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J-INVARIANT SUBMANIFOLDS OF CODIMENSION 2 IN A COMPLEX PROJECTIVE SPACE

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ABSTRACT. In this paper we prove that if M is a J-invariant submanifold of codimension 2 in a complex projective space $P_{n+1}(C)$, and the second fundamental tensor is cyclic-parallel or M has harmonic curvature, then M is locally complex quadric $Q_n(C)$ or $P_n(C)$.

1. Introduction

Let $\overline{M}$ be a Kaehlerian manifold of real dimension $2n+2$ equipped with almost complex structure J. Then for any vector field X on M, we have $J^2X = -X$, $\overline{\nabla}J = 0$, where $\overline{\nabla}$ denotes the Riemannian connection of $\overline{M}$ (cf. [1, 3–12]).

Let M be a $2n$-dimensional J-invariant submanifold of codimension 2 covered by a system of coordinate neighborhood $\{U; x^h\}$ (cf. [2, 9]) and isometrically immersed in $\overline{M}$ by the immersion $i : M \to i(M)$. Throughout this paper the indices, i, j, k, $\cdots$ run from 1 to $2n$ and the summation convention will be used with respect those indices. We represent the immersion i locally by

$$y^A = y^A(x^h) \ (A = 1, \cdots, 2n+2)$$

and put $B_j^A = \partial_j y^A$, $(\partial_j = \frac{\partial}{\partial x^j})$ then $B_j = (B_j^A)$ are $2n$-linearly independent local tangent vector fields of M. We choose two mutually orthogonal unit vectors C and D normal to M in such a way that B_i, C and D give the positive orientation of $\overline{M}$ [3].

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By denoting ∇_j the operator of Wan der Waerden-Bortolotti covariant differentiation, equations of the Gauss and Weingarten for the submanifolds M are respectively given by

$$\nabla_j B_i = H_{ji}C + K_{ji}D, \tag{1.1}$$

$$\nabla_j C = -H_j^h B_h + l_j D, \quad \nabla_j D = -K_j^h B_h - l_j C, \tag{1.2}$$

where H_{ji} and K_{ji} are components of the second fundamental tensors and the shape operator H and K are related by

$$H = (H_{ji}g^{ih}),\ K = (K_{ji}g^{ih}),\ g^{ji} = (g_{ji})^{-1},$$

and l_j is the so-called third fundamental tensor.

Since M is assumed to be J-invariant submanifold, the transformation of B_i, C and D by J are respectively represented in each coordinate neighborhood as follows :

$$JB_i = \phi_j^h B_h,\ JC = \lambda D,\ JD = -\lambda C, \tag{1.3}$$

where ϕ defines a skew-symmetric transformation on the tangent bundle of M, λ is a function on M.

By the properties of the Kaehlerian structure J, it follows from (1.3) that

$$\phi^2 = -I, \quad \lambda^2 = 1$$

where I denotes the identity transformation on M. Since we have $\overline{\nabla} J = 0$ by using (1.1), (1.2) and (1.3), we have

$$\nabla_k \phi_i^h = 0, \tag{1.4}$$

$$\lambda H_{ji} = K_{jt}\phi_i^t, \quad -\lambda K_{ji} = H_{jt}\phi_i^t. \tag{1.5}$$

From the last equation, it is verify that

$$\mathrm{Tr}H = \mathrm{Tr}K = 0, \tag{1.6}$$

$$H^2 = K^2, \quad HK + KH = 0. \tag{1.7}$$

We suppose that the ambient space is a complex projective space $P_{n+1}(C)$ of constant holomorphic sectional curvature 4. Then the Gauss, Codazzi and Ricci of M are given respectively by

$$\begin{aligned} R_{kjih} = {} & g_{kh}g_{ji} - g_{jh}g_{ki} + \phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih} \\ & + H_{kh}H_{ji} - H_{jh}H_{ki} + K_{kh}K_{ji} - K_{jh}K_{ki}, \end{aligned} \tag{1.8}$$

$$\begin{cases} \nabla_k H_{ji} - \nabla_j H_{ki} - l_k K_{ji} + l_j K_{ki} = 0, \\ \nabla_k K_{ji} - \nabla_j K_{ki} - l_k H_{ji} - l_j H_{ki} = 0, \end{cases} \tag{1.9}$$

$$\nabla_k l_j - \nabla_j l_k + H_{kt}K_j^t - H_{jt}K_k^t = 2\lambda\phi_{kj}, \tag{1.10}$$

where R_{kjih} are components of the Riemannian curvature tensor of M.

By (1.7) and (1.8), the Ricci tensor of M can be written as

$$S_{ji} = 2(n+1)g_{ji} - 2H_{ji}^2, \tag{1.11}$$

where $H_{ji}^2 = H_{jr}H_i^r$.

2. Theorems

Let z_0, z_1, $\cdots$, z_{n+1} be a homogeneous coordinate system of $P_{n+1}(C)$ and let

$$Q_n(C) = \{(z_0, Z_1, \cdots, z_{n+1}) \in P_{n+1}(C) \mid \sum z_i^2 = 0\}.$$

Then $Q_n(C)$ is called a complex quadric [7].

First of all we prove

LEMMA 1. *Let M be a J-invariant submanifold of codimension 2 in $P_{n+1}(C)$. If the normal connection of M is trivial, then $M = Q_n(C)$.*

Proof. From the second equation of (1.7) and (1.10), we see that $H_{kr}K_j^r = \lambda\phi_{kj}$ or using (1.5) $H_{ji}^2 = g_{ji}$. Thus (1.11) becomes

$$S_{ji} = 2ng_{ji}$$

which means $M = Q_n(C)$. □

THEOREM 2. *Let M be a J-invariant submanifold of codimension 2 in $P_{n+1}(C)$. If the second fundamental tensor H is cyclic-parallel [4], then $M = Q_n(C)$ or P_nC.*

Proof. Since $\nabla_k H_{ji} + \nabla_j H_{ik} + \nabla_i H_{kj} = 0$, by (1.5) we have

$$(\nabla_k K_{jt})\phi_i^t + (\nabla_j K_{it})\phi_k^t + (\nabla_i K_{kt})\phi_j^t = 0,$$

or using (1.9), we have

$$(\nabla_k K_{jt})\phi_i^t + (\nabla_k K_{jt} - l_j H_{kt} + l_k H_{jt})\phi_i^t + (\nabla_k K_{jt})\phi_i^t - l_i H_{kt}\phi_j^t + l_k H_{it}\phi_j^t = 0.$$

Since ϕ is parallel, it turns out to be

$$3\nabla_k H_{ji} = 2l_k K_{ji} - l_j K_{ki} - l_i K_{jk}, \tag{2.1}$$

where we have used (1.5). Taking the trace, we see, using (1.6), that

$$K_{kt}l^t = 0, \tag{2.2}$$

which implies $H_{jt}l^t = 0$ because of (1.5). Differentiating the second equation of (1.5) along M, we find

$$(\nabla_k H_{jt})\phi_i^t + \lambda \nabla_k K_{ji} = 0.$$

Substituting (2.1) into this and making use of (1.6), we obtain

$$\lambda\{3\nabla_k K_{ji} + 2l_k H_{ji} - l_j H_{kt}\} - K_{jk}\phi_{it}l^t = 0,$$

form which, taking the skew-symmetric part with respect to indices j and i,

$$K_{ki}\phi_{jt}l^t - K_{jk}\phi_{it}l^t + \lambda(l_i H_k j - l_j H_{ki}) = 0.$$

ϕ_{ji} being skew-symmetric, by using (2.2) it follows that

$$(l_t l^t)H_{ki} = 0.$$

By Lemma 1, $M = Q_n(C)$ or $P_n(C)$. Thus, the proof is finished. □

THEOREM 3. *Let M be a J-invariant submanifold of codimension 2 in $P_{n+1}(C)$. If M has harmonic curvature, then $M = Q_n(C)$ or $P_n(C)$ is locally symmetric.*

Proof. Since M has harmonic curvature (cf. [2, 5]), the Ricci Tensor S is of Codazzi type. Namely we have $\nabla_k S_{ji} = \nabla_j S_{ki}$. Thus by using (1.9) and (1.11), we have

$$H_{jr}\nabla_k H_i^r - H_{kr}\nabla_j H_i^r + H_{ir}(l_k K_j^r - l_j K_k^r) = 0. \tag{2.3}$$

Now, we put

$$\nabla_k h_{ji} = \nabla_k H_{ji} - l_k K_{ji}, \tag{2.4}$$

then we see, using (1.9), that $h_j i$ is of Codazzi type and

$$\nabla_t h_j^t = 0. \tag{2.5}$$

Thus (2.3) is reduced to

$$H_{jr}(\nabla_k h_i^r) - H_{kr}(\nabla_j h_i^r) = 0,$$

where we have used the second equation of (1.7). Differentiating this covariantly and taking account of (2.4), we find

$$\begin{aligned}(\nabla_l h_{jr})(\nabla_k h_i^r) - (\nabla_l h_{kr})(\nabla_j h_i^r) + H_{jr}\nabla_l\nabla_k h_i^r \\ -H_{kr}\nabla_l\nabla_j h_i^r + l_l(K_{jr}\nabla_k h_i^r - K_{kr}\nabla_j h_i^r) = 0.\end{aligned} \tag{2.6}$$

Since we can write

$$S_{ji} = 2(n+1)g_{ji} - 2K_{ji}^2,$$

we also have

$$K_{jr}\nabla_k K_i^r - K_{kr}\nabla_j K_i^r + K_{ir}(l_j H_k^r - l_k H_j^r) = 0.$$

Applying by ϕ_l^i and making use of (1.4), (1.5) and (2.4), we find

$$K_{jr}\nabla_k h_i^r - K_{kr}\nabla_j h_i^r = 0.$$

Therefore (2.6) turns out to be

$$(\nabla_l h_{jr})(\nabla_k h_i^r) - (\nabla_l h_{kr})(\nabla_j h_i^r) + H_{jr}\nabla_l\nabla_k h_i^r - H_{kr}\nabla_l\nabla_j h_i^r = 0.$$

If we take the skew-symmetric put with respect to l and i, and use the Ricci identity, then we obtain

$$\begin{aligned}(2.7)\qquad & 2\{(\nabla_l h_{jr})(\nabla_k h_i^r) - (\nabla_j h_{ir})(\nabla_k h_l^r)\} \\ & = R_l iks H_{jr}h^{rs} + R_{lirs}H_j^r h_k^s - R_{lijs}H_{kr}h^{sr} - R_{lirs}h_j^s H_k^r.\end{aligned}$$

Summing up (2.7) with respect to k and i, and using (2.5), we get

$$S_{ls}H_{jr}h^{sr} - R_{tlrs}H_j^r h^{ts} + R_{tljs}H_r^t h^{sr} + R_{tlrs}H^{tr}h_j^s = 0,$$

which an able us to obtain

$$S_{ji}H_r^j h^{ir} = R_{kjih}h^{ji}H^{kh}.$$

Summing up (2.7) $k = l$ and $i = j$ and taking account of the last equation, we obtain $(\nabla_k h_{ji})(\nabla^k h^{ji}) = 0$ and hence

$$(2.8)\qquad \nabla_k H_{ji} = l_k K_{ji}.$$

So we have

$$(2.9)\qquad \nabla_k K_{ji} = -l_k H_{ji}$$

because of (1.4) and (1.5). Differentiating (2.8) covariantly and using (2.9), we find

$$\nabla_k\nabla_j H_{ik} = (\nabla_k l_j)K_{ih} - l_j l_k H_{ik},$$

or using the Ricci identity

$$-R_{kjir}H_h^r - R_{kjhr}H_i^r = K_{ih}(\nabla_k l_j - \nabla_j l_k).$$

If we apply by K^{ih}, then we have

$$-2R_{kjih}H_t^h K_{ti} = h(2)(\nabla_k l_j - \nabla_j l_k)$$

by virtue of the first equation of (1.7), which together with (1.7) yields

$$h(2)(\nabla_k l_j - \nabla_j l_k + 4\lambda\phi_{kj}) = 0.$$

Since we have from (1.7) and (1.10)

$$\nabla_k l_j - \nabla_j l_k = H_{jr}K_k^r + 2\lambda\phi_{kj},$$

it follows that we obtain $H_{jt}K_k^t = 3\lambda\phi_{kj}$ and hence

$$(2.10)\qquad H_{ji}^2 = 3g_{ji} \quad \text{or} \quad h(s) = 0.$$

Thus (1.11) turns out to be

$$S_{ji} = 2(n-2)g_{ji}. \tag{2.11}$$

This completes the proof of the theorem by Lemma 1. □

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