

## Analysis of a Conducting Crack in an Electrostrictive Ceramic Under Combined Electric and Mechanical Loading

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A conducting crack in an electrostrictive ceramic under combined electric and mechanical loading is investigated. Analysis based on linear dielectric model predicts that the surfaces of the crack are not open completely but they are contact near the crack tip. The complete solution for the crack with a contact zone in a linear electrostrictive ceramic under combined electric and mechanical loading is obtained by using the complex variable formula. The asymptotic problems for a semi-infinite crack with a partial opening zone as well as for a fully open semi-infinite crack in a nonlinear electrostrictive ceramic are analyzed in order to investigate the effect of the electrical nonlinearity on the stress intensity factor under small scale nonlinear conditions. Particular attention is devoted to a finite crack in the nonlinear electrostrictive ceramic subjected to combined electric and mechanical loading. The stress intensity factor for the finite crack under small scale nonlinear conditions is obtained from the asymptotic analysis.

**Key Words :** Electrostrictive Ceramic, Conducting Crack, Contact, Electrical Nonlinearity, Asymptotic Analysis, Stress Intensity Factor

### 1. Introduction

Electrostrictive ceramics are the subject of intense research studies due to their wide engineering applications in advanced technological areas. An issue of extreme concern is their reliability (Winzer et al., 1989; Yang and Suo, 1994) in environments where complex physical phenomena take place. Typical examples are multilayer electromechanical devices, electronic components, and sensors and actuators embedded in intelligent structures. In these and other applications loading conditions of mechanical and electrical nature tend to produce high stresses and large deformations, which may cause deleterious processes and even catastrophic failure. Therefore, to prevent

failure during service and to secure the structural integrity of electromechanical devices using the materials, understanding of fracture behavior of electrostrictive ceramics is of great importance.

These issues regarding the effect of an electrical field on crack growth in electrostrictive ceramics have been raised in the literature (Yang and Suo, 1994; Hao et al., 1996; Ru et al., 1998). The electric field around inhomogeneities such as inclusions and internal electrodes is intense and nonuniform and the induced incompatible strain fields generate stresses. Stress concentrations at these inhomogeneities can lead to critical crack growth and subsequent mechanical failure. Therefore, the analysis of cracks in electrostrictive ceramics has attracted the attention of many researchers (Smith and Warren, 1966, 1968; McMeeking, 1987, 1989, 1990; Suo, 1993). Analysis based on linear dielectric model predicts that the stress field at the crack tip has physically inadmissible singularity. Furthermore, an electrical field does not induce any non-zero stress intensity factor for a conducting crack due to

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contact of the crack surfaces near the crack tip. However, there is a zone near the crack tip for which the linear electric solution fails due to material nonlinearity. The dielectric nonlinearity is most pronounced in the vicinity of the tip of a crack in electroceramics subjected to electric fields. The dielectric nonlinearity near the crack tip plays an important role in the behaviour of electric and elastic fields near the crack tip.

Recently, significant progress has been made in determining complete crack-tip fields for nonlinear electrostrictive ceramics. Beom (1999a, 1999b) showed that the dominant crack-tip field of the stress for a nonlinear electrostrictive ceramic has inverse square root singularity. He also verified from the nonlinear analysis that the surfaces of the conducting crack under electric loading are partially open in the vicinity of the crack tip due to the nonlinear zone. These analyses, however, only addressed the effects of electrical loading on the stress intensity factor for a crack in a nonlinear electrostrictive material. The effect of combined electrical and mechanical loading on the stress intensity factor has not been investigated as yet.

The purpose of this study is to investigate the problem of a crack with electrically conducting surfaces in an electrostrictive ceramic under combined electric and mechanical loading. Of particular interest is the effect of combined electrical and mechanical loading on the stress intensity factors for the conducting crack. The complete solution for the conducting crack is obtained by using the complex variable formula for a linear electrostrictive ceramic. The linear dielectric theory predicts that the surfaces near the tip of the conducting crack under combined electric and mechanical loading are closed. Due to the nonlinear zone near the crack tip, the surfaces of the conducting crack, however, are partially open in the vicinity of the crack tip. In order to investigate the effect of the electrical nonlinearity on the stress intensity factor under small scale nonlinear conditions, we consider the asymptotic problems for a semi-infinite crack with a partial opening zone as well as for a fully open semi-infinite crack in a nonlinear electrostrictive ceramic. The

stress intensity factors for the asymptotic problems are evaluated from the solution of the corresponding conducting sheet by applying superposition. For the small mechanical load applied at infinity, the size of the partial opening zone near the tip of the conducting crack under combined electric and mechanical loading is small. Sufficiently large mechanical loading applied on the external boundary opens the entire surface of the conducting crack, in contrast to the result obtained from the linear dielectric theory. The stress intensity factors for the conducting crack in the nonlinear electrostrictive ceramic subjected to combined electric and mechanical loading under small scale nonlinear conditions are obtained from the asymptotic analysis.

## 2. Linear Electromechanical Analysis

Consider a crack lying on the interval  $(-a, a)$  in an electrostrictive ceramic with strain quadratic in electric displacement under combined electric and mechanical loading, as shown in Fig. 1. Uniform electric fields  $E_1^\infty$  and  $E_2^\infty$  are prescribed at infinity. Constant stresses  $\sigma_{22}^\infty$  and

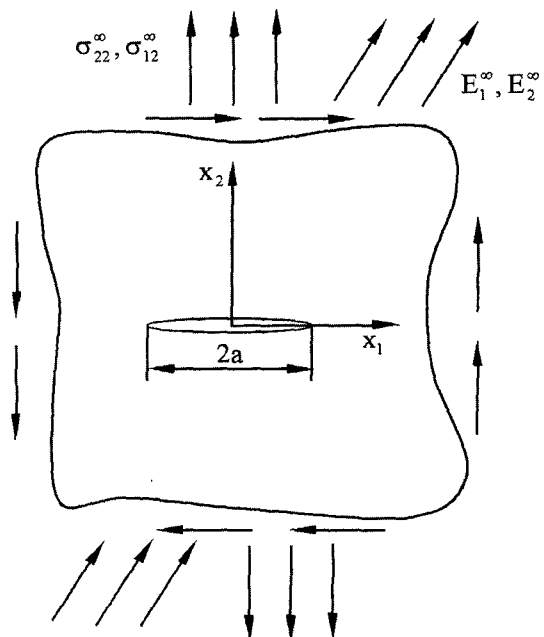


Fig. 1 Crack in an electrostrictive ceramic under combined electric and mechanical loading

$\sigma_{22}^\infty$  are applied at infinity. The surfaces of the crack are assumed to be traction free and electrically conducting. The two in-plane components of displacements and the electric potential are assumed to depend only on the in-plane Cartesian coordinates,  $x_1$  and  $x_2$ . Under the plane strain conditions, the constitutive laws for an isotropic electrostrictive ceramic are (Yang and Suo, 1994)

$$\begin{aligned} \gamma_{ij} &= \frac{1+\nu}{Y} (\sigma_{ij} - \nu \sigma_{kk} \delta_{ij}) \\ &+ Q(1+q) D_i D_j - Qq(1+\nu) D_k D_k \delta_{ij}, \quad (1) \\ D_i &= \epsilon E_i. \end{aligned}$$

Here  $\gamma_{ij}$ ,  $\sigma_{ij}$ ,  $D_i$  and  $E_i$  are the strain, the stress, the electric displacement and the electric field, respectively.  $Y$  and  $\nu$  are the Young's modulus and the Poisson's ratio, respectively,  $Q$  and  $q$  are the electrostrictive coefficients,  $\epsilon$  is the dielectric permittivity, and  $\delta_{ij}$  is the Kronecker delta. The repetition of an index in a term denotes a summation with respect to that index over its range 1 to 2 for a Roman letter lowercase.

General solutions for the electric field and the displacement field that satisfy the equilibrium equations for a homogeneous isotropic linear electrostrictive solid, and the corresponding stress components may be written in terms of three analytic functions as (Knops, 1963) :

$$\begin{aligned} -E_1 + iE_2 &= \Omega'(z), \\ 2G(u_1 + iu_2) &= (3-4\nu) \varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \\ &+ S\Omega(z)\overline{\Omega'(z)} \\ &+ \frac{4(1-\nu)}{\beta} S \int \overline{\Omega'(z)^2} dz, \quad (2) \\ \frac{\sigma_{22} + \sigma_{11}}{2} &= \varphi'(z) + \overline{\varphi'(z)} - S\Omega'(z)\overline{\Omega'(z)}, \\ \frac{\sigma_{22} - \sigma_{11}}{2} + i\sigma_{12} &= \overline{z}\varphi''(z) + \psi'(z) \\ &- S\Omega''(z)\overline{\Omega(z)}. \end{aligned}$$

Here  $u_i$  is the displacement and  $z = x_1 + ix_2$ .  $\varphi(z)$ ,  $\psi(z)$  and  $\Omega(z)$  are analytic functions, overbar ( $\bar{\phantom{x}}$ ) denotes the complex conjugate and prime ( $'$ ) implies the derivative with respect to the argument  $z$ .  $G$ ,  $S$  and  $\beta$  are constants given by

$$\begin{aligned} G &= \frac{Y}{2(1+\nu)}, \\ S &= \frac{1-(1+2\nu)q}{4} \frac{Y}{1-\nu^2} \epsilon^2 Q, \quad (3) \\ \beta &= \frac{2-2(1+2\nu)q}{1+q}. \end{aligned}$$

The complex variable formula enables us to formulate the boundary value problem in terms of the complex functions. The solution to a problem of the crack in the electrostrictive material is reduced to finding the functions  $\Omega(z)$ ,  $\varphi(z)$  and  $\psi(z)$ , which satisfy the boundary conditions of the problem. Once the three complex functions are determined, the complete electric and elastic fields are evaluated from Eqs. (1) and (2).

We consider separately the electric problem for the conducting crack in a linear electrostrictive ceramic under electric loading since the electric field for the linear electrostrictive problem does not depend on the elastic field. The electric boundary conditions on the conducting crack surface and at infinity are

$$\begin{aligned} E_1(x_1, 0) &= 0, \quad |x_1| < a, \\ E_1 &= E_1^\infty, \quad E_2 = E_2^\infty, \quad z \rightarrow \infty. \end{aligned} \quad (4)$$

The complex function generating the electric field for the problem can be written as (Smith and Warren, 1966, 1968)

$$\Omega(z) = -E_1^\infty \sqrt{z^2 - a^2} + iE_2^\infty z. \quad (5)$$

The intensity factor of electric field for a conducting crack is defined such that  $E_1 = K_1^E / \sqrt{2\pi r}$  on the prolongation of the crack at a distance  $r$  ahead of the crack tip. The intensity factor of electric field for a crack is evaluated from Eqs. (2) and (5), which yields

$$K_1^E = E_1^\infty \sqrt{\pi a}. \quad (6)$$

Next, we consider the elastic problem for the conducting crack in a linear electrostrictive ceramic under combined electric and mechanical loading. According to Beom (1999b), the traction free boundary conditions on the surfaces of the conducting crack under electric loading lead to overlapping of the conducting crack surfaces. The surfaces of the crack cannot be thus open completely but they are contact near the crack tip. We

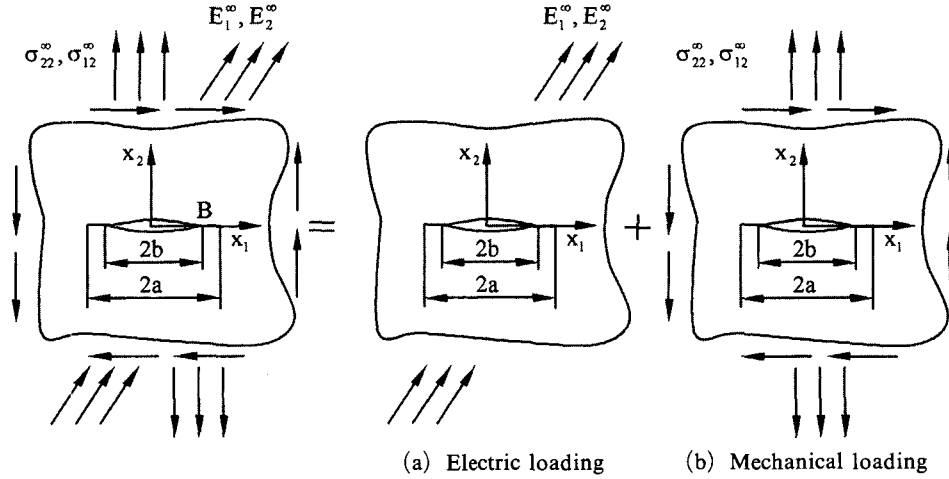


Fig. 2 Application of superposition to obtain the solution for a conducting crack with a contact zone

consider a crack with a contact zone as shown in Fig. 2. The mechanical boundary conditions on the surfaces of the crack with contact zone are

$$\begin{aligned}
 \sigma_{21}(x_1, 0) &= 0, \quad |x_1| < a, \\
 \sigma_{22}(x_1, 0) &= 0, \quad |x_1| < b, \\
 u_2(x_1, 0^+) &= u_2(x_1, 0^-), \\
 \sigma_{22}(x_1, 0^+) &= \sigma_{22}(x_1, 0^-), \quad b < |x_1| < a.
 \end{aligned}
 \tag{7}$$

Here  $b$  is the half size of the opening zone for the conducting crack, which will be determined exactly later. We are only concerned with frictionless contact of the crack surfaces in this paper. It is sufficient to consider separately the problem under electric loading or mechanical loading since the superposition principle can be applied to purely elastic fields for the electrostrictive problem. The complete solution for the electrostrictive crack under combined electric and mechanical loading is obtained by superposition. The superposition applied to the crack in an electrostrictive material under combined electric loading and mechanical loading is illustrated in Fig. 2. The complex functions for the conducting crack are written as

$$\begin{aligned}
 \varphi(z) &= \varphi^e(z) + \varphi^m(z), \\
 \psi(z) &= \psi^e(z) + \psi^m(z),
 \end{aligned}
 \tag{8}$$

where the superscripts  $e$  and  $m$  indicate the quantities associated with the problems of the crack under electric loading and mechanical loading, respectively. The complex functions for

the conducting crack with the contact zone are written as

$$\begin{aligned}
 \varphi^e(z) &= \frac{1}{2}S(E^\infty)^2 z + iSE_1^\infty E_2^\infty \sqrt{z^2 - a^2} + GC \ln \frac{z-a}{z+a} \\
 &\quad + GC \left\{ \frac{2\beta}{a} [\sqrt{z^2 - b^2} - z] \right. \\
 &\quad \left. - \ln \left[ \frac{(z-a)\sqrt{z^2 - b^2} + \sqrt{a^2 - b^2}}{(z+a)\sqrt{z^2 - b^2} - \sqrt{a^2 - b^2}} \right] \right\}, \\
 \psi^e(z) &= GC \left[ \ln \frac{z-a}{z+a} \frac{2az}{z^2 - a^2} \right] \\
 &\quad + GC \left\{ \frac{2\beta}{a} [\sqrt{z^2 - b^2} - z] - \ln \left[ \frac{(z-a)\sqrt{z^2 - b^2} + \sqrt{a^2 - b^2}}{(z+a)\sqrt{z^2 - b^2} - \sqrt{a^2 - b^2}} \right] \right\} \\
 &\quad - GC \left\{ \frac{2\beta}{a} \left[ \frac{z}{\sqrt{z^2 - b^2}} - 1 \right] + \frac{2a}{z^2 - a^2} \left[ -1 + \frac{z\sqrt{a^2 - b^2}}{\sqrt{z^2 - b^2}} \right] \right\} z,
 \end{aligned}
 \tag{9}$$

where  $C = \frac{(1+q)\epsilon^2 Q(K_1^E)^2}{8\pi(1-\nu)}$ . The solution of the complex functions for the problem of the conducting crack with a contact zone under the mechanical loading is well known as (Muskhelishvili, 1963)

$$\begin{aligned}
 \varphi^m(z) &= -\frac{1}{4}\sigma_{22}^\infty z + \frac{1}{2}\sigma_{22}^\infty \sqrt{z^2 - b^2} \\
 &\quad - \frac{1}{2}i\sigma_{12}^\infty (\sqrt{z^2 - a^2} - z), \\
 \psi^m(z) &= \frac{1}{2}\sigma_{22}^\infty \left[ z - \frac{b^2}{\sqrt{z^2 - b^2}} \right] \\
 &\quad + \frac{1}{2}i\sigma_{12}^\infty \left[ \sqrt{z^2 - a^2} + \frac{z^2}{\sqrt{z^2 - a^2}} \right].
 \end{aligned}
 \tag{10}$$

The opening size can be determined from the condition  $K_I^{(B)}=0$  where  $K_I^{(B)}$  is the stress intensity factor for the right tip of the partially opening crack located at the point  $B(x_1=b, x_2=0)$ . The stress intensity factor  $K_I^{(B)}$  is obtained from Eqs. (2), (5) and (8) as

$$K_I^{(B)} = \frac{4GC}{a} \sqrt{\pi b} \left[ \beta^* - \frac{a}{\sqrt{a^2 - b^2}} \right], \quad (11)$$

where  $\beta^* = \beta + \frac{a\sigma_{22}^0}{4GC}$ . Solving  $K_I^{(B)}=0$  for  $b$ , the opening size is given by

$$\begin{aligned} b &= a \sqrt{1 - \frac{1}{\beta^{*2}}} \text{ for } \beta^* > 1, \\ b &= 0 \text{ for } \beta^* \leq 1. \end{aligned} \quad (12)$$

It is noted that the opening size depends on the ratio of the external stress to the external electric field applied at infinity. Since the complex functions for the conducting crack with the contact zone are determined as above, the stresses and displacements can be evaluated from Eq. (2). The stresses on the closed crack surface and the prolongation of the crack are

$$\begin{aligned} \sigma_{22} + i\sigma_{12} &= -4GC \frac{x_1}{a^2 - x_1^2} \frac{\sqrt{x_1^2 - b^2}}{\sqrt{a^2 - b^2}} \text{ for } b < x_1 < a, \\ \sigma_{22} + i\sigma_{12} &= \frac{4GC}{a} \left[ \beta^* \left\{ \frac{x_1}{\sqrt{x_1^2 - b^2}} - 1 \right\} - \frac{ax_1}{a^2 - x_1^2} \frac{\sqrt{a^2 - b^2}}{\sqrt{x_1^2 - b^2}} \right] \\ &\quad + \sigma_{22}^0 + i\sigma_{12}^0 \frac{x_1}{\sqrt{x_1^2 - a^2}} \text{ for } x_1 > a. \end{aligned} \quad (13)$$

It is noted that the normal stress on the closed surface of the crack is compressive. The jump in the normal displacement on the crack surfaces for  $|x_1| < b$  is

$$\Delta u_2 = 8C(1-\nu) \left[ \frac{\sqrt{b^2 - x_1^2}}{\sqrt{a^2 - b^2}} - \tan^{-1} \frac{\sqrt{b^2 - x_1^2}}{\sqrt{a^2 - x_1^2}} \right]. \quad (14)$$

### 3. Asymptotic Nonlinear Analysis

In order to investigate the effect of the electrical nonlinearity on the stress intensity factor under small scale nonlinear conditions, we consider the asymptotic problem of a semi-infinite crack with a traction free zone in a nonlinear electrostrictive ceramic under electric loading as shown in Fig. 3. The size of traction free zone on the semi-infinite

conducting crack is  $d$ . The electric constitutive relation for the nonlinear electrostrictive ceramic is assumed to have the following form

$$\begin{aligned} E_i &= \frac{E_0}{D_0} D_i \quad D < D_0, \\ E_i &= E_0 \left( \frac{D}{D_0} \right)^n \frac{D_i}{D} \quad D > D_0, \end{aligned} \quad (15)$$

where  $E_0$ ,  $D_0$  and  $n$  are the material constants and  $D = \sqrt{D_k D_k}$ .  $D_0/E_0$  is equal to the dielectric permittivity  $\epsilon$ . The remote electric field in the asymptotic problem is prescribed to be the near-tip electric field for the crack in the linear dielectric material. Introducing cylindrical coordinates  $r$  and  $\theta$  centered at the tip of the crack as shown in Fig. 3, the electric fields at infinity are written as

$$E_r = \frac{K_I^E}{\sqrt{2\pi r}} \cos \frac{1}{2}\theta, \quad E_\theta = -\frac{K_I^E}{\sqrt{2\pi r}} \sin \frac{1}{2}\theta, \quad (16)$$

where  $K_I^E$  is the intensity factor of the electric field for the conducting crack. Dimensional considerations require that the stress intensity factors due to the small scale nonlinear zone around the crack tip are related to the material constants and the electric intensity factor by

$$\begin{aligned} K_I^N &= \Lambda(d/R_0; \nu, q, n) \sigma_0 \sqrt{\pi R_0}, \\ K_I^{N(P)} &= \Lambda^{(P)}(d/R_0; \nu, q, n) \sigma_0 \sqrt{\pi R_0}. \end{aligned} \quad (17)$$

Here  $K_I^N$  and  $K_I^{N(P)}$  are the stress intensity factors due to the small scale nonlinear zone for the original tip of the conducting crack and the left tip of the partial opening crack located at the point  $P(r=d)$ , respectively.  $\Lambda$  and  $\Lambda^{(P)}$  are the normalized stress intensity factors,  $\sigma_0 = YQD_0^2$

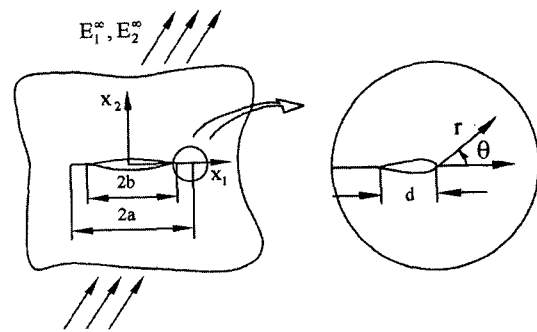


Fig. 3 Asymptotic problem for an electrostrictive crack under electric loading

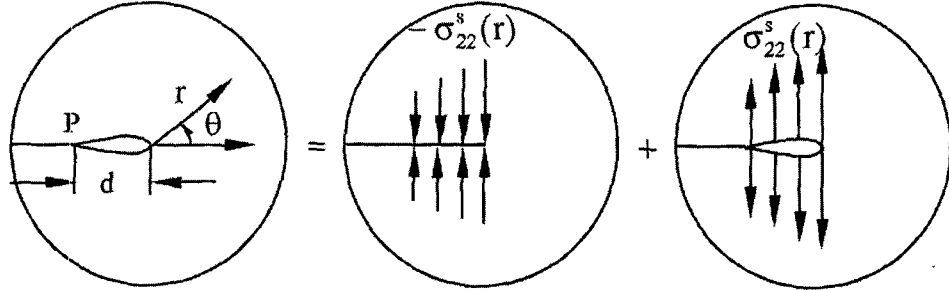


Fig. 4 Application of superposition to obtain the stress intensity factor for a conducting crack

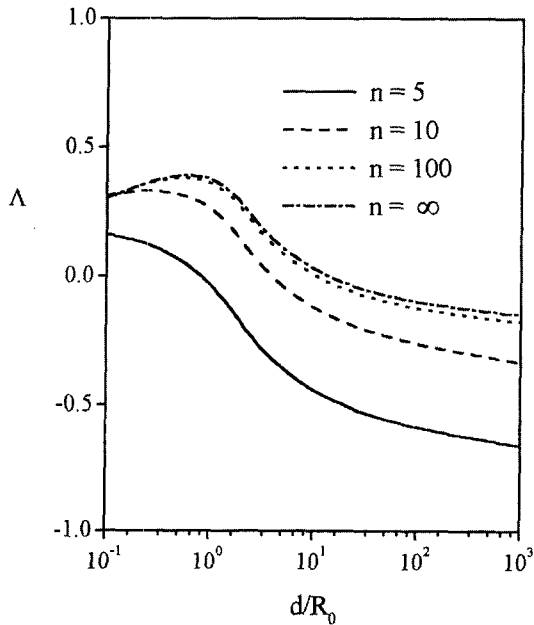


Fig. 5 Normalized stress intensity factor  $\Lambda$  for the original tip of the conducting crack as a function of  $d/R_0$

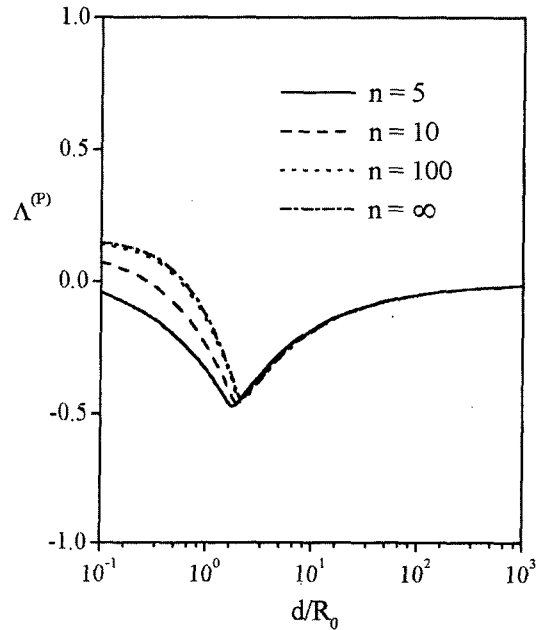


Fig. 6 Normalized stress intensity factor  $\Lambda^{(P)}$  for the left tip of the partial opening crack as a function of  $d/R_0$

and  $R_0 = \frac{1}{2\pi} \left( \frac{K_I^E}{E_0} \right)^2$ . We can evaluate the stress intensity factor for the electrostrictive crack from the solutions of the corresponding conducting sheet as shown in Fig. 4, by applying superposition. The stress on the conducting sheet induces the stress intensity factors given by Murakami (1987) :

$$K_I^N = \sqrt{\frac{2}{\pi d}} \int_0^d \sigma_{22}^s(r) \sqrt{\frac{d-r}{r}} dr, \tag{18}$$

$$K_I^{N(P)} = \sqrt{\frac{2}{\pi d}} \int_0^d \sigma_{22}^s(r) \sqrt{\frac{r}{d-r}} dr,$$

where  $\sigma_{22}^s(r)$  is the stress on the conducting sheet

for the asymptotic problem of a conducting semi-infinite sheet, which has been presented in Beom (1999b). Performing the integrals in Eq. (18) numerically, we obtain the stress intensity factors. Figures 5 and 6 illustrate the effect of the electric displacement hardening exponent on the normalized stress intensity factors  $\Lambda$  and  $\Lambda^{(P)}$  as  $d$  varies. Here  $\nu=q=0.3$  has been used in the numerical calculations. A high hardening exponent of the electric displacement results in a higher stress intensity factor  $K_I^N$ . In the limiting case as  $n \rightarrow \infty$ , the stress on the conducting sheet is (Hao et al., 1996)

$$\begin{aligned} \sigma_{22}^s(r) &= -\frac{GC}{R_0} \left[ 2 \ln \frac{r}{R_0} + 1 + \beta \right] (0 < r \leq 2R_0), \\ \sigma_{22}^s(r) &= \frac{GC}{R_0} \left[ 2 \ln \left( \frac{r-R_0}{r} \right) - (1+\beta) \left( \frac{R_0}{r-R_0} \right)^2 \right] (r > 2R_0). \end{aligned} \quad (19)$$

Substituting Eq. (19) into Eq. (18), it can be shown that the normalized stress intensity factors for  $0 < d \leq 2R_0$  are written in closed form as

$$\begin{aligned} \Lambda(d/R_0; \nu, q, \infty) &= \frac{1}{8\sqrt{2}} \frac{1+q}{1-\nu^2} \sqrt{\frac{d}{R_0}} \left[ 1 + 4 \ln 2 - \beta - 2 \ln \frac{d}{R_0} \right], \\ \Lambda^{(p)}(d/R_0; \nu, q, \infty) &= \frac{1}{8\sqrt{2}} \frac{1+q}{1-\nu^2} \sqrt{\frac{d}{R_0}} \left[ -3 + 4 \ln 2 - \beta - 2 \ln \frac{d}{R_0} \right]. \end{aligned} \quad (20)$$

It is noted that the stress intensity factor due to the small scale nonlinear zone around the crack tip has a negative value for large value of  $d$ . The negative stress intensity factor leads to overlapping of the crack surfaces. The size of the opening zone near the tip of the conducting crack under combined electric and mechanical loading will be determined later.

Sufficiently large mechanical loading applied on the external boundary opens the entire surface of the conducting crack. Thus, we consider the asymptotic problem of a semi-infinite crack with traction free surfaces in a nonlinear electrostrictive ceramic under electric loading. The stress intensity factor for the semi-infinite crack can be expressed as

$$K_I^N = \Lambda^\circ(\nu, q, n) \sigma_0 \sqrt{\pi R_0}, \quad (21)$$

where  $\Lambda^\circ(\nu, q, n) = \Lambda(\infty; \nu, q, n)$  and the stress intensity factor for the original tip of the conducting crack is given by

$$K_I^N = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sigma_{22}^s(r) dr}{\sqrt{r}}.$$

In Fig. 7, the numerical result of the function  $\Lambda^\circ(\nu, q, n)$  is shown as a function of  $n$ .  $\nu = q = 0.3$  has been used in the numerical calculations. Making use of Eqs. (19) and (21), it can be shown that the normalized stress intensity factor  $\Lambda^\circ$  for the special case of  $n \rightarrow \infty$  is written in closed form as

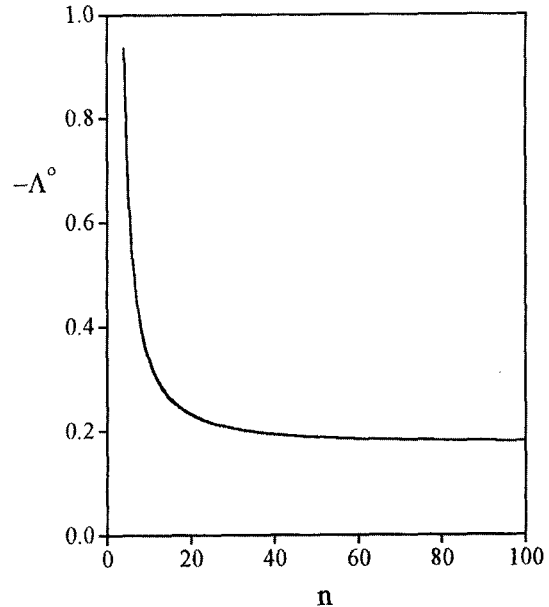


Fig. 7 Normalized stress intensity factor  $\Lambda^\circ$  as a function of  $n$

$$\begin{aligned} \Lambda^\circ(\nu, q, \infty) &= \frac{1}{8\sqrt{\pi}} \frac{1+q}{1-\nu^2} \left\{ 5\sqrt{2} + \frac{7}{2} \ln(3-2\sqrt{2}) \right. \\ &\quad \left. - \left[ 3\sqrt{2} + \frac{1}{2} \ln(3-2\sqrt{2}) \right] \beta \right\}. \end{aligned} \quad (22)$$

The numerical results shown in Figs. 5 and 6 approach this closed form solutions as  $n$  increases.

#### 4. Stress Intensity Factor

The surfaces near the tip of the conducting crack in a linear dielectric as shown in Sec. 2 are closed. Analysis based on linear dielectric model predicts that an electrical field does not induce any non zero stress intensity factor of the Mode I for the crack with a contact zone near the crack tip. However, Beom (1999b) showed from the nonlinear analysis that the surfaces of the conducting crack under electric loading are partially open in the vicinity of the crack tip due to the nonlinear zone. This result gives a suggestion that sufficiently large mechanical loading applied on the external boundary may open the entire surface of the conducting crack, in contrast to the result obtained from the linear dielectric theory.

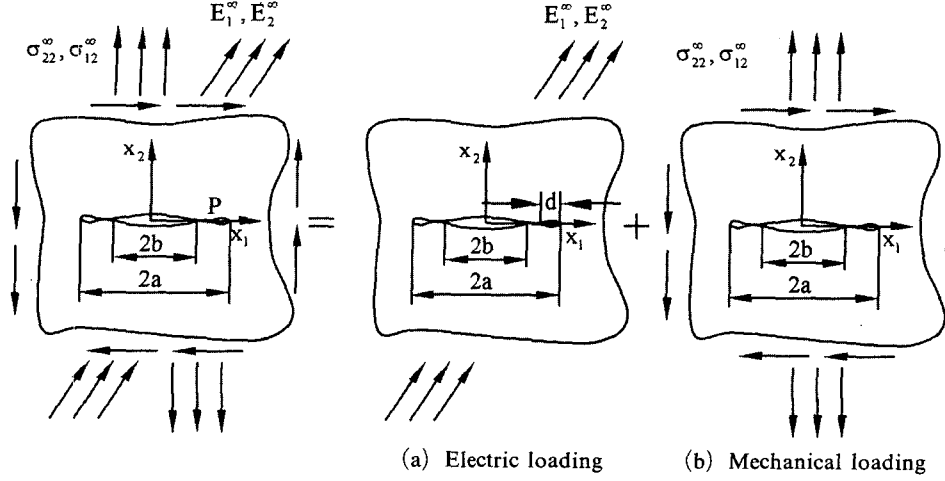


Fig. 8 Application of superposition to obtain the solution for a conducting crack under combined electric and mechanical loading

#### 4.1 Crack with a contact zone

We consider a crack in an electrostrictive ceramic under combined electric and mechanical loading. For the small mechanical load  $\sigma_{22}^{\infty}$  applied at infinity, the size of the partial opening zone near the tip of the conducting crack under combined electric and mechanical loading is small. The superposition principle can be applied to purely elastic fields for the electrostrictive problem as shown in Fig. 8. Therefore, we can obtain the complete solution for the electrostrictive crack from sum of the solutions of the crack under electric loading and the crack under mechanical loading. The total stress intensity factors for the left tip of the partial opening crack under small scale nonlinear conditions are given by

$$K_I^{(P)} = \Lambda^{(P)}(d/R_0; \nu, q, n) \sigma_0 \sqrt{\pi R_0} + \sigma^m \sqrt{\pi d}, \quad (23)$$

where  $\sigma^m = \sigma_{22}^{\infty} \frac{a}{\sqrt{a^2 - b^2}}$  for  $d/a \ll 1$  (Murakami, 1987). The size of the opening zone near the tip of the conducting crack under combined electric and mechanical loading can be determined from the condition  $K_I^{(P)} = 0$ . The total stress intensity factors for the electrostrictive crack under small scale nonlinear conditions are given by

$$\begin{aligned} K_I &= \Lambda(d/R_0; \nu, q, n) \sigma_0 \sqrt{\pi R_0} + \sigma^m \sqrt{\pi d}, \\ K_{II} &= \sigma_{12}^{\infty} \sqrt{\pi a}. \end{aligned} \quad (24)$$

The normalized stress intensity factor  $\Lambda^{(P)}$  has a

minimum value  $\Lambda_{\min}^{(P)}$  for  $d = d_{\min}$ . When  $\sigma_{22}^{\infty}$  is small such that  $d < d_{\min}$ , Eq. (24) is valid. In particular, the size of the opening zone and the stress intensity factor for  $n \rightarrow \infty$  and  $0 < d \leq 2R_0$  are expressed in closed form as

$$\begin{aligned} d &= R_0 \exp \left[ \frac{1}{2} (4 \ln 2 - 3 - \beta) + 4\sqrt{2} \frac{1 - \nu^2}{1 + q} \frac{\sigma^m}{\sigma_0} \right], \\ K_I &= \frac{1}{2\sqrt{2}} \frac{1 + q}{1 - \nu^2} \exp \left[ \frac{1}{4} (4 \ln 2 - 3 - \beta) \right. \\ &\quad \left. + 2\sqrt{2} \frac{1 - \nu^2}{1 + q} \frac{\sigma^m}{\sigma_0} \right] \sigma_0 \sqrt{\pi R_0}. \end{aligned} \quad (25)$$

#### 4.2 Open crack

Consider a fully open crack in an electrostrictive ceramic under combined electric and mechanical loading. Sufficiently large mechanical loading applied on the external boundary may open the entire surface of the conducting crack. When the mechanical loading is large such that  $b + d \geq a$ , the entire surface of the conducting crack is open. The solution of the complex functions for the problem of the crack with traction free surfaces in a linear electrostrictive ceramic under electric loading may be chosen in the form :

$$\begin{aligned} \varphi(z) &= \frac{1}{2} S(E_1^{\infty})^2 z + S((E_1^{\infty})^2 + i E_1^{\infty} E_2^{\infty}) \left( \sqrt{z^2 - a^2} - z \right), \\ \psi(z) &= -S(E_1^{\infty})^2 \frac{a^2}{\sqrt{z^2 - a^2}}, \end{aligned} \quad (26)$$



where  $E^\infty = \sqrt{(E_1^\infty)^2 + (E_2^\infty)^2}$ . It is can be easily seen from Eqs. (2), (5) and (26) that the crack surfaces near the crack tip under the electric loading overlap. However, the surfaces of the conducting crack are open due to the nonlinear zone near the crack tip and the mechanical loading. The linear dielectric theory predicts that the stress intensity factor for the crack under electric loading is

$$K_I = 2S(E_1^\infty)^2 \sqrt{\pi a}. \quad (27)$$

It is well known that the stress intensity factor for the conducting crack due to mechanical loading is given by

$$\begin{aligned} K_I &= \sigma_{22}^\infty \sqrt{\pi a}, \\ K_{II} &= \sigma_{12}^\infty \sqrt{\pi a}. \end{aligned} \quad (28)$$

Thus, the total stress intensity factors for the electrostrictive crack under small scale nonlinear conditions are given by

$$\begin{aligned} K_I &= 2S(E_2^\infty)^2 \sqrt{\pi a} + \Lambda^0(\nu, q, n) \sigma_0 \sqrt{\pi R_0} + \sigma_{22}^\infty \sqrt{\pi a}, \\ K_{II} &= \sigma_{12}^\infty \sqrt{\pi a}. \end{aligned} \quad (29)$$

### 5. Concluding Remarks

A conducting crack in an electrostrictive ceramic under combined electric and mechanical loading is analyzed. Analysis based on linear dielectric model predicts that the traction free boundary conditions on the crack surfaces lead to overlapping of the conducting crack surfaces. The surfaces of the crack cannot be thus open completely and they are contact near the crack tip. The complete solution for the electrostrictive crack with a contact zone under combined electric and mechanical loading is obtained by using the complex variable formula. The size of the opening zone for the conducting crack is determined exactly. The combined electric and mechanical loading does not induce any non zero stress intensity factor of the Mode I for a conducting crack due to contact of the crack surfaces near the crack tip.

However, it is shown from the nonlinear analysis that the surfaces of the conducting crack under combined electric and mechanical loading are partially open in the vicinity of the crack tip due

to the nonlinear zone. In order to investigate the effect of the electrical nonlinearity on the stress intensity factors under small scale nonlinear conditions, we consider the asymptotic problems for a semi-infinite crack with a partial opening zone as well as for a fully open crack in a nonlinear electrostrictive ceramic. The stress intensity factors for the asymptotic problems are evaluated from the solution of the corresponding conducting sheet by applying superposition.

Particular attention is devoted to a finite crack in the nonlinear electrostrictive ceramic subjected to combined electric and mechanical loading. For the small mechanical load applied at infinity, the size of the partial opening zone near the tip of the conducting crack under combined electric and mechanical loading is small. Sufficiently large mechanical loading applied on the external boundary opens the entire surface of the conducting crack, in contrast to the result obtained from the linear dielectric theory. The stress intensity factors for the finite crack under small scale nonlinear conditions are obtained from the asymptotic analysis.

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