Commun. Korean Math. Soc. **17** (2002), No. 3, pp. 439–449

FREE ARRANGEMENTS AND NICE PARTITIONS

HYUN-HEE CHOI AND KI-SUK LEE

ABSTRACT. If an arrangement admits a nice partition, or an arrangement is free, then the characteristic polynomial of the arrangement can be factored. It is known that a free arrangement does not always admit a nice partition. We show that even an inductively free arrangement does not always admit a nice partition.

1. Introduction

A (central) arrangement of hyperplanes is a finite collection of codimension one subspaces in a finite dimensional vector space over some field $\mathbf{K}$. For each hyperplane of an arrangement $\mathcal{A}$, a linear polynomial is defined. From the product of all corresponding linear polynomials, an algebraic property, freeness, is derived. On the other hand, a lattice $L(\mathcal{A})$ can be made by inclusion of finite intersections of hyperplanes of $\mathcal{A}$. Nice partition which is a partition of $\mathcal{A}$ satisfying some conditions is a combinatorial property of the lattice $L(\mathcal{A})$ of an arrangement. If an arrangement is free, or admits a nice partition, the **Poincaré** polynomial

$$\pi(\mathcal{A},t) = \sum_{X\in L} \mu(X)(-t)^{r(X)}.$$

($\mu(X)$ is the Möbius function and $r(X)$ is the rank of X) of the arrangement factors as

$$\pi(\mathcal{A},t) = \prod(1+d_i t)\ (d_i \in Z).$$

The fact can be found in [1].

Supersolvable arrangements, inductively free arrangements, recursively free arrangements are all free arrangements [1]. And their relation is following:

$$supersolvable \subset inductively\ free \subset recursively\ free \subset free.$$

Received September 24, 2001. Revised October 22, 2001.
2000 Mathematics Subject Classification: Primary 52B30; Secondary 32S25.
Key words and phrases: hyperplane arrangement, inductively free, nice partition.
This work was partially supported by Korea National University of Education.

Terao proved that an arrangement admits a nice partition if it is supersolvable. It is known that a free arrangement does not always admit a nice partition [3]. We will improve the above statement. In this paper we show that an example which is inductively free but admits no nice partition. So our conclusion is that inductively freeness is not a sufficient condition to get a nice partition.

REMARK 1.1. Existence of a nice partition does not always imply freeness. Let $M = AG(2,3)$ be the matroid of the affine plane of order 3. Then the corresponding hyperplane arrangement is not a free arrangement, but admits a nice partition [2].

2. Free arrangements

Let $\mathbf{K}$ be a field and let V be an ℓ-dimensional vector space over $\mathbf{K}$.

DEFINITION 2.1. A *hyperplane* H in V is a codimension one subspace of V and an *(central) arrangement* $\mathcal{A}$ in V is a finite collection of hyperplanes in V.

In this article, we only consider central arrangements. The empty arrangement in V is denoted by Φ_ℓ. Let V^* denote the dual space of V and let $x_1, \cdots, x_\ell$ be a basis for V^*. Let S be the polynomial algebra $\mathbf{K}[x_1, \cdots, x_\ell]$. Then each hyperplane H in V has a defining form

$$\alpha_H = a_1x_1 + \cdots + a_\ell x_\ell \ (a_i \in \mathbf{K})$$

unique up to a constant such that the hyperplane is $\ker\alpha_H$. The product

$$Q = Q(\mathcal{A}) = \prod_{H\in\mathcal{A}} \alpha_H$$

is called a **defining polynomial** for $\mathcal{A}$. We agree that $Q(\Phi_\ell) = 1$.

DEFINITION 2.2. Let $T(\mathcal{A}) = \cap_{H\in\mathcal{A}} H$.

EXAMPLE 2.3. Define an arrangement $\mathcal{A}$ by $Q(\mathcal{A}) = xy(x-y)$. It consists of three lines through the origin (see Figure 1).

DEFINITION 2.4. Let $\mathcal{A}$ be an arrangement and let $L = L(\mathcal{A})$ be the set of nonempty intersections of elements of $\mathcal{A}$. We define a *partial order* on L by

$$X \leq Y \Leftrightarrow Y \subseteq X.$$

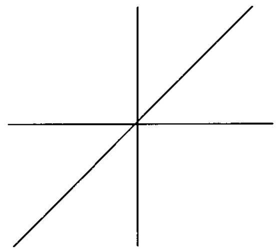

FIGURE 1. $Q(\mathcal{A}) = xy(x-y)$

DEFINITION 2.5. Let $X, Y \in L$. We define their *meet* by $X \wedge Y = \cap\{Z \in L(\mathcal{A}) | X \cup Y \subseteq Z\}$. If $X \cap Y \neq \phi$, we define their *join* by $X \vee Y = X \cap Y$.

DEFINITION 2.6. For $X \in L(\mathcal{A})$, let $r(X)$ be the codimension of X in V.

DEFINITION 2.7. Let $r(\mathcal{A}) = r(T(\mathcal{A}))$.

DEFINITION 2.8. For $X \in L(\mathcal{A})$, we define two new arrangements by

$$\mathcal{A}_X = \{H \in \mathcal{A} | X \subset H\}, \ \mathcal{A}^X = \{X \cap H | H \in \mathcal{A} - \mathcal{A}_X\}.$$

DEFINITION 2.9. Let $\mathcal{A}$ be a nonempty arrangement and let $H \in \mathcal{A}$. Let $\mathcal{A}' = \mathcal{A} - \{H\}$ and let $\mathcal{A}'' = \mathcal{A}^H$. We call $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ a *triple* of arrangements and H the distinguished hyperplane.

DEFINITION 2.10. A pair $(X, Y) \in L \times L$ is called a *modular pair* if $X \wedge Y = X + Y$.

DEFINITION 2.11. An element $X \in L$ is called *modular* if (X, Y) is a modular pair for all $Y \in L$.

DEFINITION 2.12. Let $\mathcal{A}$ be an arrangement with r($\mathcal{A}$)= ℓ. We call $\mathcal{A}$ *supersolvable* if $L(\mathcal{A})$ has a maximal chain of modular elements

$$V = X_0 < X_1 < \cdots < X_\ell = T.$$

DEFINITION 2.13. Let $\mathcal{A}$ be an arrangement and let $L = L(\mathcal{A})$. We define the Möbius function $\mu_{\mathcal{A}} = \mu : L \times L \longrightarrow \mathbb{Z}$ as follows:

$$\begin{aligned} \mu(X,X) &= 1 \quad \text{if} \quad X \in L, \\ \textstyle\sum_{X \le Z \le Y} \mu(X,Z) &= 0 \quad \text{if} \quad X,Y,Z \in L \quad \text{and} \quad X < Y, \\ \mu(X,Y) &= 0 \quad \text{otherwise.} \end{aligned}$$

DEFINITION 2.14. For $X \in L$, we define $\mu(X) = \mu(V, X)$.

DEFINITION 2.15. Let $\mathcal{A}$ be an arrangement with $L(\mathcal{A})$ and Möbius function μ. Let t be an indeterminate. We define the *Poincaré polynomial* of $\mathcal{A}$ by

$$\pi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{r(X)}.$$

EXAMPLE 2.16. If $\mathcal{A} = \phi$ is empty, then $\pi(\mathcal{A}, t) = 1$. The Poincaré polynomial in Example 2.3 is $\pi(\mathcal{A}, t) = 1 + 3t + 2t^2 = (1+t)(1+2t)$.

DEFINITION 2.17. A $\mathbf{K}$-linear map $\theta : S \to S$ is a *derivation* if for $f, g \in S$,

$$\theta(fg) = f\theta(g) + g\theta(f).$$

Let $\mathrm{Der}_{\mathbf{K}}(S)$ denote the set of $\mathbf{K}$-derivations on S. Note that $\mathrm{Der}_{\mathbf{K}}(S)$ is a free S-module with basis $D_1, \cdots, D_\ell$, where $D_i = \partial/\partial x_i$. Thus each element $\theta \in \mathrm{Der}_{\mathbf{K}}(\mathrm{S})$ can be written

$$\theta = f_i D_1 + \cdots + f_\ell D_\ell,$$

where each $f_i \in S$ $(i = 1, \cdots, \ell)$.

DEFINITION 2.18. Let $\mathcal{A}$ be an arrangement in V with defining polynomial

$$Q = Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H.$$

Then the *module of $\mathcal{A}$-derivations* is defined by

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}_{\mathrm{K}}(\mathrm{S}) | \theta(\mathrm{Q}) \in (\mathrm{Q})\}.$$

Note that $D(\mathcal{A})$ is an S-submodule of $\mathrm{Der}_{\mathrm{K}}(S)$.

DEFINITION 2.19. An arrangement $\mathcal{A}$ is called *free* if the module $D(\mathcal{A})$ is a free S-module.

DEFINITION 2.20. A nonzero element $\theta \in \mathrm{Der}_{\mathrm{K}}(\mathrm{S})$ is *homogeneous of polynomial degree* p if $\theta = \sum_{k=1}^{\ell} f_k D_k$ with each polynomial $f_k \in S$ $(i = 1, \cdots, \ell)$ homogeneous of degree p. In this case we write $\mathrm{pdeg}\theta = \mathrm{p}$. Note that $\mathrm{pdeg} D_i = 0$.

THEOREM 2.21. ([1]). *If $\mathcal{A}$ is a free arrangement, then $D(\mathcal{A})$ has a basis consisting of ℓ homogeneous elements.*

THEOREM 2.22. ([1]). *If $\mathcal{A}$ is free and $\{\theta_1, \cdots, \theta_\ell\}$ is a homogeneous basis for $D(\mathcal{A})$, the pdegrees $\{\mathrm{pdeg}\theta_1, \cdots, \mathrm{pdeg}\theta_\ell\}$ (with multiplicity but neglecting the order) depend only on $\mathcal{A}$.*

DEFINITION 2.23. Let $\mathcal{A}$ be a free arrangement and let $\{\theta_1, \cdots, \theta_\ell\}$ be a homogeneous basis for $D(\mathcal{A})$. We call $\text{pdeg}\theta_1, \cdots, \text{pdeg}\theta_\ell$ the *exponents* of $\mathcal{A}$ and write

$$\exp\mathcal{A} = \{\text{pdeg}\theta_1, \cdots, \text{pdeg}\theta_\ell\}.$$

THEOREM 2.24. ([1]). (**Factorization**) *If $\mathcal{A}$ is a free arrangement with $\exp\mathcal{A} = \{b_1, \cdots, b_\ell\}$, then*

$$\pi(\mathcal{A}, t) = \prod_{i=1}^{\ell}(1 + b_i t).$$

THEOREM 2.25. ([1]). (**Addition-Deletion**) *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple, then any two of the following statements implies the third:*

$$\mathcal{A} \text{ is free with } \exp\mathcal{A} = \{b_1, \cdots, b_{\ell-1}, b_\ell\},$$
$$\mathcal{A}' \text{ is free with } \exp\mathcal{A}' = \{b_1, \cdots, b_{\ell-1}, b_\ell - 1\},$$
$$\mathcal{A}'' \text{ is free with } \exp\mathcal{A}'' = \{b_1, \cdots, b_{\ell-1}\}.$$

DEFINITION 2.26. The class IF of *inductively free* arrangements is the smallest class of arrangements which satisfies:

1. $\Phi_\ell \in IF$ *for* $\ell \geq 0$,
2. if there exists $H \in \mathcal{A}$ such that $\mathcal{A}'' \in IF$, $\mathcal{A}' \in IF$, and $\exp\mathcal{A}'' \subset \exp\mathcal{A}'$, then $\mathcal{A} \in IF$.

By induction, we can show that the arrangement of example 2.3 is inductively free.

1. $\phi \in IF$,
2. To $Q(\mathcal{A}') = 1$, i.e., $\mathcal{A}' = \phi$, add $\alpha_H = x$. Then $Q(\mathcal{A}) = x$ and $Q(\mathcal{A}'') = 1$, i.e., $\mathcal{A}'' = \phi$. Since $\mathcal{A}' \in IF$ with $\exp\mathcal{A}' = \{0, 0\}$ and $\mathcal{A}'' \in IF$ with $\exp\mathcal{A}'' = \{0\}$, $\mathcal{A} \in IF$ and by theorem 2.25, $\exp\mathcal{A} = \{0, 1\}$.
3. To $Q(\mathcal{A}') = x$, add $\alpha_H = y$. Then $Q(\mathcal{A}) = xy$ and $Q(\mathcal{A}'') = x$. Since $\mathcal{A}' \in IF$ with $\exp\mathcal{A}' = \{0, 1\}$ and $\mathcal{A}'' \in IF$ with $\exp\mathcal{A}'' = \{1\}$, $\mathcal{A} \in IF$ with $\exp\mathcal{A} = \{1, 1\}$.
4. To $Q(\mathcal{A}') = xy$, add $\alpha_H = x - y$. Then $Q(\mathcal{A}) = xy(x - y)$ and $Q(\mathcal{A}'') = x$. Since $\mathcal{A}' \in IF$ with $\exp\mathcal{A}' = \{1, 1\}$ and $\mathcal{A}'' \in IF$ with $\exp\mathcal{A}'' = \{1\}$, $\mathcal{A} \in IF$ with $\exp\mathcal{A} = \{1, 2\}$.

Following induction table (Table 1) describes the above process.

In order to show that a given arrangement $\mathcal{A}$ is inductively free, we must start with some inductively free arrangement and add hyperplanes one at a time satisfying 2 in Definition 2.26. This process may be described conveniently in an **induction table**. Each row is one step in the

exp$\mathcal{A}'$	α_H	exp$\mathcal{A}''$
0,0	x	0
0,1	y	1
1,1	$x-y$	1
1,2		

TABLE 1. Induction Table $Q(\mathcal{A}) = xy(x-y)$

process. The first column gives exp$\mathcal{A}'$ of the arrangement which is the $\mathcal{A}'$ of that step. The second column gives α_H, where $H = \ker\alpha_H$ is the hyperplane added to $\mathcal{A}'$. The third column gives exp$\mathcal{A}''$. The last row displays exp$\mathcal{A}$ and the product of every α_H is the $Q(\mathcal{A})$. Since $Q(\mathcal{A}')$ is the product of the α_H in the rows above the row in consideration, it is easy to compute $Q(\mathcal{A}'')$. At each step the difficulty lies in showing that $\mathcal{A}''$ is free, and in computing exp$\mathcal{A}''$.

THEOREM 2.27. ([1]). *Let $\mathcal{A}$ be a supersolvable ℓ-arrangement with a maximal chain of modular elements*

$$V = X_0 < X_1 < \cdots < X_\ell = T.$$

Define $b_i = |\mathcal{A}_{X_i} \setminus \mathcal{A}_{X_{i-1}}|$ for $1 \le i \le \ell$. Then $\mathcal{A}$ is inductively free with

$$\text{exp}\mathcal{A} = \{b_1, \cdots, b_\ell\}.$$

THEOREM 2.28. ([1]). *Every 2-dimensional arrangement is inductively free with exp$\mathcal{A} = \{1, |\mathcal{A}|-1\}$ for $\mathcal{A} \neq \Phi_2$.*

THEOREM 2.29. ([1]). *For each arrangement $|\mathcal{A}| \le 3$, $\mathcal{A}$ is an inductively free arrangement.*

PROOF. If $|\mathcal{A}| = 0$, $\mathcal{A} = \phi$. It is trivially inductively free arrangement. If $|\mathcal{A}| = 1$, $\mathcal{A}' = \phi$ with exp$\mathcal{A}' = \{0^\ell\}$ and $\mathcal{A}'' = \phi$ with exp$\mathcal{A}'' = \{0^{\ell-1}\}$. Hence $\mathcal{A} \in IF$ and by theorem 2.25 exp$\mathcal{A} = \{0^{\ell-1}, 1\}$. If $|\mathcal{A}| = 2$, $Q(\mathcal{A}) = xy$ is the unique defining form. Then $Q(\mathcal{A}') = x$ with exp$\mathcal{A}' = \{0^{\ell-1}, 1\}$ and $Q(\mathcal{A}'') = x$ with exp$\mathcal{A}'' = \{0^{\ell-2}, 1\}$, where a basis of $\mathcal{A}'$ is $\{xD_1, D_2, \cdots, D_\ell\}$ and a basis of $\mathcal{A}''$ is $\{xD_1, D_2, \cdots, D_{\ell-1}\}$. Hence $\mathcal{A} \in IF$ with exp$\mathcal{A} = \{0^{\ell-2}, 1, 1\}$. If $|\mathcal{A}| = 3$, there are two defining forms. First let $Q(\mathcal{A}) = xyz$ and $\alpha_H = z$. Then $Q(\mathcal{A}') = xy$ with exp$\mathcal{A}' = \{0^{\ell-2}, 1, 1\}$ and $Q(\mathcal{A}'') = xy$ with exp$\mathcal{A}'' = \{0^{\ell-3}, 1, 1\}$ where a basis of $\mathcal{A}'$ is $\{xD_1, yD_2, D_3, \cdots, D_\ell\}$ and a basis of $\mathcal{A}''$ is $\{xD_1, yD_2, D_3, \cdots, D_{\ell-1}\}$. Second, let $Q(\mathcal{A}) = xy(x+y)$ and $\alpha_H = x+y$. Then $Q(\mathcal{A}') = xy$ with exp$\mathcal{A}' = \{0^{\ell-2}, 1, 1\}$ and $Q(\mathcal{A}'') = x$ with exp$\mathcal{A}'' = \{0^{\ell-2}, 1\}$. where a basis of $\mathcal{A}'$ is $\{xD_1, yD_2, \cdots, D_\ell\}$ and a basis of $\mathcal{A}''$ is $\{xD_1, D_2, \cdots, D_{\ell-1}\}$. For each case, by Definition 2.26 $\mathcal{A}$ is inductively free. □

3. Nice partitions

DEFINITION 3.1. A partition $\pi = (\pi_1, \cdots, \pi_s)$ of $\mathcal{A}$ is called *independent* if for every choice of hyperplanes $H_i \in \pi_i$ $(1 = 1, \cdots, s)$, the resulting set of hyperplanes are independent, i.e., $r(H_1 \vee \cdots \vee H_s) = s$.

DEFINITION 3.2. For $X \in L(\mathcal{A})$, the *induced partition* π_X is a partition of $\mathcal{A}_X$. Its blocks are the nonempty sets $\pi_i \cap \mathcal{A}_X$ $(i = 1, \cdots, s)$.

DEFINITION 3.3. A partition $\pi = (\pi_1, \cdots, \pi_s)$ of $\mathcal{A}$ is called nice if:

1. π is independent and
2. if $X \in L(\mathcal{A}) \backslash \{V\}$, then the induced partition π_X contains a block which is a singleton.

EXAMPLE 3.4. For an arrangement $\mathcal{A}$ defined by $Q(\mathcal{A}) = xy(x-y)$, if $\pi_1 = \{H_x\}$ and $\pi_2 = \{H_y, H_{x-y}\}$, $\pi = (\pi_1, \pi_2)$ is a nice partition of $\mathcal{A}$.

THEOREM 3.5. *Let $\mathcal{A}$ be a supersolvable ℓ-arrangement with a maximal chain of modular elements $V = X_0 < X_1 < \cdots < X_\ell = T$. Let $\pi_i = \mathcal{A}_{X_i} \setminus \mathcal{A}_{X_{i-1}}$. Then the partition $(\pi_1, \cdots, \pi_\ell)$ is nice.*

PROOF. Choose $H_i \in \pi_i$ for each i. First we use induction on i to prove that $r(H_1 \vee \cdots \vee H_i) = i$. This is clear when $i = 1$. Let $Y = H_1 \vee \cdots \vee H_i$. Then $Y \leq X_{i-1}$. Since $H_i \not\leq X_{i-1}$, we have $H_i \not\leq Y$. Thus $H_i \wedge Y = V$. By the inductive assumption, we have $r(Y) = i - 1$. Therefore we have

$$\begin{aligned} r(H_1 \vee \cdots \vee H_i) &= r(Y \vee H_i) = r(Y) + r(H_i) - r(Y \wedge H_i) \\ &= (i-1) + 1 - r(V) = i. \end{aligned}$$

This shows that the partition $(\pi_1, \cdots, \pi_\ell)$ is independent. Next let $X \in L(\mathcal{A}) \setminus \{V\}$. Let j be the largest integer such that $V = X \wedge X_j$. Then

$$\begin{aligned} 0 < r(X \wedge X_{j+1}) &= r(X) + r(X_{j+1}) - r(X \vee X_{j+1}) \\ &\leq r(X) + r(X_{j+1}) - r(X \vee X_j) \\ &= r(X) + r(X_{j+1}) - (r(X) + r(X_j) - r(X \wedge X_j)) \\ &= 1. \end{aligned}$$

This implies that $X \wedge X_{j+1}$ is a hyperplane belonging to $\mathcal{A}$. Thus $\mathcal{A}_X \cap \pi_j = \{X \wedge X_{j-1}\}$ is a singleton. □

COROLLARY 3.6. ([4]). *If $\mathcal{A}$ has a nice partition $\pi = (\pi_1, \cdots, \pi_s)$, then the multiset $\{|\pi_1|, \cdots, |\pi_s|\}$ depends only on $\mathcal{A}$.*

4. Conclusion

In this paper, we show that the arrangement of Coxeter group D_4 is an inductively free arrangement. We also show that the arrangement does not admit a nice partition.

THEOREM 4.1. *An arrangement $\mathcal{A}$ defined by $Q(\mathcal{A}) = (x^2-y^2)(x^2-z^2)(x^2-w^2)(y^2-z^2)(y^2-w^2)(z^2-w^2)$ is inductively free but admits no nice partition.*

PROOF. First we prove that the arrangement is inductively free. We start with $Q(\mathcal{A}) = (x+z)(y-w)(y+w)$. From theorem 2.29, $\mathcal{A}$ defined by $Q(\mathcal{A}) = (x+z)(y-w)(y+w)$ can be identified by $Q(\mathcal{A}) = x'y'z'$. Then it is inductively free with a basis $\{x'D_1, y'D_2, z'D_3, D_4\}$, and $\exp\mathcal{A} = \{0,1,1,1\}$. Now in each step, we have only to show that $\mathcal{A}''$ is inductively free and compute $\exp\mathcal{A}''$.

1. To $Q(\mathcal{A}') = (x+z)(y-w)(y+w)$, add $\alpha_H = z-w$. Then $Q(\mathcal{A}) = (x+z)(y-w)(y+w)(z-w)$ and $Q(\mathcal{A}'') = (x+z)(y-z)(y+z)$. By theorem 2.29, $\mathcal{A}'' \in IF$ with $\exp\mathcal{A}'' = \{1,1,1\}$. Hence $\mathcal{A} \in IF$ with $\exp\mathcal{A} = \{1,1,1,1\}$.
2. To $Q(\mathcal{A}') = (x+z)(y-w)(y+w)(z-w)$, add $\alpha_H = y+z$. Then $Q(\mathcal{A}) = (x+z)(y-w)(y+w)(z-w)(y+z)$ and $Q(\mathcal{A}'') = (x-y)(y-w)(y+w)$. By theorem 2.29, $\mathcal{A}'' \in IF$ with $\exp\mathcal{A}'' = \{1,1,1\}$. Hence $\mathcal{A} \in IF$ with $\exp\mathcal{A} = \{1,1,1,2\}$.
3. To $Q(\mathcal{A}') = (x+z)(y-w)(y+w)(z-w)(y+z)$, add $\alpha_H = y-z$. Then $Q(\mathcal{A}) = (x+z)(y-w)(y+w)(z-w)(y+z)(y-z)$ and $Q(\mathcal{A}'') = (x+y)(y-w)(y+w)y$. To show $\mathcal{A}'' \in IF$, let $\mathcal{A}'' = \mathcal{B}$ and $\alpha_H = y-w$. Then $Q(\mathcal{B}') = (x+y)(y+w)y$ and $Q(\mathcal{B}'') = (x+y)y$. By theorem 2.29, $\mathcal{B}'$, $\mathcal{B}'' \in IF$ and $\exp\mathcal{B}' = \{1,1,1\}$ and $\exp\mathcal{B}'' = \{1,1\}$. Hence $\mathcal{B} \in IF$ with $\exp\mathcal{B} = \{1,1,2\}$. Hence $\mathcal{A} \in IF$ with $\exp\mathcal{A} = \{1,1,2,2\}$.
4. To $Q(\mathcal{A}') = (x+z)(y-w)(y+w)(z-w)(y+z)(y-z)$, add $\alpha_H = x+y$. Then $Q(\mathcal{A}) = (x+z)(y-w)(y+w)(z-w)(y+z)(y-z)(x+y)$ and $Q(\mathcal{A}'') = (y-w)(y+w)(z-w)(y+z)(y-z)$. To show $\mathcal{A}'' \in IF$, let $\mathcal{A}'' = \mathcal{B}$ and $\alpha_H = y-z$. Then $Q(\mathcal{B}') = (y-w)(y+w)(z-w)(y+z)$ is the same type with the above $Q(\mathcal{A}'') = (x+y)(y-w)(y+w)y$, *i.e.*, $x'y'z'(y'+z')$. Thus $\mathcal{B}' \in IF$, with $\exp\mathcal{B}' = \{1,1,2\}$. And $Q(\mathcal{B}'') = (y-w)(y+w)y$ is the same type with $x'y'(x'+y')$ in theorem 2.29. It is inductively free with $\exp\mathcal{B}'' = \{1,2\}$. Then $\mathcal{B} \in IF$ with $\exp\mathcal{B} = \{1,2,2\}$. Hence $\mathcal{A} \in IF$ with $\exp\mathcal{A} = \{1,2,2,2\}$.

5. To $Q(\mathcal{A}') = (x+z)(y-w)(y+w)(z-w)(y+z)(y-z)(x+y)$, add $\alpha_H = x+w$. Then $Q(\mathcal{A}) = (x+z)(y-w)(y+w)(z-w)(y+z)(y-z)(x+y)(x+w)$ and $Q(\mathcal{A}'') = (x+z)(x+y)(x-y)(y+z)(y-z)$. The $Q(\mathcal{A}'')$ is the same type with the above $Q(\mathcal{A}'')$. Hence $\mathcal{A} \in IF$ with $\exp\mathcal{A} = \{1,2,2,3\}$.
6. To $Q(\mathcal{A}') = (x+z)(y-w)(y+w)(z-w)(y+z)(y-z)(x+y)(x+w)$, add $\alpha_H = x-y$. Then $Q(\mathcal{A}) = (x+z)(y-w)(y+w)(z-w)(y+z)(y-z)(x+y)(x+w)(x-y)$ and $Q(\mathcal{A}'') = (x+z)(x-w)(x+w)(z-w)(x-z)x$. To show that $\mathcal{A}'' \in IF$, let $\mathcal{A}'' = \mathcal{B}$ and $\alpha_H = x$. Then $Q(\mathcal{B}') = (x+z)(x-w)(x+w)(x-z)x$ is the same type with the above $\mathcal{A}''$. Thus $\mathcal{B}' \in IF$ with $\exp\mathcal{B}' = \{1,2,2\}$. And for $Q(\mathcal{B}'') = zw(z-w)$, by theorem 2.29, $\mathcal{B}'' \in IF$ with $\exp\mathcal{B}'' = \{1,2\}$. Hence $\mathcal{B} \in IF$ with $\exp\mathcal{B} = \{1,2,3\}$. Hence $\mathcal{A} \in IF$ with $\exp\mathcal{A} = \{1,2,3,3\}$.
7. To $Q(\mathcal{A}') = (x+z)(y-w)(y+w)(z-w)(y+z)(y-z)(x+y)(x+w)(x-y)$, add $\alpha_H = z+w$. Then $Q(\mathcal{A}) = (x+z)(y-w)(y+w)(z-w)(y+z)(y-z)(x+y)(x+w)(x-y)(z+w)$ and $Q(\mathcal{A}'') = (x+z)(y+z)(y-z)z(x+y)(x-z)(x-y)$. To show that $\mathcal{A}'' \in IF$, let $\mathcal{A}'' = \mathcal{B}$ and $\alpha_H = z$. Then $Q(\mathcal{B}') = (x+z)(y+z)(y-z)(x+y)(x-z)(x-y)$ and $Q(\mathcal{B}'') = x(x+y)y(x-y)$. First to show that $\mathcal{B}' \in IF$, let $\mathcal{B}' = \mathcal{C}$ and $\alpha_H = y-z$. Then $Q(\mathcal{C}') = (x+z)(y+z)(x+y)(x-z)(x-y)$ is the same type with the above $Q(\mathcal{A}'')$ and $Q(\mathcal{C}'') = (x+y)y(x-y)$. Thus $\mathcal{C}' \in IF$ with $\exp\mathcal{C}' = \{1,2,2\}$ and $\mathcal{C}'' \in IF$ with $\exp\mathcal{C}'' = \{1,2\}$. Hence $\mathcal{C} \in IF$ with $\exp\mathcal{C} = \{1,2,3\}$. By theorem 2.28, $\mathcal{B}'' \in IF$ with $\exp\mathcal{B}'' = \{1,3\}$. Thus $\mathcal{B} \in IF$ with $\exp\mathcal{B} = \{1,3,3\}$. Hence $\mathcal{A} \in IF$ with $\exp\mathcal{A} = \{1,3,3,3\}$.
8. To $Q(\mathcal{A}') = (x+z)(y-w)(y+w)(z-w)(y+z)(y-z)(x+y)(x+w)(x-y)(z+w)$, add $\alpha_H = x-z$. Then $Q(\mathcal{A}) = (x+z)(y-w)(y+w)(z-w)(y+z)(y-z)(x+y)(x+w)(x-y)(z+w)(x-z)$ and $Q(\mathcal{A}'') = x(y-w)(y+w)(x-w)(y+x)(y-x)(x+w)$ is the same type with the above $Q(\mathcal{A}'')$. Hence $\mathcal{A} \in IF$ with $\exp\mathcal{A} = \{1,3,3,4\}$.
9. To $Q(\mathcal{A}') = (x+z)(y-w)(y+w)(z-w)(y+z)(y-z)(x+y)(x+w)(x-y)(z+w)(x-z)$, add $\alpha_H = x-w$. Then $Q(\mathcal{A}) = (x+z)(y-w)(y+w)(z-w)(y+z)(y-z)(x+y)(x+w)(x-y)(z+w)(x-z)(x-w) = (x^2-y^2)(x^2-z^2)(x^2-w^2)(y^2-z^2)(y^2-w^2)(z^2-w^2)$ and $Q(\mathcal{A}'') = (x+z)(y-x)(y+z)(y-z)(x+y)x(z-x)$ is the same type with the above $Q(\mathcal{A}'')$. Hence $\mathcal{A} \in IF$ with $\exp\mathcal{A} = \{1,3,3,5\}$.

The induction table (Table 2) shows the above process.

Now we prove that the arrangement admits no nice partition. Suppose that the arrangement has a nice partition. Then by Theorem 3.6,

exp$\mathcal{A}'$	α_H	exp$\mathcal{A}''$
0,1,1,1	$z-w$	1,1,1
1,1,1,1	$y+z$	1,1,1
1,1,1,2	$y-z$	1,1,2
1,1,2,2	$x+y$	1,2,2
1,2,2,2	$x+w$	1,2,2
1,2,2,3	$x-y$	1,2,3
1,2,3,3	$z+w$	1,3,3
1,3,3,3	$x-z$	1,3,3
1,3,3,4	$x-w$	1,3,3
1,3,3,5		

TABLE 2. Induction Table D_4

the partition $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ and $\{|\pi_1|, |\pi_2|, |\pi_3|, |\pi_4|\} = \{1, 3, 3, 5\}$. Without loss of generality, we may choose one hyperplane, i.e., H_{x+y}, and let $\pi_1 = \{H_{x+y}\}$. Then $\{H_{x+z}, H_{y-z}\}, \{H_{y+z}, H_{x-z}\}, \{H_{x+w}, H_{y-w}\}$, and $\{H_{x-w}, H_{y+w}\}$ have to be contained in a block to satisfy independence condition, and H_{x-y}, H_{z+w} and H_{z-w} have to be contained in different blocks, i.e., $H_{x-y} \in \pi_2, H_{z+w} \in \pi_3, H_{z-w} \in \pi_4$. Then it can be easily checked that it is impossible to construct a partition satisfying independence condition and $\{|\pi_1|, |\pi_2|, |\pi_3|, |\pi_4|\} = \{1, 3, 3, 5\}$. □

References

[1] P. Orlik and H. Terao, *Arrangements of Hyperplanes*, Grundlehren 300, Springer-Verlag, New York, NY, 1992.

[2] G. Ziegler, *Matroid Representations and Free Arrangements*, Trans. Amer. Math. Soc. **320** (1990), 525–541.

[3] H. Terao, *Factorizations of the Orlik-Solomon Algebras*, Advances in Mathematics **91** (1992), 45–53.

[4] P. Orlik and L. Solomon, *Combinatorics and Topology of Complements of Hyperplanes*, Ivent. Math. **56** (1980), 167–189

[5] L. C. Grove and C. T. Benson, *Finite Reflection Groups*, Springer-Verlag, New York, 1985.

Department of Mathematics Education
Korea National University of Education
7 Darak-Ri Kangnae-Myun, Chungwon-Gun
Chungbuk 363-791, Korea
E-mail: gabriel_choi@hanmail.net
kslee@cc.knue.ac.kr