the first Čech cohomology group of $\Delta(A)$ with integer coefficients $H^1(\Delta(A), \mathbb{Z})$ [3]. But this description is not easy for analysts. Hence, from the winding number theory in the complex analysis, we directly evaluate the quotient group GL(A) by $\exp(A)$ for some Banach algebras $C(\mathbb{D})$, $A(\mathbb{D})$, $P(\mathbb{T})$, $C(\mathbb{T})$ and $H^\infty(U)$.

2. Banach algebra $A(\mathbb{D})$ and $P(\mathbb{T})$

Let K be a compact subset of the complex field $\mathbb{C}$. Let $C(K)$ be the Banach algebra of all continuous complex valued functions on K with pointwise definition of addition, scalar multiplication, product and the supremum norm

$$\|f\| = \sup\{|f(x)| : x \in K\}.$$

Let $\mathbb{D}$ denote the closed unit disc $\{z \in \mathbb{C} : |z| \le 1\}$ and $\mathbb{D}^0$ denote the interior of $\mathbb{D}$. The disc algebra $A(\mathbb{D})$ is the algebra of complex valued functions on $\mathbb{D}$ that are holomorphic on $\mathbb{D}^0$; that is

$$A(\mathbb{D}) = \{f \in C(\mathbb{D}) : f|\mathbb{D}^0 \text{ is holomorphic}\},$$

where $f|\mathbb{D}^0$ is the restriction of f on $\mathbb{D}^0$. Then $A(\mathbb{D})$ is a closed subalgebra of $C(\mathbb{D})$ and consequently a Banach algebra. It is well known that the carrier space $\Delta(A(\mathbb{D}))$ of $A(\mathbb{D})$ is $\mathbb{D}$. By the Gelfand representation, for each $f \in A(\mathbb{D})$, $\sigma_{A(\mathbb{D})}(f) = f(\mathbb{D})$.

Let $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. $P(\mathbb{T})$ denotes the closure of the set of all polynomials in $C(\mathbb{T})$. Let $T : A(\mathbb{D}) \to P(\mathbb{T})$ be defined by

$$T(f)(z) = f(z), \quad z \in \mathbb{T}, \quad f \in A(\mathbb{D}),$$

that is $T(f) = f|\mathbb{T}$. For any $g \in P(\mathbb{T})$, there is a sequence of polynomial $\{p_n\}$ such that $p_n \to g$ uniformly on $\mathbb{T}$. Then from the maximum modulus theorem $\{p_n\}$ is a Cauchy sequence in $A(\mathbb{D})$. Thus there is the unique $\bar{g} \in A(\mathbb{D})$ with $\bar{g}|\mathbb{T} = g$. Therefore, T is an isometric algebra isomorphism of $A(\mathbb{D})$ onto $P(\mathbb{T})$. If we identify $A(\mathbb{D})$ with $P(\mathbb{T})$ then we may regard $A(\mathbb{D})$ as a Banach subalgebra of $C(\mathbb{T})$. And $A(\mathbb{D})$ is the closure of the set of all polynomials. It is easily seen that $\Delta(C(\mathbb{T})) = \mathbb{T}$ and for $u(z) = z$, $\sigma_{C(\mathbb{T})}(u) = \mathbb{T}$. So we have a following example; $P(\mathbb{T})$ is a Banach subalgebra of $C(\mathbb{T})$ but $\Delta(C(\mathbb{T})) \subsetneq \Delta(P(\mathbb{T}))$ and $\sigma_{C(\mathbb{T})}(u) \subsetneq \sigma_{P(\mathbb{T})}(u)$.

3. The quotient group of GL(A) by exp(A)

Let U be a simply connected open subset of the complex field $\mathbb{C}$. Let $H^\infty(U)$ be the Banach algebra of functions which are holomorphic and bounded on U, with pointwise definition of addition, scalar multiplication, product and the supremum norm

$$\|f\| = \sup\{|f(x)| : x \in U\}.$$

THEOREM 1. *$GL(H^\infty(U)) = \exp(H^\infty(U))$.*

PROOF. Let $f \in \mathrm{GL}(H^\infty(U))$. Then $f(z) \neq 0$ for all $z \in U$. Let

$$h(z) = \frac{f'(z)}{f(z)} \quad \text{for all } z \in U.$$

Then h is holomorphic on simply connected open set U. Hence there is a primitive ϕ of h; that is

$$\phi'(z) = \frac{f'(z)}{f(z)} \quad \text{for all } z \in U.$$

Then

$$\begin{aligned}\frac{d}{dz}(f(z)e^{-\phi(z)}) &= f'(z)e^{-\phi(z)} - f(z)e^{-\phi(z)}\frac{f'(z)}{f(z)} \\ &= 0\end{aligned}$$

for all $z \in U$. Hence $f(z)e^{-\phi(z)}$ is a non zero constant c. If $\alpha \in \mathbb{C}$ and $e^\alpha = c$, then $f(z)e^{-\phi(z)} = e^\alpha$. Thus we have

$$f(z) = e^{\phi(z)+\alpha} \quad \text{for all } z \in U.$$

Let $g(z) = \phi(z) + \alpha$ for all $z \in U$. Then g is holomorphic and clearly bounded on U, that is $g \in H^\infty(U)$. Therefore, we have

$$e^{g(z)} = e^{\phi(f(z))+\alpha} = f(z) \quad \text{for all } \ z \in U.$$

Hence $f \in \exp(H^\infty(U))$. □

Let X, X' be n-dimensional manifolds, a map $p : X' \to X$ is said to be a *local homeomorphism* if for any $a' \in X'$ there exists an open set U' in X', $a' \in U'$, such that $p(U') = U$ is open and $p|U'$ is a homeomorphism onto U. Let Y be a topological space and $f : Y \to X$ a continuous map. A *lifting* f' of f (to X' or with respect to p) is a continuous map $f' : Y \to X'$ such that $p \circ f' = f$. If a lifting f' exists, we say that the map f can be lifted. Given a local homeomorphism $p : X' \to X$ and a continuous map $f : Y \to X$, there does not always exist a lifting of f; even when it does, it is not, in general, unique.

A continuous map $p : X' \to X$ is said to be a *covering map* (and call X' a *covering* of X) if every $a \in X$ has a neighborhood U with the following property: $p^{-1}(U)$ is a disjoint union, $p^{-1}(U) = \bigcup_{j \in J} U'_j$, of open sets U'_j in X' such that $p|U'_j$ is a homeomorphism onto U for each $j \in J$. Such an open set U is said to be *evenly covered* by p. If U is connected, the U'_j are the simply connected components of $p^{-1}(U)$. A covering map is necessarily a local homeomorphism. If $U \subset X$ is evenly covered by p and $V \subset U$ is open, then V is also evenly covered by p. The following proposition is in [4].

PROPOSITION 2. *Let $p : X' \to X$ be a covering map and let $a' \in X'$, $p(a') = a$. Let $\gamma : [0,1] \to X$ be a curve, that is γ is a continuous map from the closed unit interval $[0,1]$ to X, with $\gamma(0) = a$. Then there exists a lifting $\gamma' : [0,1] \to X'$ of γ with $\gamma'(0) = a'$.*

The map $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ defined by $\exp(z) = e^z$ is a covering map. In fact, if $V \subset \mathbb{C}$ is of the form $\tau < \operatorname{Im} z < \tau + 2\pi$, where $\tau \in \mathbb{R}$, we have

$$\exp^{-1}(\exp(V)) = \bigcup_{n \in \mathbb{Z}} V_n,$$

where $V_n = V + 2\pi i n = \{z + 2\pi i n | z \in V\}$. Hence $\exp(V)$ is evenly covered. ($\exp(V)$ is the complement in $\mathbb{C} \setminus \{0\}$ of a ray $\{\rho e^{i\tau} | \rho > 0\}$.)

Let $\gamma : [0,1] \to \mathbb{C}$ be a closed curve in $\mathbb{C}$, that is γ is a continuous map with $\gamma(0) = \gamma(1)$. And let $a \in \mathbb{C}$, with $a \notin \gamma([0,1])$. Since the map

$$\mathbb{C} \to \mathbb{C} \setminus \{a\} \quad \text{defined by} \quad z \mapsto a + e^z$$

is a covering map. There is a lifting γ' of γ with respect to this map by proposition 2. We define the *winding number* $W(\gamma, a)$ of γ with respect to a to be

$$W(\gamma, a) = \frac{1}{2\pi i}[\gamma'(1) - \gamma'(0)].$$

The winding number $W(\gamma, a)$ is an integer and dose not depend on a lifting γ' of γ. If the closed curve γ is piecewise continuously differential, then the winding number $W(\gamma, a)$ is to be the well known form

$$W(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz.$$

The following result is found in [4].

PROPOSITION 3. *Let X, X' be manifolds and $p : X' \to X$ a covering map. Let Y be a simply connected manifold. Let $a' \in X'$, $a = p(a')$. Suppose that $f : Y \to X$ is a continuous map and that there is $y_0 \in Y$ such that $f(y_0) = a$. Then there is a lifting $f' : Y \to X'$ such that $f'(y_0) = a'$.*

THEOREM 4. *$GL(C(\mathbb{D})) = \exp(C(\mathbb{D}))$.*

PROOF. Let $f \in \mathrm{GL}(C(\mathbb{D}))$. Then $f(z) \neq 0$ for all $z \in \mathbb{D}$. And hence $f : \mathbb{D} \to \mathbb{C} \setminus \{0\}$ is a continuous map. Since the map

$$\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}, \quad \exp(z) = e^z,$$

is a covering map. There is a lifting $f' : \mathbb{D} \to \mathbb{C} \setminus \{0\}$ with $e^{f'} = f$ by Proposition 3. Hence $f \in \exp(C(\mathbb{D}))$. Therefore, we have

$$\mathrm{GL}(C(\mathbb{D})) = \exp(C(\mathbb{D})). \qquad \square$$

THEOREM 5. *$GL(A(\mathbb{D})) = \exp(A(\mathbb{D}))$.*

PROOF. Let $f \in \mathrm{GL}(A(\mathbb{D}))$. Then $f(z) \neq 0$ for all $z \in \mathbb{D}$. Since f is holomorphic on the connected open set $\mathbb{D}^0$ and $f(z) \neq 0$ for all $z \in \mathbb{D}^0$. There is a holomorphic function g_1 with $e^{g_1} = f$ on $\mathbb{D}^0$ by the proof of Theorem 1. Since the map

$$\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}, \quad \exp(z) = e^z,$$

is a covering map, there is a lifting $g_2 : \mathbb{D} \to \mathbb{C} \setminus \{0\}$ with $e^{g_2} = f$ on $\mathbb{D}$ by Proposition 3. Thus $e^{g_1} = e^{g_2}$ on $\mathbb{D}^0$. Hence there is an integer $n \in \mathbb{Z}$ such that

$$g_1(z) = g_2(z) + 2\pi i n \quad \text{for all} \quad z \in \mathbb{D}^0.$$

Define the map $\bar{g} : \mathbb{D} \to \mathbb{C}$ by $\bar{g}(z) = g_2(z) + 2\pi i n$. Then $\bar{g} \in C(\mathbb{D})$ and $\bar{g}|\mathbb{D}^0 = g_1$. Hence $\bar{g} \in A(\mathbb{D})$. Then

$$e^{\bar{g}} = e^{g_2 + 2\pi i n} = e^{g_2} = f \quad \text{on} \quad \mathbb{D}.$$

Therefore, we have

$$\mathrm{GL}(A(\mathbb{D})) = \exp(A(\mathbb{D})).$$ □

Since $A(\mathbb{D})$ is isometrically isomorphic to $P(\mathbb{T})$, we have

$$\mathrm{GL}(P(\mathbb{T})) = \exp(P(\mathbb{T})).$$

THEOREM 6. *The quotient group $GL(C(\mathbb{T}))/\exp(C(\mathbb{T}))$ is isomorphic to $\mathbb{Z}$, the additive group of integers.*

PROOF. For each $f \in C(\mathbb{T})$, define the map

$$\gamma(f) : [0,1] \to \mathbb{C} \quad \text{by} \quad \gamma(f)(t) = f(e^{2\pi i t}), \quad t \in [0,1].$$

Then $\gamma(f)$ is a closed curve. If $f \in \mathrm{GL}(C(\mathbb{T}))$ then $f(z) \neq 0$ for all $z \in \mathbb{T}$. Hence $0 \notin f(\mathbb{T}) = \gamma(f)$. Define the map

$$\Psi : \mathrm{GL}(C(\mathbb{T})) \to \mathbb{Z} \quad \text{by} \quad \Psi(f) = W(\gamma(f), 0),$$

where $W(\gamma(f), 0)$ is the winding number of $\gamma(f)$ with respect to 0. We want to show that Ψ is a group homomorphism from $C(\mathbb{T})$ onto $\mathbb{Z}$ with $\ker(\Psi) = \exp(C(\mathbb{T}))$, where $\ker(\Psi)$ denotes the kernel of Ψ.

Let $f, g \in \mathrm{GL}(C(\mathbb{T}))$. Since the map

$$\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}, \quad \exp(z) = e^z,$$

is a covering map. There are liftings $\gamma(f)'$ of $\gamma(f)$ and $\gamma(g)'$ of $\gamma(g)$. That is

$$\gamma(f) = e^{\gamma(f)'} \quad \text{and} \quad \gamma(g) = e^{\gamma(g)'}.$$

Hence

$$\gamma(fg) = \gamma(f)\gamma(g) = e^{\gamma(f)' + \gamma(f)'}.$$

Hence $\gamma(f)' + \gamma(g)'$ is a lifting of $\gamma(fg)$. Since the winding number does not depend on a lifting,

$$\begin{aligned} W(\gamma(fg), 0) &= \frac{1}{2\pi i}[\gamma(fg)'(1) - \gamma(fg)'(0)] \\ &= \frac{1}{2\pi i}[(\gamma(f)' + \gamma(g)')(1) - (\gamma(f)' + \gamma(g)')(0)] \\ &= \frac{1}{2\pi i}[\gamma(f)'(1) - \gamma(f)'(0)] + \frac{1}{2\pi i}[\gamma(g)'(1) - \gamma(g)'(0)] \\ &= W(\gamma(f), 0) + W(\gamma(g), 0). \end{aligned}$$

Hence Ψ is a group homomorphism from $C(\mathbb{T})$ into $\mathbb{Z}$.

For each $n \in \mathbb{Z}$, the function $z^n \in \mathrm{GL}(C(\mathbb{T}))$. Then

$$\Psi(z^n) = \frac{1}{2\pi i} \int_{\gamma(z^n)} \frac{dz}{z} = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{nz^{n-1}}{z^n} dz = n.$$

Hence Ψ is onto.

Let $\Psi(f) = 0$. Then $W(\gamma(f), 0) = 0$. Let $\gamma(f)' : [0,1] \to \mathbb{C}$ be a lifting of $\gamma(f)$. That is $e^{\gamma(f)'} = \gamma(f)$. Then $\gamma(f)'(0) = \gamma(f)'(1)$. Define the map $g : \mathbb{T} \to \mathbb{C}$ by

$$g(e^{i\theta}) = \gamma(f)'(\frac{\theta}{2\pi}) \quad \text{for all} \quad \theta \in [0, 2\pi].$$

Then $g \in C(\mathbb{T})$. For each $\theta \in [0, 2\pi i]$,

$$\begin{aligned} e^{g(e^{i\theta})} &= e^{\gamma(f)'(\frac{\theta}{2\pi})} \\ &= \gamma(f)(\frac{\theta}{2\pi}) \\ &= f(e^{i\theta}), \end{aligned}$$

that is $e^g = f$ on $\mathbb{T}$. Hence $f \in \exp(C(\mathbb{T}))$.

Conversely, if $k \in \exp(C(\mathbb{T}))$ then there is a $h \in C(\mathbb{T})$ with $e^h = k$. Then $e^{\gamma(h)} = \gamma(k)$. Hence $\gamma(h)$ is a lifting of $\gamma(k)$. Since $\gamma(h)$ is a closed curve, $\gamma(h)(1) = \gamma(h)(0)$. Hence

$$\begin{aligned} W(\gamma(k), 0) &= \frac{1}{2\pi i}[\gamma(h)(1) - \gamma(h)(0)] \\ &= 0. \end{aligned}$$

That is $k \in \ker(\Psi)$. Therefore, we have

$$\ker(\Psi) = \exp(C(\mathbb{T})).$$

By the group isomorphism theorem, we have

$$\mathrm{GL}(C(\mathbb{T}))/\exp(C(\mathbb{T})) = \mathbb{Z}.$$

This completes the proof. □

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