Eigenvalue analysis of axisymmetric circular Mindlin plates by pseudospectral method

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ABSTRACT

A study of free vibration of axisymmetric circular plates based on Mindlin theory using a pseudospectral method is presented. The analysis is based on Chebyshev polynomials that are widely used in the fluid mechanics research community. Clamped, simply supported and free boundary conditions are considered, and numerical results are presented for various thickness-to-radius ratios.

Keywords : Axisymmetric Mindlin plate, Eigenvalue problem, Pseudospectral method, Chebyshev polynomials

1. Introduction

Plate vibration is important in many applications in mechanical, civil and aerospace engineering. Real plates may have appreciable thickness where transverse shear and rotary inertia are not negligible as assumed in the classical plate theory. As a result the thick plate model based on Mindlin theory has gained more popularity. Surveys of the literature show that considerable studies have been done on the free vibration of circular plates based on Mindlin theory[1-8].

The pseudospectral method can be considered as a spectral method that performs a collocation process. As the formulation is simple and powerful enough to produce approximate solutions close to exact solutions, this method has been used extensively in fluid mechanics researches, including meteorological studies[9-10].

The pseudospectral method can be made as spatially accurate as desired through exponential rate of convergence with mesh refinement. It also permits the choice of a wide variety of functions for the expansion. Even though this method could be used for the solution of structural mechanics problems, it has been largely unnoticed by the structural mechanics community and few articles are available where the pseudospectral method has been successfully applied. Soni and Amba-Rao[11] and Gupta and Lal[12] are among those who have applied Chebyshev collocation method to the axisymmetric vibration analyses of circular and annular plates. Mattei applied the pseudospectral method to the solution of the vibration problem of a fluid loaded plate, but it was dominated by the point of view of fluid mechanics[13]. Recently, the usefulness of pseudospectral method in the solution of structural mechanics problems has been demonstrated in an analysis of L-shaped Reissner-Mindlin plate[14].

In the present work, the pseudospectral method is applied to the free axisymmetric vibrations of circular Mindlin plates.

2. Circular Plates and Pseudospectral Method

The equations of motion of a homogeneous, isotropic axisymmetric circular plate based on Mindlin theory are [1]

$$\frac{\partial M_r}{\partial r}+\frac{1}{r}(M_r-M_\theta)-Q=\frac{\rho h^3}{12}\frac{\partial^2\Psi}{\partial t^2}\ , \quad (1)$$
$$\frac{\partial Q}{\partial r}+\frac{1}{r}Q=\rho h\frac{\partial^2 W}{\partial t^2}$$

where $W(r,t)$ and $\Psi(r,t)$ are the transverse displacement and the bending rotation normal to the midplane in the radial direction, and h and ρ are the thickness and the density of the plate, respectively. The stress resultants M_r, M_θ, and Q are defined by

$$M_r=D\left(\frac{\partial\Psi}{\partial r}+\frac{\nu}{r}\Psi\right)\ ,$$
$$M_\theta=D\left(\frac{\Psi}{r}+\nu\frac{\partial\Psi}{\partial r}\right)\ , \quad (2)$$
$$Q=x^2Gh\left(\Psi+\frac{\partial W}{\partial r}\right)$$

where $D=Eh^3/12(1-\nu^2)$ is the flexural rigidity, E is the modulus of elasticity, ν is the Poisson's ratio, $x^2=\pi^2/12$ is the shear correction factor, and G is the shear modulus.

Substitution of (2) into (1) assuming a sinusoidal motion in time

$$\Psi(r,t)=\phi(r)\cos\omega t\ ,\qquad W(r,t)=w(r)\cos\omega t \quad (3)$$

yields

$$\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}-\left(\frac{1}{r^2}+\frac{x^2Gh}{D}\right)\phi-\frac{x^2Gh}{D}\frac{dw}{dr}=-\omega^2\frac{\rho h^3}{12D}\phi\ , \quad (4)$$
$$\frac{d\phi}{dr}+\frac{\phi}{r}+\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}=-\omega^2\frac{\rho}{x^2G}w\ .$$

The boundary conditions considered in the present study are

clamped:	$w=0$,	$\phi=0$	
simply supported:	$w=0$,	$M_r=0$	(5)
free:	$M_r=0$,	$Q=0$.	

In their attemps to compute the natural frequencies of axisymmetric circular and annular plates Soni and Amba-Rao[11] and Gupta and Lal[12] formed a fourth order differential equation in terms of ϕ by eliminating w, and applied Chebyshev collocation method. The boundary conditions that did not contain the eigenvalue were combined with the governing equations to form the characteristic equations from which the eigenvalues were calculated. Their formulations, however, did not lead to a standard system equation of eigenvalue problems making it more difficult to use subroutine libraries to find the eigenvalues.

The author tried to make the most of the conceptual simplicity of the pseudospectral method and pursued the solution of (4) without eliminating w. But the efforts to compute the eigenvalues using the standard set of Chebyshev polynomials as basis functions and including the boundary conditions in the system equation as side constraints were plagued by unphysical complex eigenvalues and spurious roots. Zebib[15] reported that spurious roots occurring in the solution of hydrodynamic stability eigenvalue problems could be removed by the introduction of the Galerkin spectral method.

Lee *et al*[16] also showed the spurious roots in the solution of the Rayleigh convection problem were suppressed by the Galerkin spectral method and compared the results with those of pseudospectral method. It was decided to apply the pseudospectral method to the title problem using test functions that satisfy the boundary conditions as basis functions.

The distance from the origin, r, is normalized as

$$x=\frac{r}{R}\in[0,1] \quad (6)$$

where R is the radius of the plate, and (4) can be rewritten as

$$\frac{1}{R^2}\frac{d^2\phi}{dx^2}+\frac{1}{xR^2}\frac{d\phi}{dx}-\left(\frac{1}{x^2R^2}+\frac{x^2Gh}{D}\right)\phi-\frac{x^2Gh}{RD}\frac{dw}{dx}=-\omega^2\frac{\rho h^3}{12D}\phi\ ,$$
$$\frac{1}{R}\frac{d\phi}{dx}+\frac{\phi}{xR}+\frac{1}{R^2}\frac{d^2w}{dx^2}+\frac{1}{xR^2}\frac{dw}{dx}=-\omega^2\frac{\rho}{x^2G}w\ . \quad (7)$$

Here ϕ and w are represented by the same truncation. The eigenfunction expansions are then given by

$$\psi(x)=\sum_{n=1}^{N} a_n A_n(x) \ , \quad w(x)=\sum_{n=1}^{N} b_n B_n(x) \tag{8}$$

where a_n and b_n are the expansion coefficients, $A_n(x)$ and $B_n(x)$ are one-dimensional basis functions. It is worthwhile to note that $\psi(x)$ and $w(x)$ in the axisymmetric analysis possess odd and even parities, respectively. The eigenfunctions can be expanded using only basis functions of the same parity [10].

The pseudospectral algebraic system is formed by setting the residuals of (7) equal to zero at N collocation points

$$x_i=\cos\frac{\pi(2i-1)}{4N}, \quad i=1,\cdots,N \tag{9}$$

that are the roots of Chebyshev polynomials. When the expansions of (8) are substituted and collocated at x_i, (7) is rewritten as

$$\begin{aligned}
&\sum_{n=1}^{N} a_n\left\{\frac{A''_n(x_i)}{R^2}+\frac{A'_n(x_i)}{x_iR^2}-\left(\frac{1}{x_i^2R^2}+\frac{x^2Gh}{D}\right)A_n(x_i)\right\}\\
&\quad-\frac{x^2Gh}{RD}\sum_{n=1}^{N} b_nB'_n(x_i)\\
&\quad=-\omega^2\frac{\rho h^3}{12D}\sum_{n=1}^{N} a_nA_n(x_i)\ ,\\
&\sum_{n=1}^{N} a_n\left(\frac{A'_n(x_i)}{R}+\frac{A_n(x_i)}{x_iR}\right)+\sum_{n=1}^{N} b_n\left(\frac{B''_n(x_i)}{R^2}+\frac{B'_n(x_i)}{x_iR^2}\right)\\
&\quad=-\omega^2\frac{\rho}{x^2G}\sum_{n=1}^{N} b_nB_n(x_i)
\end{aligned} \tag{10}$$

where $'$ stands for differentiation with respect to x. The unknowns in (10) are ω and the expansion coefficients of (8). Equation (10) can also be written in matrix form

$$\boldsymbol{K}\ \boldsymbol{d}=\omega^2\ \boldsymbol{H}\ \boldsymbol{d} \tag{11}$$

where $\boldsymbol{K}$ and $\boldsymbol{H}$ are matrices of size $2N\times 2N$, and the eigenvector $\boldsymbol{d}$ is defined by

$$\boldsymbol{d}=\{a_1, a_2,\cdots, a_N, b_1, b_2,\cdots, b_N\}^T. \tag{12}$$

The pseudospectral algebraic problem is solved for the eigenvalues using Eispack GRR subroutine.

3. Numerical Examples

Table 1 Convergence test of the frequency parameter λ_i^2 (clamped, ν=0.3, h/R=0.05)

	N=2	N=4	N=6	N=9	N=12	N=15
1	10.034	10.137	10.145	10.145	10.145	10.145
2	-	37.756	38.840	38.855	38.855	38.855
3	-	139.78	83.590	84.994	84.995	84.995
4	-	-	161.70	146.26	146.40	146.40
5	-	-	494.64	218.28	220.72	220.73
6	-	-	-	321.70	305.31	305.71
7	-	-	-	577.55	396.09	399.29
8	-	-	-	1330.7	516.75	499.07
9	-	-	-	-	748.74	602.11
10	-	-	-	-	1233.9	733.70
11	-	-	-	-	-	955.21
12	-	-	-	-	-	1344.1
13	-	-	-	-	-	2084.7
14	-	-	-	-	-	-
15	-	-	-	-	-	-
	N=20	N=25	N=30	Irie	Liu	Liew
1	10.145	10.145	10.145	10.145	10.212	10.145
2	38.855	38.855	38.855	38.855	39.209	38.855
3	84.995	84.995	84.995	84.995	86.064	84.995
4	146.40	146.40	146.40	146.40	148.86	146.40
5	220.73	220.73	220.73			220.73
6	305.71	305.71	305.71			305.71
7	399.32	399.32	399.32			339.32
8	499.82	499.82	499.82			449.82
9	605.78	605.79	605.79			605.78
10	716.03	716.07	716.07			716.07
11	829.08	829.74	829.74			829.74
12	942.32	946.07	946.07			946.07
13	1070.2	1064.4	1064.5			1064.5
14	1255.1	1183.9	1184.5			1184.5
15	1533.1	1302.4	1305.7			1305.7

The clamped boundary condition

$$\psi=0, \quad w=0 \quad \text{at} \quad x=1 \tag{13}$$

is satisfied by the choice of basis functions

$$\begin{aligned} A_n(x) &= T_{2n+1}(x) - T_1(x), \\ B_n(x) &= T_{2n}(x) - T_0(x) \end{aligned} \tag{14}$$

where T_{2n} and T_{2n+1} are even and odd terms of Chebyshev polynomials of the first kind.

A preliminary run for the convergence check is carried out with thickness-to-radius ratio $h/R=0.05$ and the result is given in Table 1. This clearly shows the rapid convergence of the pseudospectral method that requires less than 15 terms for the first 6 eigenvalues and less than 30 terms for convergence of the lowest 15 modes to 5 significant digits. The results of Irie *et al*[6], Liu and Chen[7] and Liew *et al* [8] are also given in Table 1 for comparison. The Poisson's ratio ν is 0.3 throughout the paper. The given numbers in the table are nondimensionalized frequency parameters λ_i^2 defined as

$$\lambda_i^2 = \omega_i \frac{R^2}{\sqrt{D/\rho h}}. \tag{15}$$

Computational results with $N=30$ for various thickness-to-radius ratios of plates with clamped boundary condition are given in Table 2.

The simply supported boundary condition

$$\frac{d\psi}{dx} + \nu\psi = 0, \quad w=0 \quad \text{at} \tag{16}$$

is satisfied when

$$\begin{aligned} A_n(x) &= T_{2n+1}(x) - T_1(x) - \frac{4n(n+1)}{1+\nu}x \\ B_n(x) &= T_{2n}(x) - T_0(x) \end{aligned} \tag{17}$$

are selected as the basis functions for $\psi(x)$ and $w(x)$.

The correction term $-\frac{4n(n+1)}{1+\nu}x$ appearing in $A_n(x)$ is added to account for the first condition of (16). The computed eigenvalues of the circular plates with simply supported boundary condition are shown in Table 3 for various thickness-to-radius ratios.

The basis functions for $\psi(x)$ and $w(x)$ are required to satisfy

$$\frac{d\psi}{dx} + \nu\psi = 0, \quad \psi + \frac{1}{R}\frac{dw}{dx} = 0 \quad \text{at} \quad x=1$$

for the free edge. The solution of the differential condition $\frac{d\psi}{dx} + \nu\psi = 0$ at $x=1$ is the same as in the simply supported boundary condition. The mixed differential condition $\psi + \frac{1}{R}\frac{dw}{dx} = 0$ at $x=1$, however, makes it difficult to select suitable basis functions for w. Since $\psi + \frac{1}{R}\frac{dw}{dx}$ is an odd function of x and vanishes at $x=1$, it is more convenient to handle $\psi + \frac{1}{R}\frac{dw}{dx}$ rather than w. The eigenfunction expansions are redefined as

$$\begin{aligned} \psi(x) &= \sum_{n=1}^{N} a_n A_n(x), \\ \psi + \frac{1}{R}\frac{dw}{dx} &= \sum_{n=1}^{N} c_n C_n(x). \end{aligned} \tag{19}$$

Table 2 Frequency parameter λ_i^2 (clamped boundary condition, ν=0.3, N=30)

	h/R					
	0.01	0.02	0.05	0.1	0.15	0.2
1	10.213	10.204	10.145	9.941	9.629	9.240
2	39.734	39.620	38.855	36.479	33.393	30.211
3	88.926	88.401	84.995	75.664	65.551	56.682
4	157.65	156.08	146.40	123.32	102.09	85.571
5	245.74	242.06	220.73	176.52	140.93	115.56
6	352.99	345.64	305.71	232.97	180.99	145.94
7	479.19	466.04	399.32	291.71	221.62	174.97
8	624.05	602.37	499.82	351.82	262.45	178.76
9	787.27	753.72	605.79	412.77	301.11	205.32
10	968.52	919.15	716.07	474.18	305.15	210.53
11	1167.4	1097.7	829.74	535.81	336.52	237.46
12	1383.6	1288.4	946.07	597.43	345.59	248.18
13	1616.7	1490.5	1064.5	657.61	380.88	268.60
14	1866.3	1702.9	1184.5	662.37	388.16	290.67
15	2131.8	1924.8	1305.7	698.63	425.43	299.71

Table 3 Frequency parameter λ_i^2 (simply supported boundary condition, ν=0.3, N=30)

	h/R					
	0.01	0.02	0.05	0.1	0.15	0.2
1	4.934	4.933	4.925	4.894	9.629	9.240
2	29.706	29.656	29.323	28.240	33.393	30.211
3	74.050	73.751	71.756	65.942	65.551	56.682
4	137.96	136.92	130.35	113.57	102.09	85.571
5	221.31	218.65	202.81	167.53	140.93	115.56
6	323.85	318.27	286.79	225.34	180.99	145.94
7	445.52	435.05	380.13	285.44	221.62	174.97
8	585.92	568.13	480.94	346.83	262.45	178.76
9	744.86	716.61	587.65	408.91	301.11	205.32
10	921.83	879.52	698.96	471.31	305.15	210.53
11	1116.8	1056.0	813.86	533.80	336.52	237.46
12	1329.1	1245.0	931.50	596.23	345.59	248.18
13	1558.4	1445.6	1051.2	649.29	380.88	268.60
14	1804.5	1657.0	1172.6	658.55	388.16	290.67
15	2066.6	1878.2	1295.1	677.58	425.43	299.71

Table 4 Frequency parameter λ_i^2 (free boundary condition, ν=0.3, N=30)

	h/R					
	0.01	0.02	0.05	0.1	0.15	0.2
1	9.002	8.998	8.969	8.868	8.710	8.505
2	38.416	38.335	37.787	36.041	33.674	31.111
3	87.609	87.189	84.443	76.676	67.827	59.645
4	156.37	155.04	146.76	126.27	106.40	90.059
5	244.53	241.31	222.38	181.46	146.83	120.57
6	351.89	345.31	308.98	239.98	187.79	149.63
7	478.24	466.27	404.44	300.38	228.39	171.18
8	623.31	603.32	506.96	361.73	267.32	183.36
9	786.79	755.54	615.01	423.41	297.08	199.04
10	968.36	922.00	727.37	484.93	310.03	217.13
11	1167.7	1101.7	843.04	545.74	330.92	231.82
12	1384.3	1293.8	961.26	604.75	351.70	251.78
13	1617.9	1497.3	1081.4	653.92	372.16	268.69
14	1868.1	1711.3	1202.9	667.41	397.54	285.12
15	2134.3	1934.9	1325.5	695.93	416.63	308.16

The basis functions that satisfy the boundary cond\itions are

$$A_n(x) = T_{2n+1}(x) - T_1(x) - \frac{4n(n+1)}{1+\nu}x\,,$$
$$C_n(x) = T_{2n+1}(x) - T_1(x)\,. \quad (20)$$

It is also necessary to rearrange the governing equation (7) so that ψ and $\phi = \psi + \frac{1}{R}\frac{dw}{dx}$ may appear as the dependent variables as follows

$$\frac{1}{R^2}\frac{d^2\psi}{dx^2} + \frac{1}{xR^2}\frac{d\psi}{dx} - \frac{1}{x^2R^2}\psi - \frac{x^2Gh}{D}\phi = -\omega^2\frac{\rho h^3}{12D}\psi\,,$$
$$\frac{1}{R^2}\frac{d^2\phi}{dx^2} + \frac{1}{xR^2}\frac{d\phi}{dx} - \frac{1}{x^2R^2}\phi = -\omega^2\frac{\rho}{x^2G}(\phi-\psi). \quad (21)$$

The pseudospectral algebraic system is again formed by requiring the above equations are satisfied at all N collocation points (9). When (20) is substituted and collocated at x_i, (21) is rewritten as

$$\sum_{n=1}^{N} a_n\left\{\frac{A''_n(x_i)}{R^2} + \frac{A'_n(x_i)}{x_iR^2} - \frac{A_n(x_i)}{x_i^2R^2}\right\} - \frac{x^2Gh}{D}\sum_{n=1}^{N} c_nC_n(x_i) = -\omega^2\frac{\rho h^3}{12D}\sum_{n=1}^{N} a_nA_n(x_i)$$
$$\sum_{n=1}^{N} c_n\left\{\frac{C''_n(x_i)}{R^2} + \frac{C'_n(x_i)}{x_iR^2} - \frac{C_n(x_i)}{x_i^2R^2}\right\} = -\omega^2\frac{\rho}{x^2G}\sum_{n=1}^{N}\{c_nC_n(x_i) - a_nA_n(x_i)\} \quad (22)$$

The eigenvalue problem (22) is solved using Eispack GRR subroutine. The computational results with free boundary condition are given in Table 4. It is also found that, for the same vibration mode and boundary condition, smaller truncation N is required for the plate with larger thickness-to-radius ratio h/R.

4. Conclusions

The pseudospectral method using Chebyshev polynomials as basis functions is applied to the free vibration analysis of axisymmetric circular plates based on Mindlin theory.

Rapid convergence and accuracy as well as the conceptual simplicity of the pseudospectral method is achieved when it is applied to the eigenvalue problems. It is observed that the choices of basis eigenfunctions that satisfy the boundary conditions suppresses spurious eigenvalues that were present when the standard set of Chebyshev polynomials was used as basis functions with the boundary conditions included in the system equation as side constraints. Numerical examples of circular plates with clamped, simply supported and free boundary conditions are provided for various thickness-to-radius ratios.

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