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ON THE HOMOLOGY OF SCHUR COMPLEXES

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ABSTRACT. We give an upper bound for the degrees of the non-vanishing homology modules of the Schur complex $L_{\lambda/\mu}\phi$ in terms of the depths of the determinantal ideals of ϕ. Using this fact, we obtain the acyclic theorem for $L_\lambda\phi$ and the information concerning the support of the homology modules of $L_{\lambda/\mu}\phi$.

1. Introduction

Let R be a Noetherian ring with unity. Let m, n be positive integers and let us denote by S the polynomial ring over R with mn variables x_{ij} for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. Consider the $m \times n$ matrix (x_{ij}) with entries in S. We call such a matrix a generic matrix. For a positive integer t such that $1 \leq t \leq \min(m, n)$, I_t is defined to be the ideal of S generated by all t-minors of (x_{ij}). We call such ideals determinantal ideals. The rings S/I_t have been the classical objects of intensive study. For instance, by the first and the second fundamental theorems, it is well known [5, 15] that when an orthogonal group acts on a certain polynomial ring in a certain way, the ring of invariants is described in the form of S/I_t. Furthermore, M. Hochster and J. A. Eagon proved in [6] that I_t is perfect (i.e., $\mathrm{pd}_S S/I_t = \operatorname{grade} I_t = (m-t+1)(n-t+1)$). For many years there has been considerable interest in finding a minimal free resolution of S/I_ts.

It is well known [10, 14] that, in order to show the existence of minimal free resolutions of S/I_t, we have only to prove that Betti numbers are independent of the characteristic of the coefficient field. In 1978, Lascoux gave in [11] an explicit description of minimal free resolution

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of S/I_t for any m, n and t, when R contains the field of rational numbers $\mathbb{Q}$, using the classical representation theory of general linear groups. In fact, all underlying modules of Lascoux complexes give polynomial representations of general linear groups [13]. But over an arbitrary commutative ring R, the existence of minimal free resolutions is not known in general. If $t = 1$ then the Koszul complex gives us such a resolution. If $t = \min(m, n)$, such resolutions were constructed by Eagon and Northcott in [7]. In 1979, Buchsbaum gave in [3] the new construction of the Eagon-Northcott complex using multilinear algebra. If $t = \min(m, n) - 1$, then Akin, Buchsbaum and Weyman constructed a minimal free resolution of S/I_t over $\mathbb{Z}$ using the characteristic free representation theory of general linear groups in [1]. If $t = \min(m, n) - 2$, then Hashimoto proved in [9] the existence of minimal free resolutions of S/I_t. On the other hand, Hashimoto proved in [8] that there is no minimal free resolution of S/I_t over $\mathbb{Z}$ in the case $2 \leq t \leq \min(m, n) - 3$. Hence the construction of the minimal free resolution of S/I_t over $\mathbb{Z}$ is based on the characteristic free representation theory of general linear groups. But Schur functors are fundamental objects in the characteristic free representation theory of general linear groups and Schur complexes are natural generalization of Schur functors. Therefore Schur complexes play central roles in the resolution of the determinantal ideal. This forces us to further study Schur complexes.

In this paper, we gave an upper bound for the degrees of the nonvanishing homology modules of the Schur complex $L_{\lambda/\mu}\phi$ in terms of the depths of the determinantal ideals of ϕ. Now we describe the contents of this paper. In Chapter 2, we review some basic facts on characteristic free representation theory of general linear groups, including Schur functors and Schur complexes. Although Schur complex is defined for any finite free complex in the characteristic zero case, it is defined only for a morphism of finite free complex of length one in the characteristic free case. In Chapter 3, we prove an upper bound for the degrees of the nonvanishing homology modules of the Schur complex $L_{\lambda/\mu}\phi$ by induction on rank of G. Using this fact, we obtain the acyclic theorem for $L_\lambda\phi$ and the information concerning the support of the homology modules of $L_{\lambda/\mu}\phi$.

2. Preliminaries

This chapter is devoted to introducing the definitions and the basic facts on Schur complex. Therefore all proofs are omitted. As for the proofs of Theorems, we refer to Akin, Buchsbaum, and Weyman [2]. Throughout this chapter, R is a commutative ring and $\phi : G \longrightarrow F$ is an R-module homomorphism between free R-modules of rank n and m, respectively. We will denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{N}^\infty$ the set of sequence of elements of $\mathbb{N}$ of finite support.

DEFINITION 2.1. A partition is an element $\lambda = (\lambda_1, \lambda_2, \cdots)$ of $\mathbb{N}^\infty$ such that $\lambda_1 \geq \lambda_2 \geq \cdots$. The weight of partition λ, denoted by $|\lambda|$, is the sum $\sum \lambda_i$. If $|\lambda| = n$, λ is said to be a partition of n. The number of nonzero terms of λ is called the length of λ. To each partition λ of weight n, we associate its transpose $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_t)$ where $\tilde{\lambda}_k$ is the number of integers λ_l such that $\lambda_l \geq k$. If $\mu = (\mu_1, \mu_2, \cdots)$ is also a partition, we will say that μ is a sub-partition of λ, or that $\mu \subseteq \lambda$, if $\mu_i \leq \lambda_i$ for all i.

DEFINITION 2.2. Let $\lambda = (\lambda_1, \cdots, \lambda_s)$, $\mu = (\mu_1, \cdots, \mu_s)$ be partitions such that $\mu \subseteq \lambda$. If F is a finite free R-module then we define $S_{\lambda/\mu}F$, $\wedge_{\lambda/\mu}F$, $D_{\lambda/\mu}F$ as follows :

$$\begin{aligned} S_{\lambda/\mu}F &= S_{\lambda_1-\mu_1}F \otimes \cdots \otimes S_{\lambda_s-\mu_s}F, \\ \wedge_{\lambda/\mu} F &= \wedge^{\lambda_1-\mu_1}F \otimes \cdots \otimes \wedge^{\lambda_s-\mu_s}F, \\ D_{\lambda/\mu}F &= D_{\lambda_1-\mu_1}F \otimes \cdots \otimes D_{\lambda_s-\mu_s}F. \end{aligned}$$

Let $\lambda = (\lambda_1, \cdots, \lambda_s)$, $\mu = (\mu_1, \cdots, \mu_s)$ be partitions such that $\mu \subseteq \lambda$. Let $\tilde{\lambda} = (\tilde{\lambda}_1, \cdots, \tilde{\lambda}_t)$, $\tilde{\mu} = (\tilde{\mu}_1, \cdots, \tilde{\mu}_t)$ be their transposes. Let (α_{ij}) be the $s \times t$ matrix defined by

$$\alpha_{ij} = \begin{cases} 1 & \text{if } \mu_i + 1 \leq j \leq \lambda_i \\ 0 & \text{otherwise.} \end{cases}$$

We define the map $d_{\lambda/\mu}(F) : \wedge_{\lambda/\mu}F \longrightarrow S_{\tilde{\lambda}/\tilde{\mu}}F$ to be the composition

$$\begin{aligned} &\wedge^{\lambda_1-\mu_1} F \otimes \cdots \otimes \wedge^{\lambda_s-\mu_s} F \\ \longrightarrow &(\wedge^{\alpha_{11}}F \otimes \cdots \otimes \wedge^{\alpha_{1t}}F) \otimes \cdots \otimes (\wedge^{\alpha_{s1}}F \otimes \cdots \otimes \wedge^{\alpha_{st}}F) \\ \longrightarrow &(S_{\alpha_{11}}F \otimes \cdots \otimes S_{\alpha_{1t}}F) \otimes \cdots \otimes (S_{\alpha_{s1}}F \otimes \cdots \otimes S_{\alpha_{st}}F) \\ \longrightarrow &(S_{\alpha_{11}}F \otimes \cdots \otimes S_{\alpha_{s1}}F) \otimes \cdots \otimes (S_{\alpha_{1t}}F \otimes \cdots \otimes S_{\alpha_{st}}F) \\ \longrightarrow &S_{\tilde{\lambda}_1-\tilde{\mu}_1}F \otimes \cdots \otimes S_{\tilde{\lambda}_t-\tilde{\mu}_t}F \end{aligned}$$

where the first map is the tensor product of the diagonalization maps $\triangle_{\wedge F} : \wedge^{\lambda_i - \mu_i} F \rightarrow (\wedge^{\alpha_{i1}} F \otimes \cdots \otimes \wedge^{\alpha_{it}} F)$ $(i = 1, \cdots, s)$, the second map is the tensor product of the canonical isomorphisms $\wedge^{\alpha_{ij}} F \rightarrow S_{\alpha_{ij}} F$ $(\alpha_{ij} = 0$ or $1)$, the third map is the tensor product of the commutation maps, and the fourth map is multiplication. Similarly, we define the map $d'_{\lambda/\mu}(F) : D_{\lambda/\mu} F \longrightarrow \wedge_{\tilde{\lambda}/\tilde{\mu}} F$.

DEFINITION 2.3. $\mathrm{Im}(d_{\lambda/\mu}(F))$ (resp. $\mathrm{Im}(d'_{\lambda/\mu}(F))$) is denoted by $L_{\lambda/\mu}F$ (resp. $K_{\lambda/\mu}F$). $L_{\lambda/\mu}$ (resp. $K_{\lambda/\mu}$) is called Schur (resp. coSchur) functor with respect to the skew shape λ/μ.

Note that if R=$\mathbb{Q}$, then $K_{\lambda/\mu}F$ is isomorphic to $L_{\tilde{\lambda}/\tilde{\mu}}F$ as a GL(F)-module, and is irreducible if $\mu = (0)$.

THEOREM 2.4 [2]. (The Standard Basis Theorem for Schur Functor) *Let $\lambda = (\lambda_1, \cdots, \lambda_s)$, $\mu = (\mu_1, \cdots, \mu_s)$ be partitions such that $\mu \subseteq \lambda$, and let F be a free modules with ordered basis $X = \{x_1, \cdots, x_m\}$. Then $\{d_{\lambda/\mu}(X_T) | T$ is a standard tableau in $Tab_{\lambda/\mu}(X)\}$ is a free basis for $L_{\lambda/\mu}F$, and the map $\theta_{\lambda/\mu} : \bar{L}_{\lambda/\mu}F \longrightarrow L_{\lambda/\mu}F$ is an isomorphism. Hence $L_{\lambda/\mu}F$ is universally free.*

The definition of Schur complex is quite similar to that of Schur functor. Given $\phi \in \mathrm{Hom}(G, F)$ we denote by c_ϕ the element of $F \otimes G^*$ corresponding to ϕ under the canonical isomorphism $\mathrm{Hom}_R(G, F) \cong F \otimes G^*$.

DEFINITION 2.5. (1) The symmetric algebra $S\phi$ of the morphism ϕ is the R-bialgebra $SF \otimes \wedge G$ formed by taking the usual tensor product of the R-bialgebras SF and $\wedge G$. We let $m_{S\phi} : S\phi \otimes S\phi \rightarrow S\phi$, $\triangle_{S\phi} : S\phi \rightarrow S\phi \otimes S\phi$, and $T_{S\phi} : S\phi \otimes S\phi \rightarrow S\phi \otimes S\phi$ denote the multiplication, the comultiplication, and the commutation map of the R-bialgebra $S\phi$, respectively.

(2) We put a complex structure on $S\phi$ as follows: let $(S\phi)_j = \sum_{i=0}^{\infty} S_i F \otimes \wedge^j G$ be the j-th degree of the complex and we let $\partial_{S\phi} : (S\phi)_j \rightarrow (S\phi)_{j-1}$ be the R-map given by the action of $c_\phi \in SF \otimes \wedge G^*$ on $SF \otimes \wedge G$.

LEMMA 2.6. *$m_{S\phi}, \triangle_{S\phi}, T_{S\phi}$ are all compatible with the differential $\partial_{S\phi}$.*

DEFINITION 2.7. $S_k\phi$ is the subcomplex of $S\phi$ given by

$$0 \rightarrow \wedge^k G \rightarrow F \otimes \wedge^{k-1} G \rightarrow \cdots \rightarrow S_{k-j} F \otimes \wedge^j G \rightarrow \cdots \rightarrow S_k F \rightarrow 0$$

where the j-th degree component $(S_k\phi)_j$ is $S_{k-j}F \otimes \wedge^j G$.

Note that $S_0\phi$ is the complex $0 \to R \to 0$ with $(S_0\phi)_0 = R$. If ϕ is the map $0 \to F$ then $S_k\phi$ is the complex $0 \to S_kF \to 0$ where $(S_k\phi)_0 = S_kF$ and if ϕ is the map $G \to 0$ then $S_k\phi$ is the complex $0 \to \wedge^k G \to 0$ where $(S_k\phi)_k = \wedge^k G$.

DEFINITION 2.8. (1) The exterior algebra $\wedge\phi$ of the morphism ϕ is the R-bialgebra $\wedge F \dot{\otimes} DG$ formed by taking the antisymmetric tensor product of the R-bialgebras $\wedge F$ and DG. We let $m_{\wedge\phi} : \wedge\phi \otimes \wedge\phi \to \wedge\phi$, $\triangle_{\wedge\phi} : \wedge\phi \to \wedge\phi \otimes \wedge\phi$, and $T_{\wedge\phi} : \wedge\phi \otimes \wedge\phi \to \wedge\phi \otimes \wedge\phi$ denote the multiplication, the comultiplication, and the commutation map of the R-bialgebra $\wedge\phi$ respectively.

(2) We put a complex structure on $\wedge\phi$ as follows : let $(\wedge\phi)_j = \sum_{i=0}^{\infty} \wedge^i F \otimes D_j G$ be the j-th degree component of the complex and $\partial_{\wedge\phi} : (\wedge\phi)_j \to (\wedge\phi)_{j-1}$ be the R-map given by the action of $c_\phi \in \wedge F \dot{\otimes} SG^*$ on $\wedge F \otimes DG$.

LEMMA 2.9. *$m_{\wedge\phi}, \triangle_{\wedge\phi}, T_{\wedge\phi}$ are all compatible with the differential $\partial_{\wedge\phi}$.*

DEFINITION 2.10. $\wedge^k\phi$ is the subcomplex of $\wedge\phi$ given by

$$0 \to D_kG \to F \otimes D_{k-1}G \to \cdots \to \wedge^{k-j}F \otimes D_jG \to \cdots \to \wedge^k F \to 0$$

where the j-th degree component $(\wedge^k\phi)_j$ is $\wedge^{k-j}F \otimes D_jG$.

Note that $\wedge^0\phi$ is the complex $0 \to R \to 0$ with $(\wedge^0\phi)_0 = R$. If ϕ is the map $0 \to F$ then $\wedge^k\phi$ is the complex $0 \to \wedge^k F \to 0$ where $(\wedge^k\phi)_0 = \wedge^k F$ and if ϕ is the map $G \to 0$ then $\wedge^k\phi$ is the complex $0 \to D_kG \to 0$ where $(\wedge^k\phi)_k = D_kG$. It is easy to see that $\wedge\phi = \sum_{k\geq 0} \wedge^k\phi, S\phi = \sum_{k\geq 0} S_k\phi$ as direct sums of complexes.

DEFINITION 2.11. Let $\lambda = (\lambda_1, \cdots, \lambda_s)$, $\mu = (\mu_1, \cdots, \mu_s)$ be partitions such that $\mu \subseteq \lambda$. We define $S_{\lambda/\mu}\phi$ and $\wedge_{\lambda/\mu}\phi$ as follows:

$$S_{\lambda/\mu}\phi = S_{\lambda_1-\mu_1}\phi \otimes \cdots \otimes S_{\lambda_s-\mu_s}\phi,$$
$$\wedge_{\lambda/\mu}\phi = \wedge^{\lambda_1-\mu_1}\phi \otimes \cdots \otimes \wedge^{\lambda_s-\mu_s}\phi.$$

Let $\lambda = (\lambda_1, \cdots, \lambda_s)$, $\mu = (\mu_1, \cdots, \mu_s)$ be partitions such that $\mu \subseteq \lambda$. Let $\tilde{\lambda} = (\tilde{\lambda}_1, \cdots, \tilde{\lambda}_t)$, $\tilde{\mu} = (\tilde{\mu}_1, \cdots, \tilde{\mu}_t)$ be their transposes. Let (α_{ij})

be the $s \times t$ matrix defined by

$$\alpha_{ij} = \begin{cases} 1 & \text{if } \mu_i + 1 \le j \le \lambda_i \\ 0 & \text{otherwise.} \end{cases}$$

We define the Schur map $d_{\lambda/\mu}(\phi) : \wedge_{\lambda/\mu}\phi \longrightarrow S_{\tilde{\lambda}/\tilde{\mu}}\phi$ to be the composition

$$\begin{aligned} &\wedge^{\lambda_1-\mu_1}\phi \otimes \cdots \otimes \wedge^{\lambda_s-\mu_s}\phi \\ \longrightarrow &(\wedge^{\alpha_{11}}\phi \otimes \cdots \otimes \wedge^{\alpha_{1t}}\phi) \otimes \cdots \otimes (\wedge^{\alpha_{s1}}\phi \otimes \cdots \otimes \wedge^{\alpha_{st}}\phi) \\ \longrightarrow &(S_{\alpha_{11}}\phi \otimes \cdots \otimes S_{\alpha_{1t}}\phi) \otimes \cdots \otimes (S_{\alpha_{s1}}\phi \otimes \cdots \otimes S_{\alpha_{st}}\phi) \\ \longrightarrow &(S_{\alpha_{11}}\phi \otimes \cdots \otimes S_{\alpha_{s1}}\phi) \otimes \cdots \otimes (S_{\alpha_{1t}}\phi \otimes \cdots \otimes S_{\alpha_{st}}\phi) \\ \longrightarrow &S_{\tilde{\lambda}_1-\tilde{\mu}_1}\phi \otimes \cdots \otimes S_{\tilde{\lambda}_t-\tilde{\mu}_t}\phi, \end{aligned}$$

where the first map is the tensor product of the maps $\triangle_{\wedge\phi} : \wedge^{\lambda_i-\mu_i}\phi \to (\wedge^{\alpha_{i1}}\phi \otimes \cdots \otimes \wedge^{\alpha_{it}}\phi)$ $(i = 1, \cdots, s)$, the second map is the tensor product of the canonical isomorphisms $\wedge^{\alpha_{ij}}\phi \to S_{\alpha_{ij}}\phi$ $(\alpha_{ij} = 0$ or $1)$, the third map is the tensor product of the commutation maps, and the fourth map is multiplication in the algebra $\otimes^t S\phi = S\phi \otimes \cdots \otimes S\phi$.

DEFINITION 2.12. $\mathrm{Im}(d_{\lambda/\mu}(\phi))$ is denoted by $L_{\lambda/\mu}\phi$. $L_{\lambda/\mu}\phi$ is called the Schur complex on the morphism ϕ.

REMARK 2.13. (1) If $G = 0$ then $L_{\lambda/\mu}\phi = L_{\lambda/\mu}F$ in degree zero.
(2) If $F = 0$ then $L_{\lambda/\mu}\phi = K_{\lambda/\mu}G$ in degree $|\lambda| - |\mu|$.
(3) If $\lambda = (\lambda_1)$ and $\mu = 0$ then $L_{\lambda/\mu}\phi = \wedge^{\lambda_1}\phi$.
(4) If $\lambda = (\underbrace{1, \cdots, 1}_{q})$ and $\mu = 0$ then $L_{\lambda/\mu}\phi = S_q\phi$.

THEOREM 2.14 [2]. (The Standard Basis Theorem for Schur Complexes) *Let $\lambda = (\lambda_1, \cdots, \lambda_s)$, $\mu = (\mu_1, \cdots, \mu_s)$ be partitions such that $\mu \subseteq \lambda$, and let $\phi : G \to F$ be a map of free modules. Let $Y=\{y_1, \cdots, y_n\}$, $X=\{x_1, \cdots, x_m\}$ be bases for G and F, and let $S = Y \cup X$ be totally ordered so that the orders of X and Y are preserved. Then $\{d_{\lambda/\mu}(Z_T) | T$ is a standard tableau mod Y in $\mathrm{Tab}_{\lambda/\mu}(S)\}$ is a free basis for $L_{\lambda/\mu}\phi$, and the map $\theta_{\lambda/\mu} : \bar{L}_{\lambda/\mu}\phi \longrightarrow L_{\lambda/\mu}\phi$ is an isomorphism. Hence $L_{\lambda/\mu}\phi$ is universally free.*

COROLLARY 2.15 [2]. *Let $\phi : G \to F$ be a split injection. Then $L_{\lambda/\mu}\phi$ is acyclic and $H_0(L_{\lambda/\mu}\phi) = L_{\lambda/\mu}(\mathrm{coker}\phi)$.*

COROLLARY 2.16 [2]. *If $\phi = \phi_1 \oplus \phi_2$ and ϕ_1 is an isomorphism, then $L_{\lambda/\mu}\phi$ is homotopically equivalent to $L_{\lambda/\mu}\phi_2$.*

3. On the homology of $L_{\lambda/\mu}\phi$

In this chapter, R is a commutative Noetherian ring with unity. Let $\phi : G \to F$ be an R-module homomorphism between free R-modules of ranks n and m, respectively, and let $\lambda = (\lambda_1, \lambda_2, \cdots)$, $\mu = (\mu_1, \mu_2, \cdots)$ be partitions with $\mu \subseteq \lambda$. In this chapter, we give an upper bound for the degrees of the nonvanishing homology modules of the complex $L_{\lambda/\mu}\phi$ in terms of the depths of the determinantal ideals of ϕ.

DEFINITION 3.1. Let $\lambda = (\lambda_1, \lambda_2, \cdots)$, $\mu = (\mu_1, \mu_2, \cdots)$ be partitions with $\mu \subseteq \lambda$. We define, for $j = 1, \cdots$,

$$k_j = \sum_{i=1}^{\lambda_1} w_i,$$

$$w_i = \begin{cases} (\tilde{\lambda}_i - \tilde{\mu}_i) - j & \text{if } \tilde{\lambda}_i - \tilde{\mu}_i \geqslant j \\ 0 & \text{otherwise.} \end{cases}$$

For example, let $\lambda = (5, 4, 3, 2, 1)$ and $\mu = (2, 1)$. Then $k_1 = 7$, $k_2 = 3$, $k_i = 0$ for all i $(i \geq 3)$. Note that if $\mu = (0)$, then $k_{n-1} - k_n = \lambda_n$.

THEOREM 3.2. *Let $\lambda = (\lambda_1, \lambda_2, \cdots)$ and $\mu = (\mu_1, \mu_2, \cdots)$ be partitions with $\mu \subseteq \lambda$ and let $t_{\lambda/\mu} = (t_1, t_2, \cdots)$ be a sequence of nonnegative integers satisfying*

$$t_j - t_{j-1} \leqslant k_{j-1} - k_j$$

for $j = 2, \cdots$. If $\phi : G \to F$ is an R-module homomorphism between free R-modules G and F with $n = \operatorname{rank} G \leqslant \operatorname{rank} F = m$ and

$$\operatorname{depth} I_j\phi \geq t_{n-j+1}$$

for $j = 1, 2, \cdots, n$, then

$$H_*(L_{\lambda/\mu}\phi) = 0$$

whenever $* > (|\lambda| - |\mu| - k_n) - t_n$.

PROOF. We proceed by induction on rank$G = n$. If $n = 0$, then clearly $L_{\lambda/\mu}\phi$ is acyclic. Suppose $n \geq 1$. By the Standard Basis Theorem for Schur Complexes, the length of $L_{\lambda/\mu}\phi$ is at most $|\lambda| - |\mu| - k_n$. By the acyclicity lemma [4], it is sufficient to prove that the homology of the localized complex $L_{\lambda/\mu}\phi \otimes R_p$ vanishes in degree $> (|\lambda| - |\mu| - k_n) - t_n$ at each prime ideal P satisfying depth $PR_p < t_n$. But such a prime P cannot contain the ideal $I_1\phi$ by the assumptions on depth $I_j\phi$. Hence $I_1\phi$ blows up in R_p, and over R_p we have $\phi = \phi' \oplus 1$, where ϕ' is a homomorphism between a free R_p-module F' with $\dim F' = m - 1$ and a free R_p-module G' with $\dim G' = n - 1$. If we prove that the depth conditions are preserved under localization, the conclusion will follow from the induction hypothesis applied to ϕ'. Observe that

$$\begin{aligned} \operatorname{depth} I_j\phi' &= \operatorname{depth} I_{j+1}\phi_p \\ &= \operatorname{depth} (I_{j+1}\phi)_p \\ &\geq \operatorname{depth} I_{j+1}\phi \\ &\geq t_{n-j} = t_{(n-1)-j+1} \end{aligned}$$

for $j = 1, 2, \cdots, n-1$. Hence

$$H_*(L_{\lambda/\mu}\phi') = 0 \quad \text{for } * > (|\lambda| - |\mu| - k_{n-1}) - t_{n-1}.$$

Since $k_{n-1} + t_{n-1} \geqslant k_n + t_n$, one has

$$H_*(L_{\lambda/\mu}\phi') = 0 \quad \text{for } * > (|\lambda| - |\mu| - k_n) - t_n.$$

Hence we are done because $L_{\lambda/\mu}\phi'$ and $L_{\lambda/\mu}\phi_p$ are homotopically equivalent. □

EXAMPLE 3.3. Let $\lambda = (5,3,3,1)$ and $\mu = (2,1)$ be partitions. Let $\phi : R^3 \to R^5$ be an R-module homomorphism defined by

$$\begin{bmatrix} x & 0 & 0 \\ y & x & 0 \\ z & y & x \\ 0 & z & y \\ 0 & 0 & z \end{bmatrix}.$$

Then $\operatorname{depth} I_1(\phi) = \operatorname{depth} I_2(\phi) = \operatorname{depth} I_3(\phi) = 3$ and $k_1 = 4$, $k_2 = 1$, $k_3 = 0$. Hence we choose $t_{\lambda/\mu}$ such that

$$\begin{aligned} t_1 &= \min(|\lambda| - |\mu| - k_1, \operatorname{depth} I_3\phi) = 3, \\ t_2 &= \min(t_1 + (k_1 - k_2), \operatorname{depth} I_2\phi) = 3, \\ t_3 &= \min(t_2 + (k_2 - k_3), \operatorname{depth} I_1\phi) = 3. \end{aligned}$$

Then

$$H_*(L_{\lambda/\mu}\phi) = 0 \quad \text{for } * > (|\lambda| - |\mu| - k_3) - t_3 = 6.$$

The following corollary in [12] can be proved by our Theorem 3.2.

COROLLARY 3.4. *Let $\lambda = (\lambda_1, \lambda_2, \cdots)$ be a partition and $t_\lambda = (t_1, t_2, \cdots)$ a sequence of nonnegative integers satisfying*

$$t_j - t_{j-1} \leq \lambda_j$$

for $j = 2, \cdots$.

If $\phi : G \to F$ is an R-module homomorphism between free R-modules G and F with $n = \text{rank}\, G \leq \text{rank}\, F = m$ and

$$depth\ I_j\phi \geq t_{n-j+1}$$

for $j = 1, 2, \cdots, n$, then

$$H_*(L_\lambda\phi) = 0$$

whenever $* > \lambda_1 + \cdots + \lambda_n - t_n$.

PROOF. Since $\mu = (0)$, $k_{j-1} - k_j = \lambda_j$ and $|\lambda| - |\mu| - k_n = \lambda_1 + \cdots + \lambda_n$. Hence we are done by Theorem 3.2. □

COROLLARY 3.5. *Let $\lambda = (\lambda_1, \lambda_2, \cdots)$ and $\mu = (\mu_1, \mu_2, \cdots)$ be partitions with $\mu \subseteq \lambda$ and let $t_{\lambda/\mu} = (t_1, t_2, \cdots)$ be a sequence of nonnegative integers satisfying*

$$t_j - t_{j-1} \leqslant k_{j-1} - k_j$$

for $j = 2, \cdots$. If $\phi : G \to F$ is an R-module homomorphism between free R-modules G and F with $n = \text{rank}\, G \leqslant \text{rank}\, F = m$ and

$$depth I_j\phi \geqslant t_{n-j+1}$$

for $j = 1, 2, \cdots, n$ then

(1) *supp $H_*(L_{\lambda/\mu}\phi) \subseteq V(I_n\phi)$ for all $* > 0$.*

(2) *For each $j = 1, 2, \cdots, n-1$,*

$$\text{supp } H_*(L_{\lambda/\mu}\phi) \subseteq V(I_j\phi)$$

whenever $\quad * > (|\lambda| - |\mu| - k_{n-j}) - t_{n-j}$.

PROOF. (1) Suppose that P is a prime ideal but $P \notin V(I_n\phi)$. Then $I_n\phi$ is invertible locally at P, and ϕ_p is a split injection. Hence $L_{\lambda/\mu}\phi_p$ is acyclic. The isomorphism

$$H_*(L_{\lambda/\mu}\phi_p) \cong H_*(L_{\lambda/\mu}\phi \otimes R_p) \cong H_*(L_{\lambda/\mu}\phi) \otimes R_p$$

implies that $P \notin \operatorname{supp} H_*(L_{\lambda/\mu}\phi)$ for all $* > 0$. (2) Suppose that P is a prime ideal but $P \notin V(I_j\phi)$ for $j = 1, \cdots, n-1$. Then $I_j\phi$ blows up in R_p, and over R_p we have $\phi_p = \phi' \oplus 1$, where ϕ' is a homomorphism between a free R_p-module F' with $\dim F' = m - j$ and a free R_p-module G' with $\dim G' = n - j$. If the depth conditions are satisfied for ϕ', we can apply Theorem 3.2 to ϕ'. Observe that

$$\begin{aligned} \operatorname{depth} I_s\phi' &= \operatorname{depth} I_{s+j}\phi_p \\ &\geqslant \operatorname{depth} I_{s+j}\phi \\ &\geqslant t_{n-(s+j)+1} \\ &= t_{(n-j)-s+1} \end{aligned}$$

for $s = 1, 2, \cdots, n-j$. Therefore

$$H_*(L_{\lambda/\mu}\phi') = 0 \quad \text{for } * > (|\lambda| - |\mu| - k_{n-j}) - t_{n-j}.$$

Since $L_{\lambda/\mu}\phi'$ and $L_{\lambda/\mu}\phi_p$ are homotopically equivalent,

$$H_*(L_{\lambda/\mu}\phi_p) = 0 \quad \text{for } * > (|\lambda| - |\mu| - k_{n-j}) - t_{n-j}.$$

Hence $P \notin \operatorname{supp} H_*(L_{\lambda/\mu}\phi)$ for $* > (|\lambda| - |\mu| - k_{n-j}) - t_{n-j}$. □

In order to maximize the value of t_n, we introduce the quantities $\tilde{t}_j$ and θ_j. We define

$$\begin{aligned} \tilde{t}_1 &= \min(|\lambda| - |\mu| - k_1, \operatorname{depth} I_n\phi), \\ \theta_1 &= (|\lambda| - |\mu| - k_1) - \tilde{t}_1. \end{aligned}$$

Inductively, we define, for $j = 2, \cdots$

$$\tilde{t}_j = \begin{cases} \tilde{t}_{j-1} + (k_{j-1} - k_j) & \text{if depth } I_{n-j+1}\phi \geq \tilde{t}_{j-1} + (k_{j-1} - k_j) \\ \text{depth } I_{n-j+1}\phi & \text{otherwise} \end{cases}$$

and

$$\theta_j = \begin{cases} 0 & \text{if depth } I_{n-j+1}\phi \geq \tilde{t}_{j-1} + (k_{j-1} - k_j) \\ \tilde{t}_{j-1} + (k_{j-1} - k_j) - \text{depth } I_{n-j+1}\phi & \text{otherwise.} \end{cases}$$

REMARK 3.6.

(1) If depth $I_n\phi \geq |\lambda| - |\mu| - k_1$ and depth $I_{n-j+1}\phi \geq \tilde{t}_{j-1} + (k_{j-1} - k_j)$ for all $j = 2, \cdots, n$, then $\tilde{t}_n = |\lambda| - |\mu| - k_n$. Hence $L_{\lambda/\mu}\phi$ is acyclic.

(2) If $\mu = (0)$, then $|\lambda| - |\mu| - k_n = \lambda_1 + \cdots + \lambda_n$. Hence if $\tilde{t}_n = \lambda_1 + \cdots + \lambda_n$, then $L_\lambda\phi$ is acyclic.

COROLLARY 3.7. *If $\phi : G \to F$ is an R-module homomorphism between free R-modules G and F with $n = \operatorname{rank} G \leqslant \operatorname{rank} F = m$ and let $\lambda = (\lambda_1, \cdots, \lambda_p)$ be a partition, where $\lambda_1 \leq m - n + 1$. If*

$$\operatorname{depth} I_j\phi \geq (m - n + 1)(n - j + 1)$$

for each $j = 1, \cdots, n$, then $L_\lambda\phi$ is acyclic.

PROOF. It is sufficient to show that $\tilde{t}_n = \lambda_1 + \cdots + \lambda_n$. If $j = 1$, then depth $I_n\phi \geq (m - n + 1) \geq \lambda_1$. Hence $\tilde{t}_1 = \lambda_1$. Accordingly, we assume the inductive hypothesis. Namely, $\tilde{t}_j = \lambda_1 + \cdots + \lambda_j$. By hypothesis,

$$\begin{aligned} \operatorname{depth} I_{n-j}\phi &\geq (m - n + 1)(j + 1) \\ &\geq (j + 1)\lambda_1 \\ &\geq \lambda_1 + \cdots + \lambda_{j+1}. \end{aligned}$$

Hence

$$\begin{aligned} \tilde{t}_{j+1} &= \min(\operatorname{depth} I_{n-j}\phi, \quad \tilde{t}_j + (k_j - k_{j+1})) \\ &= \min(\operatorname{depth} I_{n-j}\phi, \quad \lambda_1 + \cdots + \lambda_j + \lambda_{j+1}) \\ &= \lambda_1 + \cdots + \lambda_{j+1}. \end{aligned}$$

This complete the proof. □

COROLLARY 3.8. *Let $\lambda = (\lambda_1, \lambda_2, \cdots)$ and $\mu = (\mu_1, \mu_2, \cdots)$ be partitions with $\mu \subseteq \lambda$ and let $\phi : G \to F$ be an R-module homomorphism between free R-modules G and F with $n = \operatorname{rank} G \leqslant \operatorname{rank} F = m$. Then*

(1) *$H_*(L_{\lambda/\mu}\phi) = 0$ for $* > \theta_1 + \cdots + \theta_n$.*

(2) *supp $H_*(L_{\lambda/\mu}\phi) \subseteq V(I_n\phi)$ for $* > 0$.*

(3) *For each $j = 1, \cdots, n-1$, supp $H_*(L_{\lambda/\mu}\phi) \subseteq V(I_{n-j}\phi)$ whenever $* > \theta_1 + \cdots + \theta_j$.*

PROOF. (1) Observe that if we take $t_{\lambda/\mu} = (\tilde{t}_1, \tilde{t}_2, \cdots)$ then the hypothesis of Theorem 3.2 is satisfied. Hence it is sufficient to prove that

$$\theta_1 + \cdots + \theta_s \geq (|\lambda| - |\mu| - k_s) - \tilde{t}_s$$

for each $s = 1, \cdots, n$. We proceed by induction on s. If $s = 1$, then it is trivial. Suppose $s \geq 2$. Consider the following two cases;
Case 1.We assume that depth $I_{n-s+1} < \tilde{t}_{s-1} + (k_{s-1} - k_s)$. Then

$$\begin{aligned}\theta_1 + \cdots + \theta_s &\geq (|\lambda| - |\mu| - k_{s-1}) - \tilde{t}_{s-1} + \theta_s \\ &\geq (|\lambda| - |\mu| - k_{s-1}) - \tilde{t}_{s-1} + \tilde{t}_{s-1} \\ &\quad + (k_{s-1} - k_s) - \mathrm{depth} I_{n-s+1}\phi \\ &= (|\lambda| - |\mu| - k_s) - \mathrm{depth} I_{n-s+1}\phi \\ &= (|\lambda| - |\mu| - k_s) - \tilde{t}_s\end{aligned}$$

by induction hypothesis and $\tilde{t}_s = \mathrm{depth} I_{n-s+1}\phi$.
Case 2. Assume the contrary to Case 1. Then $\theta_s = 0$. Hence

$$\begin{aligned}\theta_1 + \cdots + \theta_s &= \theta_1 + \cdots + \theta_{s-1} \\ &\geq (|\lambda| - |\mu| - k_{s-1}) - \tilde{t}_{s-1} \\ &= (|\lambda| - |\mu| - k_{s-1}) - \{\tilde{t}_s - (k_{s-1} - k_s)\} \\ &= (|\lambda| - |\mu| - k_s) - \tilde{t}_s\end{aligned}$$

by induction hypothesis and $\tilde{t}_s = \tilde{t}_{s-1} + (k_{s-1} - k_s)$. (2) and (3) are proved by Theorem 3.2 and Corollary 3.5. □

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