Commun. Korean Math. Soc. **17** (2002), No. 3, pp. 403–407

UNIFORM STRUCTURES IN BCI-ALGEBRAS

Dall Sun Yoon and Hee Sik Kim*

Abstract. In this note we discuss the uniformity in BCI-algebras using Zhang's congruence relation.

1. Introduction

K. Iséki ([4]) discussed the quasi-uniformity on BCK-algebras, and R. A. Alo and E. Y. Deeba ([1]) studied the uniformity in BCK-algebras, and Y. B. Jun and E. H. Roh ([6]) discussed the uniformity in BCK-algebras by using the concept of dual ideals. Y. B. Jun and H. S. Kim ([5]) discussed the uniformity in positive implicative algebras. In this note we discuss the uniformity in BCI-algebras using Zhang's congruence relation.

2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms for all $x, y, z \in X$:

(i) $((x * y) * (x * z)) * (z * y) = 0$,
(ii) $(x * (x * y)) * y = 0$,
(iii) $x * x = 0$,
(iv) $x * y = 0$ and $y * x = 0$ imply $x = y$.

We can define a partial ordering $\leq$ on X by $x \leq y$ if and only if $x * y = 0$. In any BCI-algebra X, the following hold:

(1) $x * 0 = x$,
(2) $(x * y) * z = (x * z) * y$,

Received October 27, 2000.
2000 Mathematics Subject Classification: 06F35.
Key words and phrases: uniformity, BCI-algebra.
* This work was supported by Hanyang University Research Fund, 2000.

(3) $x \le y$ imply $x * z \le y * z$ and $z * y \le z * x$,
(4) $(x * z) * (y * z) \le x * y$

for any $x, y, z \in X$.

In what follows X and Y would mean BCI-algebras unless otherwise specified.

An ideal of X is a subset A of X containing 0 such that if $x * y \in A$ and $y \in A$ then $x \in A$. It is well known that if A is an ideal of X and $y \in A$, $x \le y$, then $x \in A$. A map $f : X \to Y$ of BCI-algebras is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. For any elements x and y in X, let us write $x * y^n$ for $(\cdots((x * y) * y) * \cdots) * y$ where y occurs n times.

PROPOSITION 2.1 ([3]). *For any $x, y \in X$ and for any positive integer n, we have*

$$0 * (x * y)^n = (0 * x^n) * (0 * y^n).$$

Let A be an ideal of X. For every natural number n, we define a relation $\sim$ on X as follows:

$$x \sim_A y \text{ if and only if } 0 * (x * y)^n \in A \text{ and } 0 * (y * x)^n \in A.$$

Then $\sim_A$ is an equivalence relation on X (see [9, Theorem 1]).

THEOREM 2.2 ([2]). *If $x \sim_A u$ and $y \sim_A v$, then $x * y \sim_A u * v$.*

THEOREM 2.3 ([7]). *Let X be a BCK/BCI-algebra and let A be a non-empty subset of X. Then the ideal $(A]$ generated by the set A is $\{x \in X \mid (\cdots((x * a_1) * a_2) * \cdots) * a_n = 0$ for some $a_1, \cdots, a_n \in A\}$.*

3. Uniformity in BCI-algebras

Let X be a non-empty set and let U and V be any subsets of $X \times X$. Define

$$U \circ V = \{(x, y) \in X \times X \mid \text{ for some } z \in X, (x, z) \in U \text{ and } (z, y) \in V\},$$
$$U^{-1} = \{(x, y) \in X \times X \mid (y, x) \in U\},$$
$$\triangle = \{(x, x) \in X \times X \mid x \in X\}.$$

DEFINITION 3.1 ([8]). By a *uniformity* on X we shall mean a non-empty collection $\mathcal{K}$ of subsets of $X \times X$ which satisfies the following conditions:

(U_1) $\triangle \subseteq U$ for any $U \in \mathcal{K}$,
(U_2) if $U \in \mathcal{K}$, then $U^{-1} \in \mathcal{K}$,
(U_3) if $U \in \mathcal{K}$, then there exists a $V \in \mathcal{K}$ such that $V \circ V \subseteq U$,
(U_4) if $U, V \in \mathcal{K}$, then $U \cap V \in \mathcal{K}$,
(U_5) if $U \in \mathcal{K}$ and $U \subseteq V \subseteq A \times A$, then $V \in \mathcal{K}$.

The pair $(X, \mathcal{K})$ is called a *uniform structure.*

THEOREM 3.2. *Let X be a BCI-algebra and let A be an ideal of X. If we define $U_A := \{(x,y) \in X \times X \mid 0*(x*y)^n \in A$ and $0*(y*x)^n \in A$ for any positive integer $n\}$ and let*

$$\mathcal{K}^* = \{U_A \mid A \text{ is an ideal of } X\}.$$

Then $\mathcal{K}^*$ satisfies the conditions $(U_1) \sim (U_4)$.

PROOF. (U_1). If $(x,x) \in A$, then $(x,x) \in U_A$ since $0*(x*x)^n = 0 \in A$ for any positive integer n. Hence $A \subseteq U_A$ for any $U_A \in \mathcal{K}^*$.

(U_2). For any $U_A \in \mathcal{K}^*$,

$$\begin{aligned}(x,y) \in U_A &\Longleftrightarrow 0*(x*y)^n \in A \text{ and } 0*(y*x)^n \in A \\ &\qquad \text{for any positive integer } n \\ &\Longleftrightarrow y \sim_A x \\ &\Longleftrightarrow (y,x) \in U_A \\ &\Longleftrightarrow (x,y) \in {U_A}^{-1}.\end{aligned}$$

Hence ${U_A}^{-1} = U_A \in \mathcal{K}^*$.

(U_3). For any $U_A \in \mathcal{K}^*$, let $\mathcal{B} := \{A_\alpha \mid A_\alpha :$ an ideal of X such that $A_\alpha \subseteq A\}$. Then $\mathcal{B}$ is non-empty, since $A \in \mathcal{B}$. If we define J the ideal of X generated by the set $\cup\mathcal{B}$ i.e., $J = (\cup\mathcal{B}]$, then $U_J \in \mathcal{K}^*$. We claim that $U_J \circ U_J \subseteq U_A$. If $(x,y) \in U_J \circ U_J$ then there exists $z \in X$ such that $(x,z),(z,y) \in U_J$. This means that $x \sim_J z$ and $z \sim_J y$. Since $\sim_J$ is an equivalence relation, we have $x \sim_J y$, i.e., $0*(x*y)^n \in J$ and $0*(y*x)^n \in J$ for any positive integer n. Since $\cup\mathcal{B} \subseteq A, J \subseteq A$. Hence $0*(x*y)^n, 0*(y*x)^n \in A$ for any positive integer n, i.e., $(x,y) \in U_A$.

(U_4). For any U_I and U_J in $\mathcal{K}^*$, we claim that $U_I \cap U_J = U_{I\cap J}$.

$$\begin{aligned}(x,y) \in U_I \cap U_J &\iff (x,y) \in U_I,\ (x,y) \in U_J \\ &\iff 0*(x*y)^n, 0*(y*x)^n \in I \cap J \\ &\qquad \text{for any positive integer } n \\ &\iff x \sim_{I\cap J} y \\ &\iff (x,y) \in U_{I\cap J}.\end{aligned}$$

Since $I \cap J$ is an ideal of X, $U_I \cap U_J = U_{I\cap J} \in \mathcal{K}^*$. This proves the theorem. □

THEOREM 3.3. *Let $\mathcal{K} := \{U \subseteq X \times X \mid U_A \subseteq U \text{ for some } U_A \in \mathcal{K}^*\}$. Then $\mathcal{K}$ satisfies a uniformity on X and hence the pair $(X, \mathcal{K})$ is a uniform structure.*

PROOF. By applying Theorem 3.2 we can show that $\mathcal{K}$ satisfies the conditions $(U_1) \sim (U_4)$. Let $U \in \mathcal{K}$ and $U \subseteq V \subseteq X \times X$. Then there exists a $U_A \subseteq U \subseteq V$, which means that $V \in \mathcal{K}$. This proves the theorem. □

Given a $x \in A$ and $U \in \mathcal{K}$, we define

$$U[x] := \{y \in X \mid (x,y) \in U\}.$$

THEOREM 3.4. *The collection $\mathcal{U}_x := \{U[x] \mid U \in \mathcal{K}\}$, $x \in X$, forms a neighborhood base at x, making X a topological space.*

PROOF. Note that $x \in U[x]$ for each $x \in X$. Since $U_1[x] \cap U_2[x] = (U_1 \cap U_2)[x]$, the intersection of neighborhoods is also a neighborhood. Finally, if $U[x] \in \mathcal{U}_x$ then by (U_3) there exists a $V \in \mathcal{K}$ such that $V \circ V \subseteq U$. Hence for any $y \in V[x]$, $V[y] \subseteq U[x]$, proving the theorem.□

References

[1] R. A. Alo and E. Y. Deeba, *A note on uniformities of a BCK-algebra*, Math. Japo. **30** (1985), 237–240.

[2] S. M. Hong and Y. B. Jun, *Quotient BCI-algebras via fuzzy ideals*, Far East J. Math. Sci. **4** (1996), no. 3, 343–351.

[3] W. P. Huang, *Nil-radical in BCI-algebras*, Math. Japo. **37** (1992), 363–366.
[4] K. Iséki, *On a quasi-uniformity on BCK-algebras*, Math. Seminar Notes (1976), 225–226.
[5] Y. B. Jun and H. S. Kim, *Uniform structures in positive implicative algebras*, International Mathematical Journal **2** (2002), 215–218.
[6] Y. B. Jun and E. H. Roh, *On uniformities of BCK-algebras*, Comm. Korean Math. Soc. **10** (1995), 11–14.
[7] J. Meng and Y. B. Jun, *BCK-algebras*, Kyung Moon Sa, Seoul, 1994.
[8] S. Willard, *General Topology*, Addison-Wesley Pub. Co., New York, 1970.
[9] F. L. Zhang, *A class of quotient algebras in BCI-algebras*, Selected papers on BCK- and BCI-algebras (in P. R. China) **1** (1992), 89–90.

Department of Mathematics
Hanyang University
Seoul 133-791, Korea
E-mail: dsyun@hanyang.ac.kr
heekim@hanyang.ac.kr