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KÄHLER SUBMANIFOLDS WITH LOWER BOUNDED TOTALLY REAL BISECTIONAL CURVATURE TENSOR II

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ABSTRACT. In this paper, we prove that if every totally real bisectional curvature of an $n(\geqq 3)$-dimensional complete Kähler submanifold of a complex projective space of constant holomorphic sectional curvature c is greater than $\frac{c}{6n(n+1)}(3n^2+2n-2)$, then it is totally geodesic and compact.

1. Introduction

The theory of Kähler submanifolds is one of fruitful fields in Riemannian geometry. For the curvatures of a Kähler manifold M, we can consider two kinds of sectional curvatures which are related to almost complex structure J and different from usual sectional curvatures, holomorphic sectional curvatures and totally real bisectional curvatures. The pinching problem for these three kinds, the sectional curvature, the holomorphic sectional curvature and the totally real bisectional curvature, is an interesting topic in Kähler geometry.

For a complex submanifold $M = M^n$ of a complex space form $M' = M^{n+p}(c)$, the set $B(M)$ of totally real bisectional curvatures satisfies $B(M) \leqq \frac{c}{2}$ by the Gauss equation. It is easily seen that a totally geodesic complex submanifold $M = M^n(c)$ of $M' = M^{n+p}(c)$ satisfies $B(M) = \frac{c}{2}$ again by the Gauss equation. On the other hand, a complex quadric $M = Q^n$ of $M' = M^{n+p}(c)$, $c > 0$, satisfies $0 \leqq B(M) \leqq \frac{c}{2}$ by Kobayashi

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and Nomizu [6]. By paying attention to this fact, the following theorem was proved by Ki and Suh [5] for totally real bisectional curvatures.

THEOREM A. *Let $M = M^n$ be an $n(\geqq 3)$-dimensional complete Kähler submanifold of an $(n+p)$-dimensional Kähler manifold $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(>0)$. If $a(M) > a_1$, then M is totally geodesic, where*

$$a_1 = \frac{c}{2n(n^2+2^n+3)}(n^3+2n^2+2n-2)$$

and $a(M)$ is the infimum of the set $B(M)$.

The purpose of this paper is to prove the following theorem for an improvement on the above estimation.

THEOREM. *Let $M = M^n$ be an $n(\geqq 3)$-dimensional complete Kähler submanifold of an $(n+p)$-dimensional Kähler manifold $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(>0)$. Then there exists a constant a_2 depending only upon n and c so that if $a(M) > a_2$, then M is totally geodesic, where $a_2 < a_1$.*

2. Kähler manifolds

This section is concerned with recalling basic formulas on Kähler manifolds. Let M be a complex $n(\geqq 2)$-dimensional Kähler manifold equipped with Kähler metric tensor g and almost complex structure J. We can choose a local field $\{E_\alpha\} = \{E_j, E_{j^*}\} = \{E_1, \cdots, E_n, E_{1^*}, \cdots, E_{n^*}\}$ of orthonormal frames on a neighborhood of M, where $E_{j^*} = JE_j$ and $j^* = n+j$. Here and in the sequel, the Latin small indices $i, j, \cdots$ run from 1 to n and the small Greek indices $\alpha, \beta, \cdots$ run from 1 to $2n = n^*$. We set $U_j = \frac{1}{\sqrt{2}}(E_j - iE_{j^*})$ and $\bar{U}_j = \frac{1}{\sqrt{2}}(E_j + iE_{j^*})$, where i denotes the imaginary unit. Then $\{U_j\}$ constitutes a local field of unitary frames on the neighborhood of M. With respect to the Kähler metric, we have $g(U_j, \bar{U}_k) = \delta_{jk}$.

Now let $\{\omega_j\}$ be the canonical form with respect to the local field $\{U_j\}$ of unitary frames on the neighborhood of M. Then $\{\omega_j\} = \{\omega_1, \cdots, \omega_n\}$ consists of complex valued 1-forms of type (1,0) on M such that $\omega_j(U_k) = \delta_{jk}$ and $\omega_1, \cdots, \omega_n, \bar{\omega}_1, \cdots, \bar{\omega}_n$ are linearly independent. The Kähler metric g of M can be expressed as $g = 2\sum_j \omega_j \otimes \bar{\omega}_j$. Associated with the frame field $\{U_j\}$, there exist complex-valued 1-forms

$\omega_{jk,}$ which are usually called *complex connection forms* on M such that they satisfy the structure equations of M

$$
\begin{aligned}
&d\omega_i + \sum_k \omega_{ik} \wedge \omega_k = 0, \qquad \omega_{ij} + \bar{\omega}_{ji} = 0, \\
(2.1) \qquad &d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \\
&\Omega_{ij} = \sum_{k,l} K_{\bar{i}jk\bar{l}}\, \omega_k \wedge \bar{\omega}_l,
\end{aligned}
$$

where Ω_{ij} (resp. $K_{\bar{i}jk\bar{l}}$) the curvature form (resp. the components of the Riemannian curvature tensor R) of M. From the structure equations, the components of the curvature tensor satisfy

$$
\begin{aligned}
(2.2) \qquad &K_{\bar{i}jk\bar{l}} = \bar{K}_{\bar{j}il\bar{k}}, \\
&K_{\bar{i}jk\bar{l}} = K_{\bar{i}kj\bar{l}} = K_{\bar{l}jk\bar{i}} = K_{\bar{l}kj\bar{i}}.
\end{aligned}
$$

For a local field $\{E_\alpha\} = \{E_j, E_{j^*}\} = \{E_1, \cdots, E_n, E_{1^*}, \cdots, E_{n^*}\}$ of orthonormal frame on a neighborhood of M, we denote by $R_{\alpha\beta\gamma\delta}$ the components of the Riemannian curvature tensor R. Then we have

$$(2.3) \qquad K_{\bar{i}jk\bar{l}} = -\{(R_{ijkl} + R_{i^*jk^*l}) + i(R_{i^*jkl} - R_{ijk^*l})\}.$$

Next, relative to the frame field chosen above, the Ricci tensor S of M can be expressed as follows:

$$(2.4) \qquad S = \sum_{i,j} (S_{i\bar{j}}\omega_i \otimes \bar{\omega}_j + S_{\bar{i}j}\bar{\omega}_i \otimes \omega_j),$$

where $S_{i\bar{j}} = \sum_k K_{\bar{k}ki\bar{j}} = S_{\bar{j}i} = \bar{S}_{\bar{i}j}$. The scalar curvature r of M is also given by

$$(2.5) \qquad r = 2\sum_j S_{j\bar{j}}.$$

An n-dimensional Kähler manifold M is said to be *Einstein*, if the Ricci tensor S satisfies the condition

$$(2.6) \qquad S_{i\bar{j}} = \frac{r}{2n}\delta_{ij}.$$

The components $K_{\bar{i}jk\bar{l}m}$ and $K_{\bar{i}jk\bar{l}\bar{m}}$ (resp. $S_{i\bar{j}k}$ and $S_{i\bar{j}\bar{k}}$) of the covariant derivative of the Riemannian curvature tensor R (resp. the Ricci tensor S) are given by

$$\begin{aligned}
&\sum_m (K_{\bar{i}jk\bar{l}m}\omega_m + K_{\bar{i}jk\bar{l}\bar{m}}\bar{\omega}_m) \\
&= dK_{\bar{i}jk\bar{l}} - \sum_m (K_{\bar{m}jk\bar{l}}\bar{\omega}_{mi} + K_{\bar{i}mk\bar{l}}\omega_{mj} \\
&\qquad + K_{\bar{i}jm\bar{l}}\omega_{mk} + K_{\bar{i}jk\bar{m}}\bar{\omega}_{ml}),
\end{aligned} \tag{2.7}$$

$$\sum_k (S_{i\bar{j}k}\omega_k + S_{i\bar{j}\bar{k}}\bar{\omega}_k) = dS_{i\bar{j}} - \sum_k (S_{k\bar{j}}\omega_{ki} + S_{i\bar{k}}\bar{\omega}_{kj}). \tag{2.8}$$

The second Bianchi identity is given as follows:

$$K_{\bar{i}jk\bar{l}m} = K_{\bar{i}jm\bar{l}k}. \tag{2.9}$$

And hence we have

$$S_{i\bar{j}k} = S_{k\bar{j}i} = \sum_m K_{\bar{j}ik\bar{m}m}. \tag{2.10}$$

A Kähler manifold of constant holomorphic sectional curvature is called a *complex space form*. The components $K_{\bar{i}jk\bar{l}}$ of the Riemannian curvature tensor R of an n-dimensional complex space form of constant holomorphic sectional curvature c is given by

$$K_{\bar{i}jk\bar{l}} = \frac{c}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}). \tag{2.11}$$

Now let (M, g) be an n-dimensional Kähler manifold with almost complex structure J.

DEFINITION 2.1. For a totally real plane section $P = [X, Y]$ spanned by orthonormal vectors X and Y, the *totally real bisectional curvature* $B(X, Y)$ is defined by

$$B(X, Y) = g(R(X, JX)JY, Y). \tag{2.12}$$

Then, using the first Bianchi identity to (2.12) and the fundamental properties of the Riemannian curvature tensor of Kähler manifolds, we get

$$\begin{aligned} B(X,Y) &= g(R(X,Y)Y,X) + g(R(X,JY)JY,X) \\ &= K(X,Y) + K(X,JY), \end{aligned} \tag{2.13}$$

where $K(X,Y)$ means the sectional curvature of the plane spanned by X and Y.

In the rest of this section, we suppose that X and Y are orthonormal vectors in a non-degenerate totally real plane section. If $X' = \frac{1}{\sqrt{2}}(X+Y)$ and $Y' = \frac{1}{\sqrt{2}}(X-Y)$, then it is easily seen that we get

$$g(X',X') = g(Y',Y') = 1, \qquad g(X',Y') = 0.$$

Then we have

$$\begin{aligned} B(X',Y') &= g(R(X',JX')JY',Y') \\ &= \frac{1}{4}\{H(X) + H(Y) + 2B(X,Y) - 4K(X,JY)\}, \end{aligned}$$

where $H(X) = K(X,JX)$ means the holomorphic sectional curvature of the holomorphic plane spanned by X and JX. Hence we have

$$4B(X',Y') - 2B(X,Y) = H(X) + H(Y) - 4K(X,JY). \tag{2.14}$$

If we put $X'' = \frac{1}{\sqrt{2}}(X+JY)$ and $Y'' = \frac{1}{\sqrt{2}}(JX+Y)$, then we get

$$g(X'',X'') = g(Y'',Y'') = 1, \qquad g(X'',Y'') = 0.$$

Using the similar method as in (2.14), we have

$$4B(X'',Y'') - 2B(X,Y) = H(X) + H(Y) - 4K(X,Y). \tag{2.15}$$

Summing up (2.14) and (2.15) and taking account of (2.13), we obtain

$$2B(X',Y') + 2B(X'',Y'') = H(X) + H(Y). \tag{2.16}$$

Next, we calculate here the totally real bisectional curvatures of a Kähler manifold. Let $M = M^n$ be an $n(\geqq 3)$-dimensional complex submanifold of an $(n+p)$-dimensional Kähler manifold $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature c. Assume that the totally real bisectional curvatures on M is bounded from below by a constant a and let $a(M)$ be the infimum of the set $B(M)$ of the totally real bisectional curvatures on M. Then we see $a \leqq a(M)$. From (2.16), we have

$$H(X) + H(Y) \geqq 4a, \tag{2.17}$$

For an orthonormal frame field $\{E_1, \cdots, E_n, E_{1^*}, \cdots, E_{n^*}\}$, the holomorphic sectional curvature $H(E_j)$ for E_j can be expressed as

$$H(E_j) = g(R(E_j, JE_j)JE_j, E_j) = R_{jj^*j^*j} = K_{\bar{j}j j\bar{j}}, \tag{2.18}$$

where the last equality follows from (2.3). On the other hand, it is easily seen that the plane sections $P_j = [E_j, JE_j]$ and $P_k = [E_k, JE_k]$, $j \neq k$, are orthogonal and the totally real bisectional curvature $B(E_j, E_k)$ is given by

$$B(E_j, E_k) = g(R(E_j, JE_j)JE_k, E_k) = R_{jj^*k^*k} = K_{\bar{j}jk\bar{k}}, \quad j \neq k. \tag{2.19}$$

From the inequality (2.17) for $X = E_j$ and $Y = E_k$, we have

$$K_{\bar{j}jj\bar{j}} + K_{\bar{k}kk\bar{k}} \geqq 4a, \quad j \neq k. \tag{2.20}$$

Thus we have

$$\sum_{j<k}(K_{\bar{j}jj\bar{j}} + K_{\bar{k}kk\bar{k}}) \geqq 2an(n-1), \tag{2.21}$$

which implies that

$$\sum_j K_{\bar{j}jj\bar{j}} \geqq 2an, \tag{2.22}$$

where the equality holds if and only if $K_{\bar{j}jj\bar{j}} = 2a$ for any index j.

Since the scalar curvature r is given by

$$r = 2\sum_{j,k} K_{\bar{j}jk\bar{k}} = 2(\sum_j K_{\bar{j}jj\bar{j}} + \sum_{j\neq k} K_{\bar{j}jk\bar{k}}),$$

we have by the condition

$$r \geqq 2\sum_j K_{\bar{j}jj\bar{j}} + 2an(n-1),$$

from which it follows that

$$\sum_j K_{\bar{j}jj\bar{j}} \leqq \frac{r}{2} - an(n-1), \tag{2.23}$$

where the equality holds if and only if $K_{\bar{j}jk\bar{k}} = a$ for any distinct indices j and k. In this case, M is locally congruent to $M^n(a)$ due to Houh [4]. Also by (2.20) gives us

$$\sum_{k(\neq j)} (K_{\bar{j}jj\bar{j}} + K_{\bar{k}kk\bar{k}}) \geqq 4a(n-1)$$

for each j, so that

$$(n-2)K_{\bar{j}jj\bar{j}} + \sum_k K_{\bar{k}kk\bar{k}} \geqq 4a(n-1).$$

From this inequality together with (2.23), it follows that

$$(n-2)K_{\bar{j}jj\bar{j}} \geqq a(n-1)(n+4) - \frac{r}{2} \tag{2.24}$$

for any index j, so that the holomorphic sectional curvature $K_{\bar{j}jj\bar{j}}$ is bounded from below for $n \geqq 3$. Moreover, the equality holds for some index j if and only if M is locally congruent to $M^n(2a)$.

Since the Ricci curvature $S_{j\bar{j}}$ is given by

$$S_{j\bar{j}} = K_{\bar{j}jj\bar{j}} + \sum_{j(\neq k)} K_{\bar{j}jk\bar{k}},$$

we have by the assumption

$$S_{j\bar{j}} \geqq K_{\bar{j}jj\bar{j}} + a(n-1)$$

and hence by (2.24), we have

$$S_{j\bar{j}} \geqq \frac{1}{2(n-2)}\{4a(n-1)(n+1) - r\}. \tag{2.25}$$

3. Complex submanifolds

This section is recalled complex submanifolds of a Kähler manifold. First of all, the basic formulas for the theory of complex submanifolds are prepared.

Let $M' = M^{n+p}$ be an $(n+p)$-dimensional Kähler manifold with Kähler structure (g', J'). Let M be an n-dimensional complex submanifold of M' and let g be the induced Kähler metric tensor on M from g'. We can choose a local field $\{U_A\} = \{U_i, U_x\} = \{U_1, \cdots, U_{n+p}\}$ of unitary frames on a neighborhood of M' in such a way that, restricted to M, $U_1, \cdots, U_n$ are tangent to M and the others are normal to M. Here and in the sequel, the following convention on the range of indices is used throughout this paper, unless otherwise stated:

$$\begin{aligned} A, B, \cdots &= 1, \cdots, n, n+1, \cdots, n+p, \\ i, j, \cdots &= 1, \cdots, n, \\ x, y, \cdots &= n+1, \cdots, n+p. \end{aligned}$$

With respect to the frame field, let $\{\omega_A\} = \{\omega_i, \omega_x\}$ be its dual frame fields. Then the Kähler metric tensor g' of M' is given $g' = 2\sum_A \omega_A \otimes \bar{\omega}_A$. The canonical forms ω_A, the connection forms ω_{AB} of the ambient space M' satisfy the structure equations

$$\tag{3.1} \begin{aligned} & d\omega_A + \sum \omega_{AC} \wedge \omega_C = 0, \qquad \omega_{AB} + \bar{\omega}_{BA} = 0, \\ & d\omega_{AB} + \sum_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB}, \\ & \Omega'_{AB} = \sum_{C,D} K'_{\bar{A}BC\bar{D}} \omega_C \wedge \bar{\omega}_D, \end{aligned}$$

where Ω'_{AB} (resp. $K'_{\bar{A}BC\bar{D}}$) denotes the curvature form (resp. the components of the Riemannian curvature tensor R') of M'.

Restricting these forms to the submanifold M, we have

$$\tag{3.2} \omega_x = 0,$$

and the induced Kähler metric tensor g of M is given by $g = 2\sum_j \omega_j \otimes \bar{\omega}_j$. Then $\{U_j\}$ is a local unitary frame field with respect to the induced metric and $\{\omega_j\}$ is a local dual frame filed due to $\{U_j\}$, which consists of complex-valued 1-forms of type (1,0) on M. Moreover, $\omega_1, \cdots, \omega_n, \bar{\omega}_1,$

$\cdots, \bar{\omega}_n$ are linearly independent, and $\{\omega_j\}$ is the canonical forms on M. It follows from (3.2) and Cartan's lemma that the exterior derivatives of (3.2) give rise to

$$\omega_{xi} = \sum_j h^x_{ij}\omega_j, \qquad h^x_{ij} = h^x_{ji}. \tag{3.3}$$

The quadratic form $\alpha = \sum_{i,j,x} h^x_{ij}\omega_i \otimes \omega_j \otimes U_x$ with values in the normal bundle on M in M' is called the *second fundamental form* of the submanifold M. From the structure equations for M', it follows that the structure equations for M are similarly given by

$$\begin{aligned} &d\omega_i + \sum_k \omega_{ik} \wedge \omega_k = 0, \qquad \omega_{ij} + \bar{\omega}_{ji} = 0, \\ &d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \\ &\Omega_{ij} = \sum_{k,l} K_{\bar{i}jk\bar{l}}\omega_k \wedge \bar{\omega}_l. \end{aligned} \tag{3.4}$$

For the Riemannian curvature tensors R and R' of M and M', respectively, it follows from (3.1), (3.3) and (3.4) that

$$K_{\bar{i}jk\bar{l}} = K'_{\bar{i}jk\bar{l}} - \sum_x h^x_{jk}\bar{h}^x_{il}. \tag{3.5}$$

The components $S_{i\bar{j}}$ of the Ricci tensor S and the scalar curvature r on M are given by

$$S_{i\bar{j}} = \sum_k K'_{\bar{k}ki\bar{j}} - h_{i\bar{j}}{}^2, \tag{3.6}$$

$$r = 2(\sum_{j,k} K'_{\bar{k}kj\bar{j}} - h_2), \tag{3.7}$$

where $h_{i\bar{j}}{}^2 = h_{\bar{j}i}{}^2 = \sum_{m,x} h^x_{im}\bar{h}^x_{mj}$ and $h_2 = \sum_j h_{j\bar{j}}{}^2$.

Now the components h^x_{ijk} and $h^x_{ij\bar{k}}$ of the covariant derivative of the second fundamental form on M are given by

$$\begin{aligned} &\sum_k (h^x_{ijk}\omega_k + h^x_{ij\bar{k}}\bar{\omega}_k) \\ &= dh^x_{ij} - \sum_k (h^x_{jk}\omega_{ki} + h^x_{ik}\omega_{kj}) + \sum_y h^y_{ij}\omega_{xy}. \end{aligned} \tag{3.8}$$

Then, substituting dh^x_{ij} in this definition into the exterior derivative

$$d\omega_{xi} = \sum_j (dh^x_{ij} \wedge \omega_j + h^x_{ij} d\omega_j)$$

of (3.3) and using (3.1) $\sim$ (3.4) and (3.6), we have

$$h^x_{ijk} = h^x_{ikj}, \qquad h^x_{ij\bar{k}} = -K'_{\bar{x}ij\bar{k}}. \tag{3.9}$$

In particular, let the ambient space $M' = M^{n+p}(c)$ be an $(n+p)$-dimensional complex space form of constant holomorphic sectional curvature c. Then, by (2.11) and (3.5) $\sim$ (3.7), we get

$$K_{\bar{i}jk\bar{l}} = \frac{c}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) - \sum_x h^x_{jk}\bar{h}^x_{il}, \tag{3.10}$$

$$S_{i\bar{j}} = \frac{c}{2}(n+1)\delta_{ij} - h_{i\bar{j}}{}^2, \tag{3.11}$$

$$r = cn(n+1) - 2h_2, \tag{3.12}$$

$$h^x_{ij\bar{k}} = 0. \tag{3.13}$$

Next, let $M' = M^{n+p}$ be an $(n+p)$-dimensional Kähler manifold and let M be an n-dimensional complex submanifold of M'. Then the Laplacian of the squared norm h_2 of the second fundamental form α on M is given by Aiyama, Kwon and Nakagawa [1] as follows:

$$\Delta h_2 = 2\|\nabla\alpha\|_2 + c(n+2)h_2 - 4h_4 - 2Tr\ A^2, \tag{3.14}$$

where $h_4 = \sum_{i,j} h_{i\bar{j}}{}^2 h_{j\bar{i}}{}^2$ and A is a Hermitian matrix of order p with entry $A^x_y = \sum_{i,j} h^x_{ij}\bar{h}^y_{ij}$.

4. Totally real bisectional curvatures

Let M^n be an n-dimensional Kähler submanifold of an $(n+p)$-dimensional complex space form $M^{n+p}(c)$, $c > 0$. Let S and r be the Ricci tensor and the scalar curvature of M, respectively. The Ricci curvature of the complex quadric Q^n of $P^{n+1}(c)$ is equal to $\frac{c}{2}n$ and furthermore the set $B(M)$ is less than or equal to $\frac{c}{2}$ and if M is totally geodesic, then $B(M) = \frac{c}{2}$. Paying attention to this fact, we consider whether or not in the value distribution of $B(M)$ the maximal value is discrete.

At the beginning of this section, we shall consider the following a generalized maximum principle.

THEOREM 4.1. *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. Let f be a non-negative function on M satisfies*

$$\Delta f \geqq kf, \tag{4.1}$$

where k is a positive constant. If f is bounded, then f vanishes identically.

PROOF. Under the assumption of the theorem, we can apply the generalized maximum principle due to Omori [8] and Yau [11] for the function f bounded from above. So, for any positive number ϵ, there exists a point $\{p\}$ in M which satisfies the following properties:

$$\|\nabla f(p)\| < \epsilon, \qquad \Delta f(p) < \epsilon, \qquad (\sup\ f) - \epsilon < f(p).$$

Thus, for any positive sequence $\{\epsilon_m\}$ in such a way that the sequence converges to zero as m tends to infinity, there exists a point sequence $\{p_m\}$ in M which satisfies the following properties:

$$\|\nabla f(p_m)\| < \epsilon_m, \qquad \Delta f(p_m) < \epsilon_m, \qquad (\sup\ f) - \epsilon_m < f(p_m). \tag{4.2}$$

By (4.1) and the above property (4.2), we have

$$\epsilon_m > \Delta f(p_m) \geqq kf(p_m) > k((\sup\ f) - \epsilon_m),$$

which implies that we have $0 \geqq k \sup f$. It turns out to be $\sup f = 0$, because k is positive and f is non-negative. Accordingly, we see that the function f vanishes identically on M. It completes the proof. □

REMARK 4.1. We do not know whether or not Theorem 4.1 holds without the condition that the function is bounded from above. Although it may be the difficult problem, it seems to be very interesting to wrestle with the problem.

Let M be an $n(\geqq 3)$-dimensional Kähler submanifold of an $(n+p)$-dimensional complex space form $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature c. Then by the equation (3.5) of Gauss, we have

$$K_{\bar{j}jk\bar{k}} = \frac{c}{2} - \sum_x h^x_{jk}\bar{h}^x_{jk} \leqq \frac{c}{2}, \qquad j \neq k.$$

Thus we see that for any totally real plane section $[X, Y]$, the totally real bisectional curvature $B(X,Y)$ satisfies

$$B(X,Y) \leqq \frac{c}{2}.$$

Now let $a(M)$ be the infimum of the set $B(M)$ of totally real bisectional curvatures of M. Though the set B is bounded from above, we have no information on $a(M)$. In their paper [5], Ki and Suh proved the following

THEOREM 4.2. *Let $M = M^n$ be an $n(\geqq 3)$-dimensional complete Kähler submanifold of an $(n+p)$-dimensional Kähler manifold $M' = M^{n+p}(c)$ of constant holomorphic sectional curvature $c(>0)$. If $a(M) > a_1$, then M is totally geodesic, where*

$$a_1 = \frac{c}{2n(n^2+2n+3)}(n^3+2n^2+2n-2).$$

Since the matrix $H = (h_{j\bar{k}}{}^2)$ defined by $h_{j\bar{k}}{}^2 = \sum_{m,x} h^x_{jm}\bar{h}^x_{mk}$ and the matrix $A = (A^x_y)$ defined by $A^x_y = \sum_{j,k} h^x_{jk}\bar{h}^y_{jk}$ are both positive Hermitian ones, the eigenvalues λ_j of H and the eigenvalues λ_x of A are non-negative real valued functions on M. Thus it is easily seen that

$$\begin{aligned} &\sum_j \lambda_j = Tr\ H = h_2, \quad \sum_x \lambda_x = Tr\ A = -h_2, \\ &h_2{}^2 \geqq h_4 = \sum_j \lambda_j{}^2 \geqq \frac{1}{n} h_2{}^2, \\ &h_2{}^2 \geqq Tr\ A^2 = \sum_x \lambda_x{}^2 \geqq \frac{1}{p} h_2{}^2, \end{aligned} \tag{4.3}$$

where the second equality in the second relationship holds if and only if all eigenvalues of the matrix H are equal, and the second equality in the last relationship holds if and only if all eigenvalues of the matrix A are equal. It means that each equality holds if and only if the rank of matrices H and A are at most one.

The following result is proved.

THEOREM 4.3. *Let $M = M^n$ be an $n(\geqq 3)$-dimensional complete Kähler submanifold of an $(n+p)$-dimensional Kähler manifold $M' =$*

$M^{n+p}(c)$ of constant holomorphic sectional curvature $c(>0)$. Then there exists a constant a_2 depending only upon n and c so that if $a(M) > a_2$, then M is totally geodesic, where $a_2 < a_1$.

PROOF. By (3.14), we have

$$\Delta h_2 \geqq c(n+2)h_2 - 4h_4 - 2Tr\ A^2,$$

where the equality holds if and only if the second fundamental form α of M is parallel. Together the above equality with the above properties about eigenvalues (4.3), it follows that

$$\Delta h_2 \geqq c(n+2)h_2 - 6{h_2}^2,$$

where the equality holds if and only if the second fundamental form of M is parallel and the rank of the matrices H and A are at most one. A non-negative function f is defined by h_2. Then the above inequality is reduced to

$$\Delta f \geqq -6f^2 + c(n+2)f, \tag{4.4}$$

where the equality holds if and only if the second fundamental form of M is parallel and the rank of the matrices H and A are at most one. By (2.22), (2.23) and (3.12), we have

$$2na(M) \leqq \frac{c}{2}n(n+1) - h_2 - n(n-1)a(M).$$

This yields that

$$f = \sum_j \lambda_j = h_2 \leqq \frac{1}{2}n(n+1)(c - 2a(M)), \quad \lambda_j \geqq 0, \tag{4.5}$$

where the first equality holds if and only if $K_{\bar{j}jj\bar{j}} = 2a(M)$ and $K_{\bar{j}jk\bar{k}} = a(M)$ for any indices $j \neq k$. This means that each eigenvalue λ_j is bounded. On the other hand, since the Ricci curvature $S_{j\bar{j}}$ of M is given by (3.11) as

$$S_{j\bar{j}} = \frac{c}{2}(n+1) - \lambda_j,$$

it is also bounded. Applying the generalized maximum principle due to Omori [8] and Yau [11] to the bounded function f, we see that for any sequence $\{\epsilon_m\}$ of positive numbers which converges to 0 as m tends to infinity, there exists a point sequence $\{p_m\}$ such that

$$\|\nabla f(p_m)\| < \epsilon_m, \qquad \Delta f(p_m) < \epsilon_m, \qquad \sup f - \epsilon_m < f(p_m).$$

Thus, we have

$$\lim_{m\to\infty} \Delta f(p_m) \leqq \lim_{m\to\infty} \epsilon_m = 0, \qquad \lim_{m\to\infty} f(p_m) = \sup f. \tag{4.6}$$

By (4.4) and (4.6), we see

$$\sup f\{\sup f - \frac{c}{6}(n+2)\} \geqq 0,$$

which means that

$$\sup f = 0 \quad \text{or} \quad \sup f \geqq \frac{c}{6}(n+2).$$

If $\sup f = 0$, then f vanishes identically on M, because f is non-negative. Then M is totally geodesic.

Suppose that M is not totally geodesic. So, f satisfies *sup* $f \geqq \frac{c}{6}(n+2)$. On the other hand, by (4.5), we have

$$\sup f \leqq \frac{1}{2}n(n+1)(c - 2a(M)).$$

Thus, we see that

$$a(M) \leqq \frac{c}{6n(n+1)}(3n^2 + 2n - 2).$$

We denote the right hand side of the above inequality by a_2, which is the constant depending on n and c, which implies that if $a(M) > a_2$, then M is totally geodesic. It completes the proof. □

REMARK 4.2. By the straightforward calculation, we can easily show that $a_1 > a_2$. Since the estimation is rough and it is not the best possible, there may be room for further improvement. For the holomorphic pinching, Ros [9] determined the best possibility under the compact submanifolds. Under the condition $a(M) > a_2$, we have by (2.24) and (3.12)

$$\begin{aligned}(n-2)K_{\bar{j}j j\bar{j}} &\geqq (n-1)(n+4)a(M) - \frac{r}{2} \\ &= (n-1)(n+4)a(M) - \frac{c}{2}n(n+1) + h_2,\end{aligned}$$

and hence we have

$$K_{\bar{j}jj\bar{j}} > \frac{1}{2(n-2)}\{2(n-1)(n+4)a_2 - cn(n+1)\} > 0$$

for $n \geqq 3$. Therefore, M is compact because the holomorphic sectional curvatures of M are positive (see [2] and [10]).

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References

[1] R. Aiyama, J.-H. Kwon, and H. Nakagawa, *Complex submanifolds of an indefinite complex space form*, J. Ramanujan Math. Soc. **1** (1987), 43–67.
[2] R. L. Bishop and S. I. Goldberg, *Some implications of the generalized Gauss-Bonnet theorem*, Trans. Amer. Math. Soc. **112** (1964), 508–535.
[3] S. I. Goldberg and S. Kobayashi, *Holomorphic bisectional curvature*, J. Differential Geometry **1** (1967), 225–234.
[4] B. S. Houh, *On totally real bisectional curvature*, Proc. Amer. Math. Soc. **56** (1976), 261–263.
[5] U-H. Ki and Y. J. Suh, *On semi-Kaehler manifolds whose totally real bisectional curvature is bounded from below*, J. Kor. Math. Soc. **33** (1996), 1009–1038.
[6] S. Kobayashi and K. Nomizu, *Foundation of Differential Geometry, I and II*, Interscience Publ. 1963 and 1969.
[7] K. Ogiue, *Differential geometry of Kaehler submanifolds*, Advances in Math. **13** (1974), 73–114.
[8] H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), 205–214.
[9] A. Ros, *Kaehler submanifolds in the complex projective space*, Differential Geometry, Peñiscola 1985, 259–274, Lecture Notes in Math. 1209, Springer Berlin, 1986.
[10] Y. Tsukamoto, *On Kaehlerian manifolds with positive holomorphic sectional curvature*, Proc. Japan Acad. **33** (1957), 333–335.
[11] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure and Appl. Math. **28** (1975), 201–228.

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