Commun. Korean Math. Soc. **17** (2002), No. 3, pp. 383–387

POLYNOMIALS THAT GENERATE A ROW OF PASCAL'S TRIANGLE

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ABSTRACT. Let p be an odd prime, and let $f(x)$ be the interpolating polynomial associated with a table of data points $(j+1, \binom{p}{j})$ for $0 \le j \le p$. In this article, we find congruence identities modulo p of $(p-1)!\,f(x)$, $(p-2)!\,f(x)$, and $(p-3)!\,f(x)$. Moreover we present some conjectures of these types.

1. Introduction

Pascal's triangle was originally developed by the ancient Chinese, but Blaise Pascal was the first person who discovered all of the patterns it contained. Assume that Pascal's triangle starts from the zeroth row and the zeroth element of a row. In the row numbered n there appear the coefficients of the expansion of the polynomial $(x+1)^n$.

Pascal's triangle has many patterns. Some of them are as follows.

(1) The sum of the numbers in any row is equal to 2^n, where n is the number of the row.
(2) If the first element in a row is a prime number, all numbers in that row (excluding the $1'$s) are divisible by it.
(3) If a diagonal of numbers of any of the $1'$s bordering the sides of the triangle and ending on any number inside the selection is equal to the number below the end of the selection that is not on the same diagonal itself.

Besides above, there are many interesting patterns relating to so called magic $11'$s, Fibonacci's sequence, polygonal numbers. Also it is connected to Sierpinski's triangle. For the details, see [1].

Received July 18, 2000.
2000 Mathematics Subject Classification: Primary 11A07; Secondary 41A05.
Key words and phrases: Pascal's triangle, Lagrange interpolation.
This work was supported by the Brain Korea 21 Project.

In this article, we consider some congruence identities modulo an odd prime p of the interpolation formula from the p^{th} row of Pascal's triangle, i.e., the coefficients of $(x+1)^p$. In fact, let p be an odd prime and $f(x)$ the interpolating polynomial associated with a table of data points $(j+1, \binom{p}{j})$ for $0 \le j \le p$. Then we show in Section 2 that

$$(p-1)!\, f(x) \equiv x^{p-1} + x^{p-2} + \cdots + x \pmod{p}.$$

It easily follows that

$$\begin{aligned}(p-2)!\, f(x) &\equiv (p-1)\left(x^{p-1} + x^{p-2} + \cdots + x\right) \pmod{p},\\ (p-3)!\, f(x) &\equiv \frac{p+1}{2}\left(x^{p-1} + x^{p-2} + \cdots + x\right) \pmod{p}\end{aligned}$$

provided all coefficients of the left sides are rational integers. Moreover we present some conjectures of these types.

2. Main result

In this section, we show the following proposition.

PROPOSITION 1. *Let p be an odd prime and $f(x)$ the interpolating polynomial associated with a table of data points $(j+1, \binom{p}{j})$ for $0 \le j \le p$. Then we have*

$$(p-1)!\, f(x) \equiv x^{p-1} + x^{p-2} + \cdots + x \pmod{p}.$$

PROOF. Let p be an odd prime and $f(x)$ the interpolating polynomial associated with a table of data points $(j+1, \binom{p}{j})$ for $0 \le j \le p$. We use Lagrange interpolation. Then

$$\begin{aligned}f(x) &= \sum_{j=0}^{p} \Big(\prod_{\substack{i=0\\ i\neq j}}^{p} \frac{x-(i+1)}{j-i} \Big) \binom{p}{j}\\ &= \sum_{j=0}^{p} \Big(\frac{x-1}{j}\, \frac{x-2}{j-1} \cdots \frac{x-j}{1}\, \frac{x-(j+2)}{-1}\, \frac{x-(j+3)}{-2}\\ &\qquad \cdots \frac{x-(p+1)}{-(p-j)} \Big) \frac{p!}{j!(p-j)!}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{p} (-1)^{p-j} \frac{p\,!}{(j\,!(p-j)\,!)^2} \frac{(x-1)(x-2)\cdots(x-(p+1))}{x-(j+1)} \\
&= \frac{1}{p\,!} \sum_{j=0}^{p} (-1)^{j+1} \binom{p}{j}^2 \frac{(x-1)(x-2)\cdots(x-(p+1))}{x-(j+1)} \\
&= \frac{1}{p\,!} \Bigg((x-2)\cdots(x-p)(x-1-(x-(p+1))) \\
&\qquad + \sum_{j=1}^{\frac{p-1}{2}} (-1)^j (p-2j) \binom{p}{j}^2 R_j(x) \Bigg) \\
&= \frac{1}{(p-1)\,!} \Bigg((x-2)\cdots(x-p) \\
&\qquad + \sum_{j=1}^{\frac{p-1}{2}} (-1)^j \binom{p}{j}^2 R_j(x) - \frac{2}{p} \sum_{j=1}^{\frac{p-1}{2}} (-1)^j j \binom{p}{j}^2 R_j(x) \Bigg),
\end{aligned}
$$

where $R_j(x)$ is an integral polynomial depending on j. Hence

$$(p-1)\,!f(x) \equiv (x-2)(x-3)\cdots(x-p) \pmod{p}.$$

Since $x^p - x \equiv 0 \pmod{p}$ for all $0 \le x \le p-1$, we have

$$x(x-1)\cdots(x-(p-1)) \equiv x^p - x \pmod{p}$$

and

$$x(x-2)(x-3)\cdots(x-(p-1)) \equiv x^{p-1} + \cdots + x \pmod{p}.$$

Hence

$$(x-p)(x-2)(x-3)\cdots(x-(p-1)) \equiv x^{p-1} + \cdots + x \pmod{p},$$

which proves the result. □

The next easily follows from Proposition 1.

Corollary 2. *With the same assumptions of Proposition 1, we have*

$$(p-2)!\,f(x) \equiv (p-1)\left(x^{p-1}+x^{p-2}+\cdots+x\right) \pmod{p},$$
$$(p-3)!\,f(x) \equiv \frac{p+1}{2}\left(x^{p-1}+x^{p-2}+\cdots+x\right) \pmod{p}$$

provided all coefficients of the left sides are rational integers.

Proof. By Proposition 1,

$$\begin{aligned}(p-1)!\,f(x) &\equiv x^{p-1}+x^{p-2}+\cdots+x \pmod{p}\\ &\equiv (p-1)^2\left(x^{p-1}+x^{p-2}+\cdots+x\right) \pmod{p}.\end{aligned}$$

Hence we have

$$\begin{aligned}(p-2)!\,f(x) &\equiv (p-1)\left(x^{p-1}+x^{p-2}+\cdots+x\right) \pmod{p}\\ &\equiv (p-2)\frac{p+1}{2}\left(x^{p-1}+x^{p-2}+\cdots+x\right) \pmod{p}\end{aligned}$$

and

$$(p-3)!\,f(x) \equiv \frac{p+1}{2}\left(x^{p-1}+x^{p-2}+\cdots+x\right) \pmod{p}. \qquad \square$$

Remark 3. With the same assumptions of Proposition 1, for $k \geq 4$, it seems that $(p-k)!\,f(x) \equiv h\left(x^{p-1}+x^{p-2}+\cdots+x\right) \pmod{p}$ for some integer $h = h(p)$ in terms of p. Finding suitable h's remains a question.

Now we consider congruence identities in the case of modulo $n-1$. A number k is called hexagonal if $k = h(2h-1)$ for some $h \geq 1$. The author has investigated extensive interpolation formulas modulo $n-1$ from the coefficients of $(x+1)^n$ and formulated the conjecture below. But it remains an open problem.

Conjecture. *Let n be an integer ≥ 3 and $f(x)$ the interpolating polynomial associated with a table of data points $(j+1, \binom{n}{j})$ for $0 \leq j \leq n$. Then on modulo $n-1$ we have that $(n-1)!\,f(x)$ is congruent to 0 if $n = 2k+1$ and k is hexagonal, and $x^2((2(k-1)+4x^{k-1}+2(k-1)x^{2(k-1)})$ if $n = 2k+1$ and k is odd prime, respectively.*

REMARK 4. One might think that there are "better" and "simpler" forms of congruence identities modulo an integer of the type of congruence identities in this article. However, due to author's search, not many interpolation formulas from the coefficients of $(x+1)^n$ by multiplying a constant have good and simple forms of congruence identities modulo an integer. We could only find congruence identities modulo an integer in this article having simple forms and regular patterns.

References

[1] V. A. Uspenskii, *Pascal's Triangle*, Moscow, 1979. (In Russian)

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