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Topological Design Sensitivity on the Air Bearing Surface of Head Slider

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In this study, a topological design sensitivity of the air bearing surface (ABS) is suggested by using an adjoint variable method. The discrete form of the generalized lubrication equation based on a control volume formulation is used as a compatible condition. A residual function of the slider is considered as an equality constraint function, which represents the slider in equilibrium. The slider thickness parameters at all grid cells are chosen as design variables since they are the topological parameters determining the ABS shape. Then, a complicated adjoint variable equation is formulated to directly handle the highly nonlinear and asymmetric coefficient matrix and vector in the discrete system equation of air-lubricated slider bearings. An alternating direction implicit (ADI) scheme is utilized for the numerical calculation. This is an efficient iterative solver to solve large-scale problem in special band storage. Then, a computer program is developed and applied to a slider model of a sophisticated shape. The simulation results of design sensitivity analysis (DSA) are directly compared with those of FDM at the randomly selected grid cells to show the effectiveness of the proposed approach. The overall distribution of DSA results are reported, clearly showing the region on the ABS where special attention should be given during the manufacturing process.

Key Words : Design Sensitivity Analysis(DSA), Air Bearing Surface(ABS)

1. Introduction

Trends in magnetic storage emphasize the need for reduced head/disk spacing, which is coupled directly to the achievement of higher recording densities in hard disk drive (HDD) systems. Typical modern HDD operates with a flying height of about 15 to 20 nm corresponding to an areal density of about 15 Gbit/in^2. The head disk interface (HDI) performance of the high recording density systems directly depends on the air bearing surface (ABS) design.

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In order to meet the HDI performance requirements, several optimization studies for ABS design have been carried out. However, most of them have been concentrated on changing the basic shapes (Yoon and Choi, 1995 and 1997 ; O'Hara and Bogy, 1995 ; Lu et al., 1996 ; O'Hara et al., 1996 ; Choi and Kang, 1999 and 2001). They also dealt with only the problems of symmetric sliders with a few configuration parameters as design variables.

To overcome these restricted designs, a topology optimization should be employed. However, it has too many design variables since the finite mesh should be fine enough to represent the geometry of the slider. If an ABS is represented in 100×100 grid system, then its topology optimization problem has at least 10,000 design variables. Most of the total computing time on optimization procedure is spent to obtain the design sensitivity since the number of design variables is much larger than that of the objective and constraints functions. Hence, the analytical design sensitivity analysis (DSA), which can effectively reduce a large portion of the total computing time and ensure its accuracy, should be employed.

The adjoint sensitivity analysis is a way to effecfively obtain the derivatives when the derivatives of state variable are aifficult to obtain. However, unlike the linear structural analysis such as FEM, the discrete system equations of air-lubricated slider bearings have highly nonlinear asymmetric coefficient matrix and vector. Hence, a special iterative solver has been employed to solve them. It is known that the conventional DSA method cannot be directly applied for air-lubricated slider bearing systems.

This study presents an extended DSA for air-lubricated slider bearings by introducing the adjoint variable method. In Sec. 2, we review the discrete system equations and the computational procedure of an air-lubricated slider bearing system. In Sec. 3, the proposed design sensitivity formulation is clearly described. Section 4 examines the numerical results of the proposed DSA method. Finally, concluding remarks are mentioned in Sec. 5.

2. Review of Air-Lubricated Slider Bearing Analysis

The generalized lubrication equation for air-lubricated slider bearings has the following form (Fukui and Kaneko, 1988)

$$\sigma\frac{\partial}{\partial T}(PH)+\nabla\cdot(\Lambda PH)=\nabla\cdot(\bar{Q}_P PH\nabla P) \quad (1)$$

where P is the pressure, H is the flying height, Λ is the bearing number vector, σ is the squeeze number, and $\bar{Q}_P$ is the flow factor. To solve the generalized lubrication equation numerically, an implicit control volume scheme based on the convection-diffusion formulation is employed. The control volume is assigned as the rectangular region surrounding each grid point with its boundaries located at the midpoint between the neighboring grids. After approximating the total fluxes by using the central difference scheme, it then is regrouped by the dependant variable at the neighboring grid points into the following discrete form (Hu and Bogy, 1998)

$$a_P P_P+(a_E P_E+a_W P_W+a_N P_N+a_S P_S)=b \quad (2)$$

where the complete expressions of the coefficients of the pressure are given in the Appendix.

As an iterative solver, the alternating direction implicit (ADI) scheme is applied for effectively solving the linearized implicit equations. The equation is solved in two separate steps. First the equation is solved implicitly in the x-direction and explicitly in the y-direction. Then, reversing the process, it is solved explicitly in the x-direction and implicitly in the y-direction.

3. DSA of Air-Lubricated Slider Bearings

The optimal design of an air-lubricated slider bearing needs an inverse analysis as a solver. The inverse analysis again calls for the forward analysis to find the residuals obtained by solving the generalized lubrication equation numerically. Since the inverse solution represents the slider attitude at zero residuals, the inverse problem can be transformed to a forward problem with the equality constraints of residuals.

3.1 The complicated adjoint variable formulation

To convert Eq. (2) to a matrix form, the total equilibrium equations in terms of the discrete pressure vector $\mathbf{p}(\mathbf{z})=R^n$ are generated as

$$\mathbf{Kp}=\mathbf{f} \quad (3)$$

where $\mathbf{K}(\mathbf{p},\ \mathbf{z})\in R^{mxn}$ refers to the coefficient matrix of the discrete governing equation, which is highly nonlinear and asymmetric unlike that of the conventional structural analysis. $\mathbf{f}(\mathbf{p},\ \mathbf{z})\in R^m$ is the coefficient vector generated by the boundary condition, which is also nonlinear. $\mathbf{z}$ is a head/disk clearance vector defined as functions of the flying attitude and the thickness parameters of the slider. Since the flying attitude parameters are given as an initial data in case of the forward analysis, $\mathbf{z}$ is replaced with a thickness parameter vector $\mathbf{t}\in R^k$, and employed as the design variables in this study.

Consider now a residual function $\phi(\mathbf{p})$, which represents the slider in equilibrium of lifting and suction forces, as an equality constraint. The issue can be summarized to find the derivatives of the residual with respect to the thickness parameters of the slider while satisfying a compatible condition.

To effectively satisfy the compatible condition of Eq. (3), we add the equilibrium equations multiplied by an adjoint variable vector or Lagrangian multiplier $\boldsymbol{\lambda}\in R^m$. Then, this adjoint vector will be selected to satisfy equation that leads to the elimination of the implicit differentiation through the dependence on $\mathbf{p}$.

$$\Psi(\mathbf{t}, \boldsymbol{\lambda}) = \phi(\mathbf{p}) + \boldsymbol{\lambda}^T\{\mathbf{Kp}-\mathbf{f}\} \tag{4}$$

Using the chain rule of differentiation, we obtain the total derivative of the functional Ψ with respect to the kth design variable in the following form

$$\begin{aligned}\frac{d\Psi}{dt_k} &= \left(\frac{d\phi}{d\mathbf{p}}\right)^T\frac{d\mathbf{p}}{dt_k} + \left(\frac{d\boldsymbol{\lambda}}{dt_k}\right)^T\{\mathbf{Kp}-\mathbf{f}\}\\ &+\boldsymbol{\lambda}^T\left\{\left(\left(\frac{\partial\mathbf{K}}{\partial\mathbf{p}}\right)^T\frac{d\mathbf{p}}{dt_k}+\frac{\partial\mathbf{K}}{\partial t_k}\right)\mathbf{p}\right.\\ &\left.+\mathbf{K}\frac{d\mathbf{p}}{dt_k}-\left(\frac{\partial\mathbf{f}}{\partial\mathbf{p}}\right)^T\frac{d\mathbf{p}}{dt_k}-\frac{\partial\mathbf{f}}{\partial t_k}\right\}\end{aligned} \tag{5}$$

where the third and fourth additional terms are concerned with the adjoint vector.

In the optimization procedure, an initial condition of DSA always satisfies the compatible condition. Thus, the third term of the above equation can be eliminated. In order to represent the total derivative of the functional as the implicit form of $d\mathbf{p}/dt_k$, we rearrange it as follows ;

$$\begin{aligned}\frac{d\Psi}{dt_k} &= \boldsymbol{\lambda}^T\left(\frac{\partial\mathbf{K}}{\partial t_k}\mathbf{p}-\frac{\partial\mathbf{f}}{\partial t_k}\right)\\ &+\left\{\boldsymbol{\lambda}^T\left(\left[\frac{\partial\mathbf{K}}{\partial P_1}\mathbf{p}, \frac{\partial\mathbf{K}}{\partial P_2}\mathbf{p}, \cdots, \frac{\partial\mathbf{K}}{\partial P_n}\mathbf{p}\right]\right.\right.\\ &\left.\left.+\mathbf{K}-\left(\frac{\partial\mathbf{f}}{\partial\mathbf{p}}\right)^T\right)+\left(\frac{d\phi}{d\mathbf{p}}\right)^T\right\}\frac{d\mathbf{p}}{dt_k}\end{aligned} \tag{6}$$

As the condition to eliminate the derivative of $\mathbf{p}$, the following adjoint equation is generated.

$$\begin{aligned}&\left(\left[\frac{\partial\mathbf{K}}{\partial P_1}\mathbf{p}, \frac{\partial\mathbf{K}}{\partial P_2}\mathbf{p}, \frac{\partial\mathbf{K}}{\partial P_3}\mathbf{p}, \cdots, \frac{\partial\mathbf{K}}{\partial P_n}\mathbf{p}\right]\right.\\ &\left.+\mathbf{K}-\left(\frac{\partial\mathbf{f}}{\partial\mathbf{p}}\right)^T\right)^T\boldsymbol{\lambda} = -\frac{d\phi}{d\mathbf{p}}\end{aligned} \tag{7}$$

The remaining terms represent the derivatives of the constraint as shown in the form

$$\frac{d\Psi}{dt_k} = \boldsymbol{\lambda}^T\left(\frac{\partial\mathbf{K}}{\partial t_k}\mathbf{p}-\frac{\partial\mathbf{f}}{\partial t_k}\right) \tag{8}$$

Note that the matrix $\mathbf{K}$ in Eq. (7) is a block tridiagonal, in which each nonzero block consists of tridiagonal and diagonal with a special rule (Hu and Bogy, 1998). Therefore, a complicated adjoint equation can be briefly rewritten as

$$\tilde{\mathbf{K}}\boldsymbol{\lambda} = -\frac{d\phi}{d\mathbf{p}} \tag{9}$$

where the complicated coefficient matrix $\tilde{\mathbf{K}}$ will be described in the next section.

Only by just solving the adjoint equation for $\boldsymbol{\lambda}$ from Eq. (9), we can effectively obtain the design sensitivity of the functional Ψ with respect to a lot of design variables without having to calculate the derivative of $\mathbf{p}$, since the adjoint equation is not a function of the design variable any more.

3.2 Numerical analysis of the extended DSA

The complicated matrix $\tilde{\mathbf{K}}$ consists of the terms such as a_E, a_W, a_N, a_S, a_P which are the functions of pressure. Denoting the pressure coefficient at control volume (p, q) as $a_{p,q}$, and the pressure at grid (i, j) as $P_{i,j}$, we can find the derivative of a_E with respect to $P_{i,j}$ as

$$\begin{aligned}\frac{\partial(a_E)_{p,q}}{\partial P_{i,j}} &= 0 && : if\ |C_e|<10\\ &= -\Delta Y\left[\frac{\partial D_e}{\partial P_{i,j}}(1-0.1\varepsilon C_e)^5\right.\\ &\left.-0.5\varepsilon D_e(1-0.1\varepsilon C_e)^4\frac{\partial C_e}{\partial P_{i,j}}\right]_{p,q}\\ & && : if\ -10\le C_e\le 10\end{aligned} \tag{10}$$

where $\varepsilon = 1$, *if* $C_e \ge 0$
$\varepsilon = -1$, *if* $C_e < 0$

$$\frac{\partial(F_e)_{p,q}}{\partial P_{i,j}}=0$$
$$\frac{\partial(C_e)_{p,q}}{\partial P_{i,j}}=-\frac{F_e}{D_e^2}\frac{\partial D_e}{\partial P_{i,j}} \tag{11}$$
$$\frac{\partial(D_e)_{p,q}}{\partial P_{i,j}}=\frac{H_e^2}{\delta X_e}\left(\frac{\partial QN_e}{\partial P_{i,j}}\right) \quad : if\ i=p\ or\ p+1,\ j=q$$
$$=0 \quad : \text{otherwise}$$

The above ($i=p$ *or* $p+1$, $j=q$) represents when the pressure grid point is consistent with the control volume point or its neighboring point in the right-hand side. Since QN is a diffusion term obtained by interpolation from the flow rate database (Fukui & Kaneko, 1990), its derivative can be approximated by the central difference scheme. Following the same idea, the derivatives of a_W, a_S, a_N with respect to $P_{i,j}$ can be generated. On the other hand, the derivative of a_P is

$$\frac{\partial(a_P)_{p,q}}{\partial P_{i,j}}=-\frac{\partial}{\partial P_{i,j}}(a_E+a_W+a_N+a_S)_{p,q} \tag{12}$$

Substituting (i, j) for (p, q) in the above equations, and then considering the relation to the neighboring points over the control volume faces, the following equations can be derived.

$$\begin{aligned}
\frac{\partial(a_P)_{i,j-1}}{\partial P_{i,j}}&=-\frac{\partial(a_N)_{i,j-1}}{\partial P_{i,j}}\\
\frac{\partial(a_P)_{i-1,j}}{\partial P_{i,j}}&=-\frac{\partial(a_E)_{i-1,j}}{\partial P_{i,j}}\\
\frac{\partial(a_P)_{i+1,j}}{\partial P_{i,j}}&=-\frac{\partial(a_W)_{i+1,j}}{\partial P_{i,j}}\\
\frac{\partial(a_P)_{i,j+1}}{\partial P_{i,j}}&=-\frac{\partial(a_S)_{i,j+1}}{\partial P_{i,j}}
\end{aligned} \tag{13}$$

Finally, considering Eqs. (10) ~ (13) over the control volume faces, the components of the complicated matrix $\tilde{\mathbf{K}}$ are rearranged as

$$\begin{aligned}
(\tilde{a}_S)_{i,j}&=\frac{\partial(a_N)_{i,j-1}}{\partial P_{i,j}}(P_{i,j}-P_{i,j-1})+(a_N)_{i,j-1}\\
(\tilde{a}_W)_{i,j}&=\frac{\partial(a_E)_{i-1,j}}{\partial P_{i,j}}(P_{i,j}-P_{i-1,j})+(a_E)_{i-1,j}\\
(\tilde{a}_P)_{i,j}&=\frac{\partial(a_S)_{i,j}}{\partial P_{i,j}}P_{i,j-1}+\frac{\partial(a_W)_{i,j}}{\partial P_{i,j}}P_{i-1,j}+\frac{\partial(a_P)_{i,j}}{\partial P_{i,j}}P_{i,j}\\
&\quad+\frac{\partial(a_E)_{i,j}}{\partial P_{i,j}}P_{i+1,j}+\frac{\partial(a_N)_{i,j}}{\partial P_{i,j}}P_{i,j+1}+(a_P)_{i,j}\\
(\tilde{a}_E)_{i,j}&=\frac{\partial(a_W)_{i+1,j}}{\partial P_{i,j}}(P_{i,j}-P_{i+1,j})+(a_W)_{i+1,j}\\
(\tilde{a}_N)_{i,j}&=\frac{\partial(a_S)_{i,j+1}}{\partial P_{i,j}}(P_{i,j}-P_{i,j+1})+(a_S)_{i,j+1}
\end{aligned} \tag{14}$$

Now consider the derivatives of the residual functions of force and moments (Choi and Yoon, 1994) as follows;

$$\phi(\mathbf{p})=1-W/F \tag{15}$$

where

$$W=\int(\mathbf{p}-1)\,d\mathbf{A}=\Sigma(P_{i,j}-1)\Delta x_i\Delta y_j \tag{16}$$

Note that W is a function of $\mathbf{p}$ only. Thus, differentiating Eq. (16) with respect to a discrete pressure $P_{i,j}$ we can obtain

$$\frac{dW}{dP_{i,j}}=\Delta x_i\Delta y_j \tag{17}$$

Using the chain rule of differentiation, we can obtain the derivative of the constraint ϕ with respect to $\mathbf{p}$ in form

$$\frac{d\phi}{d\mathbf{p}}=-\frac{1}{F}\Delta\mathbf{x}\Delta\mathbf{y} \tag{18}$$

Considering FEM based structural optimization, the adjoint method typically requires the derivatives of the stiffness matrix and load vectors with respect to the design variables. Since these derivatives are often difficult to calculate analytically, a semi-analytical approach has been popular (Haftka and Gurdal, 1992). Therefore, in Eq. (8), the enormous effort required to implement the analytical derivatives of $\mathbf{K}$ and $\mathbf{f}$ can be circumvented by using the first-order finite differences approximations, so that they are approximated as

$$\frac{\partial\mathbf{K}}{\partial t_k}\approx\frac{\mathbf{K}(t_k+\Delta t_k)-\mathbf{K}(t_k)}{\Delta t_k} \tag{19}$$
$$\frac{\partial\mathbf{f}}{\partial t_k}\approx\frac{\mathbf{f}(t_k+\Delta t_k)-\mathbf{f}(t_k)}{\Delta t_k} \tag{20}$$

The complicated adjoint equation consists of the simultaneous linear equations of order n, which is a large-scale problem. Note that the matrix $\tilde{\mathbf{K}}$ is asymmetric and banded. More specially, the matrix is a block tridiagonal, with each nonzero block being either tridiagonal or diagonal, as the pattern of a matrix arising from a typical two-dimensional elliptic boundary value problem. Hence, to solve the adjoint equation in λ, we appropriately introduce the ADI scheme that is efficient in the large-scale problem in the special band storage mentioned above.

4. Computational Results

A computer program was developed based on an adjoint variable method described in the previous section, and then applied to a model slider having the pressure distribution shown in Fig. 1. The grid system for numerical analysis is 98×98. Thus, the number of grid cells or design variables is 9409 ($=97 \times 97$). Several grid cells over the recess, shallow step, and rail regions are randomly selected in order to directly compare the results of the proposed approach with those of the finite difference method.

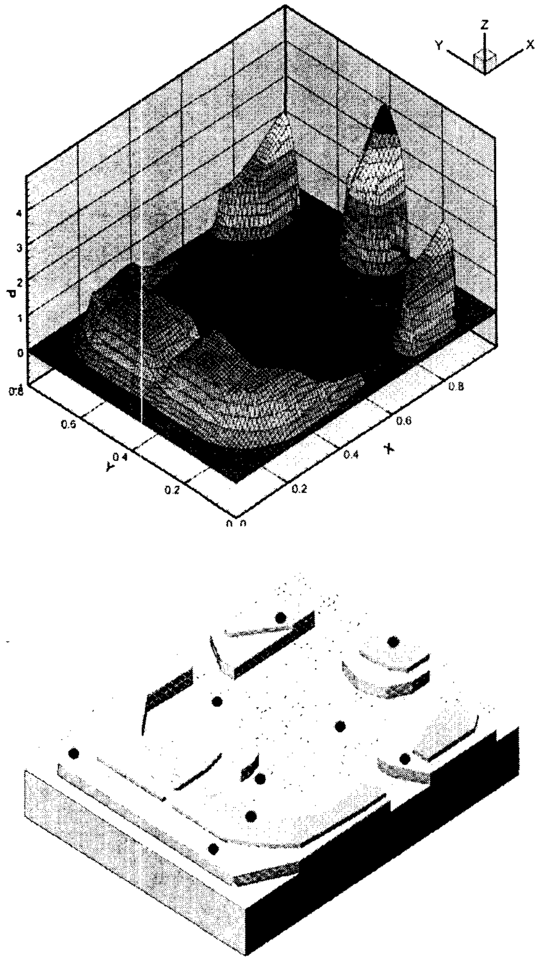

Fig. 1 Randomly selected grid cells

Table 1 lists the value of $d\Psi/dt_k$ at the given grid cell together with their discrepancies. Very good agreement between them clearly verifies the accuracy and effectiveness of our approach for air-lubricated slider bearings.

At an ultra low spacing, disturbances such as occasional head/disk contacts will most likely occur. It is desirable to consider the sensitivity to the geometric variation generated by wear and manufacturing tolerances. Figure 2 shows the overall magnitudes of DSA results with respect to topological parameters of the slider. The dark shading in the figure refers to the magnitude of the design sensitivity. It seems that the boundaries of the rail, shallow step and recess regions as well as the trailing rail zones are considerably sensitive to the variation of the ABS shape.

Table 1 Comparison of the results between DSA and FDM

Region	Grid Cell No.	FDM	DSA	$\left\|\frac{DSA-FDM}{DSA}\right\| \cdot 100(\%)$
Recess	(50, 70)	-1.249e-6	-1.249e-6	0.00
	(30, 40)	2.221e-7	2.222e-7	0.04
	(60, 40)	4.761e-7	4.761e-7	0.00
Shallow Step	(10, 20)	1.150e-5	1.151e-5	0.09
	(10, 90)	-9.608e-4	-9.609e-4	0.01
	(70, 10)	-1.465e-4	-1.466e-4	0.07
Rail	(90, 50)	1.491e-3	1.494e-3	0.02
	(20, 20)	3.075e-4	3.077e-4	0.07
	(80, 80)	5.123e-4	5.138e-4	0.30

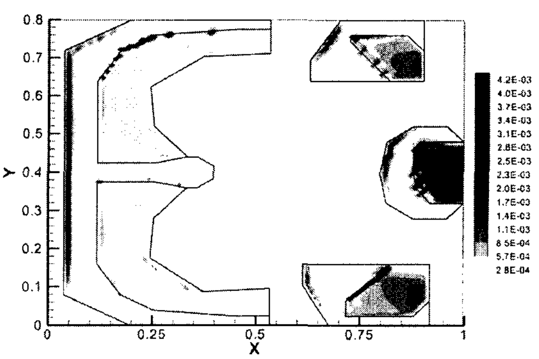

Fig. 2 Overall magnitudes of DSA results

As for the CPU time needed to calculate the total derivatives over the ABS of the slider, the proposed DSA needed 288 seconds, while the FDM took 12609 seconds, on AMD Athlon 1.6 GHz personal computer.

5. Concluding Remarks

An extended DSA is presented to solve the design sensitivity with respect to the topological parameters of air-lubricated slider bearings. The discrete form of the generalized lubrication equation was used as a compatible condition, the residual of the slider was transformed to an equality constraint function, and the slider thickness parameters at all grid cells were chosen as the design variables. A complicated adjoint equation was generated and the ADI scheme was subsequently utilized for the numerical calculation.

The comparison of the simulation results of DSA with those of FDM at several grid cells clearly showed the effectiveness of our approach. It is believed that DSA can give more accurate results than FDM since it analytically calculates the derivatives instead of calculating the difference of the functional with respect to the perturbation of design variables. Also, it is more efficient since it does not require much computing time in proportion to the number of design variables like FDM. The overall distribution of the design sensitivities clearly showed which parts over the ABS are sensitive to the geometric variations.

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Appendix

In the discrete form of the generalized lubrication equation, the coefficients at control volume (p, q) are

$$\begin{aligned} a_E &= -\Delta Y\{D_e A(|C_e|) + \max(-F_e, 0)\} \\ a_W &= -\Delta Y\{D_w A(|C_w|) + \max(F_w, 0)\} \\ a_N &= -\Delta X\{D_n A(|C_n|) + \max(-F_n, 0)\} \\ a_S &= -\Delta X\{D_s A(|C_s|) + \max(F_s, 0)\} \\ b &= \frac{\sigma H_P{}^0 P_P{}^0 \Delta X \Delta Y}{\Delta T} \\ a_P &= -(a_E + a_W + a_N + a_S) \\ &\quad + (\Delta Y F_e - \Delta Y F_w + \Delta X F_n - \Delta X F_s) \\ &\quad + \frac{\sigma H_P \Delta X \Delta Y}{\Delta T} \end{aligned} \tag{A.1}$$

$$\begin{aligned} F_e &= (\Lambda_x H)_e,\ D_e = \left(\frac{\Gamma}{\delta X}\right)_e,\ C_e = \left(\frac{\Lambda_x H \delta X}{\Gamma}\right)_e, \\ F_w &= (\Lambda_x H)_w,\ D_w = \left(\frac{\Gamma}{\delta X}\right)_w,\ C_w = \left(\frac{\Lambda_x H \delta X}{\Gamma}\right)_w, \\ F_n &= (\Lambda_y H)_n,\ D_n = \left(\frac{\Gamma}{\delta Y}\right)_n,\ C_n = \left(\frac{\Lambda_y H \delta Y}{\Gamma}\right)_n, \\ F_s &= (\Lambda_y H)_s,\ D_s = \left(\frac{\Gamma}{\delta Y}\right)_s,\ C_s = \left(\frac{\Lambda_y H \delta Y}{\Gamma}\right)_s. \end{aligned} \tag{A.2}$$

where F, D, C are the convection strength, diffusion conductance and Peclet number, respectively. Various schemes can be reached by adopting different functions for $A(|C|)$. Here the power-law scheme recommended by Patankar (1980) is used, which is

$$A(|C|) = \max\{0, (1-0.1|C|)^5\} \tag{A.3}$$

For solving the problem of clearance discontinuities, the technique introduced by Kogure et al (1983) is implemented. This involves averaging the mass flow across the discontinuity by appropriately weighting the mass flow contribution from different clearance heights.