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NOTE ON NORMAL EMBEDDING

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ABSTRACT. It was shown by L. Polterovich ([3]) that if L is a totally real submanifold of a symplectic manifold (M, ω) and L is parallelizable then L is normal. So we try to find an answer to the question of whether there is a compatible almost complex structure J on the symplectic vector bundle $TM|_L$ such that $TL \cap JTL = 0$ assuming L is normal and parallelizable. Although we could not reach an answer, we observed that the claim holds at the vector space level. And related to the question, we showed that for a symplectic vector bundle (E, ω) of rank $2n$ and $E = E_1 \oplus E_2$, where E_1, E_2 are Lagrangian subbundles of E, there is an almost complex structure J on E compatible with ω and $JE_1 = E_2$. And finally we provide a necessary and sufficient condition for a given embedding into a symplectic manifold to be normal.

1. Introduction

The notion of normal submanifold was introduced by J. C. Sikorav ([4]) as a weaker version of Lagrangian submanifold. He proved among others that if a sequence of Lagrangian embeddings converges to a normal embedding, then in fact the limit is Lagrangian.

For the basic notions such as 'Lagrangian', 'normal' and for any other unfamiliar notions or notations in this introduction, one may refer to Section 2 below.

On the other hand, L. Polterovich ([3]) showed that if L is a totally real submanifold of a symplectic manifold (M, ω) and L is parallelizable, then L is normal. So we have a question whether a parallelizable normal submanifold is totally real or not. In fact this is not true if we fix the almost complex structure, which we show by providing a counter

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example in Section 3 below. Therefore we ask the question of whether there is a compatible almost complex structure J on the symplectic vector bundle $TM|_L$ such that $TL \cap JTL = 0$. Although we could not reach an answer, we observed that the claim holds at the vector space level as the following shows:

PROPOSITION 1. *Let* (V, ω) *be a symplectic vector space of dimension* $2n$ *and* T *be any* n*-dimensional subspace for which a Lagrangian subspace* $\mathbb{L} \subset V$ *exists so that* $\mathbb{L} \cap T = 0$. *Then there exists a complex structure* J *on* V *such that* $T \cap JT = 0$.

For the question of whether the converse of the result by Polterovich holds, in the form of asking the existence of an appropriate almost complex structure as above, let us consider the following:

THEOREM 1. *Let* (E, ω) *be a symplectic vector bundle of rank* $2n$ *and* $E = E_1 \oplus E_2$, *where* E_1, E_2 *are Lagrangian subbundles of* E. *Then there is an almost complex structure* J *on* E *compatible with* ω *such that* $JE_1 = E_2$.

This must be a well-known standard fact. Nevertheless it is not easy to find a reference for a proof and we include one in Section 3 below.

Even if we are unable to resolve the question of whether a normal embedding of parallelizable manifold into a symplectic manifold is totally real with respect to some almost complex structure, we provide a special case in which it is true below the proof of Proposition 1.

And we provide a necessary and sufficient condition for a given embedding into $\mathbb{R}^{2n}$ to be normal as follows:

THEOREM 2. *Let* $f : M^n \longrightarrow \mathbb{R}^{2n}$ *be an embedding of an* n*-dimensional smooth manifold* M. *Then* f *is a normal embedding if and only if there exists a Lagrangian subbundle* $\mathbb{L}$ *of* $f^*T\mathbb{R}^{2n}$ *such that* $\mathbb{L} \cap TM$ *is a real vector bundle over* M *and, writing* $A = (\mathbb{L} \cap TM)^{\perp_g}$ *in* $\mathbb{L}$, $A^{\perp_\omega} + TM = f^*T\mathbb{R}^{2n}$.

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2. Basic notions and facts

A smooth manifold M is called *symplectic* if there is a nondegenerate closed 2-form ω on M. Such a 2-form is called a *symplectic form* or a *symplectic structure* on M.

In a similar vein, a vector bundle of finite rank is referred to as a *symplectic vector bundle* if it is considered with a fixed nondegenerate 2-form, which is usually referred to as a symplectic form. Nevertheless one should note that the closedness condition for the symplectic form of a manifold cannot be stated in this more general setting. A subbundle η of a symplectic vector bundle ξ is a *Lagrangian subbundle* if $2(\text{rank } \eta) = \text{rank } \xi$ and the restriction of the symplectic form to η is the zero form.

Let M be a symplectic manifold. Note that dim M is even. Let L be a smooth manifold of dimension $\frac{1}{2}\dim M$ and let $f : L \to M$ be an embedding (resp. immersion) of L into a symplectic manifold M with a symplectic structure ω. We call f a *Lagrangian embedding* (resp. *immersion*) if the tangent bundle TL of L is a Lagrangian subbundle of the symplectic vector bundle f^*TM with the symplectic form $f^*\omega$. We call f a *normal embedding* (resp. *immersion*) if there is a Lagrangian subbundle $\mathbb{L}$ of f^*TM which is transverse to TL. Note that every Lagrangian submanifold L of M is normal.

We will consider $\mathbb{C}^n$ with the usual symplectic structure. A Lagrangian embedding or a normal embedding must be understood as 'into $\mathbb{C}^n$' unless otherwise specified.

Let (V, ω) be a symplectic vector space. A complex structure J is said to be *compatible with* ω if $\omega(Jv, Jw) = \omega(v, w)$ for all $v, w \in V$ and $\omega(v, Jv) > 0$ for all nonzero $v \in V$. That is, a complex structure J is compatible with ω if and only if the bilinear form $g_J : V \times V \to \mathbb{R}$ defined by $g_J(v, w) = \omega(v, Jw)$ is symmetric, positive definite, and J-invariant. Let $\mathcal{J}(V, \omega)$ be the space of complex structures on (V, ω) compatible with ω. Then it is well known that $\mathcal{J}(V, \omega)$ is homeomorphic to the space $\mathcal{P}$ of symmetric positive definite symplectic matrices and thus $\mathcal{J}(V, \omega)$ is contractible (See [1]). These notions transfer to a symplectic bundle and one refers to the smooth family of compatible complex structures on the fibers as the compatible almost complex structures. And a subbundle η of rank n of a symplectic bundle ξ of rank $2n$ is totally real with respect to an almost complex structure J if $\eta \cap J\eta = 0$, where 0 denotes the zero section.

For a subspace A of a symplectic vector space (V, ω), denote $A^{\perp_\omega}$ and $A^{\perp_g}$ as follows.

$$A^{\perp_\omega} = \{v \in V \,|\, \omega(v, a) = 0, \text{ for any } a \in A\},$$
$$A^{\perp_g} = \{v \in V \,|\, g_J(v, a) = 0, \text{ for any } a \in A\}.$$

Here J denotes an almost complex structure compatible with ω and

g_J the Riemannian metric induced by ω and J. Note that if A is a subbundle of a symplectic vector bundle (V, ω) then $A^{\perp_\omega}$ and $A^{\perp_g}$ form subbundles of (V, ω).

3. Proofs of the main results

It is shown that if L is a totally real submanifold of a symplectic manifold (M, ω) and L is parallelizable, then L is normal (See [3]). So we have a question whether the converse holds true or not.

Consider the symplectic vector space $\mathbb{C}^{2n}$ with the standard symplectic structure ω_0. We also consider the complex structure J given by the multiplication by $i = \sqrt{-1}$. Denote by e_i, $i = 1, 2, \cdots, 2n$, the standard basis for the 'real' part of $\mathbb{C}^{2n}$ and f_i, $i = 1, 2, \cdots, 2n$, for the 'imaginary' part of $\mathbb{C}^{2n}$.

Then $T = \mathbb{C}^n \times 0$ is a subspace as far as possible from being totally real; it is a complex vector space. However there is a Lagrangian subspace $\mathbb{L}_0$ transverse to T which in fact we may construct as a subspace of $\mathbb{C}^{2n}$ spanned by the following $2n$ vectors

$$e_i + e_{n+i}, f_i - f_{n+i}, \; i = 1, 2, \cdots, n.$$

Thus we have an example of a symplectic submanifold which is also a normal submanifold by regarding $\mathbb{C}^{2n}$ as a symplectic manifold and T as a submanifold and considering $\bar{\mathbb{L}}_0$ the subbundle of $T\mathbb{C}^{2n}|_T$ whose fibers are parallel translations of $\mathbb{L}_0$.

The question still remaining is whether it is possible to find an almost complex structure J by which a given normal immersion(or embedding) becomes totally real. For example one may ask whether the standard almost complex structure can be replaced by a new one so that T is totally real in the above, which can be indeed done as the Proposition 1 shows. Now we provide:

Proof of Proposition 1. It is straightforward to construct a basis $v_1, v_2, \cdots, v_n$ for T such that, for some k, we have: $\omega(v_i, v_j) = 0$ if $\{i, j\} \neq \{2\iota - 1, 2\iota\}$ and $\omega(v_{2\iota-1}, v_{2\iota}) = 1$ for $\iota = 1, 2, \cdots, k$. We need the following fact to complete the proof.

LEMMA 1. *There is a basis $w_1, w_2, \cdots, w_n$ for $\mathbb{L}$ so that $\omega(v_i, w_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker index.*

Proof. Let $\mathbb{L}^i$ be the space defined as the kernel of the linear functional $\omega(v_i, \cdot) : \mathbb{L} \to \mathbb{R}$.

Note that, for any $i = 1, 2, \cdots, n$, the dimension of

$$\mathbb{L}^1 \cap \mathbb{L}^2 \cap \cdots \cap \widehat{\mathbb{L}^i} \cap \cdots \cap \mathbb{L}^n$$

is greater than or equal to 1, where the hat means the one under it is deleted. Further note that if w is a non-zero element of this subspace, then $\omega(v_i, w)$ cannot be zero, for, otherwise $\omega(\cdot, w) : V \to \mathbb{R}$ is the zero map.

Thus we can choose non-zero elements w_i for $i = 1, 2, \cdots, n$ in

$$\mathbb{L}^1 \cap \mathbb{L}^2 \cap \cdots \cap \widehat{\mathbb{L}^i} \cap \cdots \cap \mathbb{L}^n$$

so that $\omega(v_i, w_i) = 1$. Then $w_1, w_2, \ldots, w_n$ is a basis for $\mathbb{L}$ so that $\omega(v_i, w_j) = \delta_{ij}$. □

Now it is easy to see that $u_1, u_2, \ldots, u_n$ defined by $u_{2i-1} = v_{2i-1}$, $u_{2i} = v_{2i} - w_{2i-1}$ for $i \leq k$ and $u_i = v_i$ for $2k < i \leq n$, together with $w_1, w_2, \cdots, w_n$ form a symplectic basis. Consider the almost complex structure J defined by

$$Ju_i = w_i, Jw_i = -u_i, \; i = 1, 2, \ldots, n.$$

It must be clear that J is compatible with ω.

Then we have $Jv_{2i} = J(u_{2i} + w_{2i-1}) = w_{2i} - u_{2i-1} = w_{2i} - v_{2i-1}$ for $i \leq k$. And we have $Jv_i = w_i$ if i is odd or greater than $2k$. Note that $T + JT$ contains $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$, which means $T + JT = V$. We conclude that $T \cap JT = 0$ as desired. □

From the vector space case, we obtain an obvious partial result, even if not very much significant, concerning the question of whether a submanifold which is both normal and parallelizable is necessarily totally real with respect to an appropriate almost complex structure. Before proceeding any further, we provide a terminology which nevertheless very much standard: If E is a real vector bundle of rank $2n$ with a nondegenerate 2-form ω over M, then by a symplectic frame $e_1, e_2, \cdots, e_n$, $f_1, f_2, \cdots, f_n$ of E, we mean a frame such that $\omega(e_i, f_j) = \delta_{ij}$ and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ for any i, j. Note that it is possible to extend an almost complex structure on f^*TN to the whole TN since the space of complex structure of a symplectic vector space compatible with the symplectic form is contractible. Then we summarize as follows:

Proposition 2. *Let $f : M^n \to N^{2n}$ be a normal embedding of a smooth manifold M into a symplectic manifold N and assume the subbundle $TM \subset f^*TN$ decomposes as $E_0 \oplus E_1$, where E_0, E_1 are subbundles on which $f^*\omega$ is respectively zero and nondegenerate and $\omega(v_x, w_x) = 0$ for any $v_x \in E_{0x}, w_x \in E_{1x}$ and for any $x \in M$. If E_0 and E_1 are trivial as real vector bundles and in particular E_1 admits a symplectic frame, then f is totally real with respect to an almost complex structure of N.*

Proof. Note that in the proof of Proposition 1 and in Lemma 1, $w_1, w_2, \cdots, w_n \in \mathbb{L}$ are in fact uniquely determined once $v_1, v_2, \cdots, v_n \in T$ are given. □

For instance it is not easy to remove from the above the condition that E_1 admit a symplectic frame: First we note that a symplectic bundle is a complex vector bundle with respect an almost complex structure compatible with the symplectic structure and it is not hard to show that the complex bundle structure does not depend on the choice of such an almost complex structure. In this sense we may say that a symplectic bundle admits a symplectic frame if and only if it is trivial as a complex vector bundle. Now assume E is a trivial real vector bundle of rank $2n$ with a nondegenerate 2-form ω over M and try to prove there is a symplectic frame for E by an inductive process. First choose an almost complex structure J on E compatible with ω and let g denote the Riemannian metric determined by J and ω. Let e_1 be any nowhere vanishing section of E. Set $f_1 = Je_1$. Then $\bar{E}_1 = \{v_x \in E | \omega(e_1(x), v_x) = 0\}$ is a subbundle of E of rank $2n-1$. Write $E^1 = \{v_x \in \bar{E}_1 | g(f_1(x), v_x) = 0\}$. Then it is easy to see that ω is nondegenerate on E^1. Nevertheless, in general E^1 is not trivial as a real vector bundle, which fails our inductive attempt. Indeed there is an example in which a bundle ξ is a symplectic bundle which is not a trivial complex vector bundle while it is trivial as a real vector bundle. Consider the Whitney sum $\xi = TS^2 + \epsilon^1_{\mathbb{C}}$, where we regard S^2 as a symplectic manifold and $\epsilon^1_{\mathbb{C}}$ as the trivial symplectic bundle whose fiber is $\mathbb{C}$. Then ξ is not a trivial complex vector bundle since its first Chern class is nonzero while trivial as a real bundle: the real bundle TS^2 is well-known to be stably trivial.

Now let (E, ω) be a symplectic vector bundle of rank $2n$ over M and $E = E_1 \oplus E_2$, where E_1 is a Lagrangian subbundle of E, and J be an almost complex structure on E compatible with ω and $JE_1 = E_2$. Then for any $v_{2x}, w_{2x} \in E_{2x}$, $x \in M$, we have $\omega(v_{2x}, w_{2x}) = \omega(Jv_{2x}, Jw_{2x}) =$

0. So E_2 should be a Lagrangian subbundle of E. Thus we obtain an observation that for a bundle $E = E_1 \oplus E_2$, where E_1 is Lagrangian and E_2 is not Lagrangian, there is no almost complex structure J compatible with ω and $JE_1 = E_2$. Now we consider the case that L_1 and L_2 are Lagrangian, that is, we prove Theorem 1. For the proof we need Lemmas 2 and 3 below.

LEMMA 2. *Let (V, ω) be a symplectic vector space such that $V = L_1 \oplus L_2$, where L_1, L_2 are Lagrangian subspaces of V. Then there exists a symplectomorphism $\Phi : (V, \omega) \to (\mathbb{R}^{2n}, \omega_0)$ such that $\Phi(L_1) = \mathbb{R}^n \times 0, \Phi(L_2) = 0 \times \mathbb{R}^n$.*

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be any basis for L_1 and W be the subspace spanned by $\{e_1, e_2, \cdots, \widehat{e_i}, \ldots, e_n\}$, where the hat means the one under it is deleted. Then $L_1 \subset W^{\perp_\omega}$ and the dimension of $W^{\perp_\omega}$ is $n+1$. And since $W^{\perp_\omega} + L_2 = \mathbb{R}^{2n}$ we have the dimension of $W^{\perp_\omega} \cap L_2$ is 1.

Let v be any vector in $W^{\perp_\omega} \cap L_2$ such that $\omega(v, e_i) = 0$. Then we have: $v \in (\mathrm{Span}(V \cup \{e_i\}))^{\perp_\omega} = L_1^{\perp_\omega} = L_1$ and we conclude that $v = 0$. Therefore we may choose a unique nonzero $v \in W^{\perp_\omega} \cap L_2$ so that $\omega(e_i, v) = 1$, which we call f_i. Then we have that $\omega(e_i, f_j) = \delta_{ij}, i, j = 1, 2, \cdots, n$, where δ_{ij} denotes the Kronecker index.

Now define a map $\Phi : (V, \omega) \to (\mathbb{R}^{2n}, \omega_0)$ by

$$p_j \Phi(e_i) = \delta_{ij},\ q_j \Phi(f_i) = \delta_{ij},\ \text{and } q_j \Phi(e_i) = 0,\ p_j \Phi(f_i) = 0$$

for any $i, j = 1, 2, \cdots, n$, where $p_j, q_j : \mathbb{R}^{2n} \to \mathbb{R}$ are the projections to the real and the imaginary coordinates respectively. Then Φ is a symplectomorphism. □

The following is a key step to show that the observation above persists to the bundle case.

LEMMA 3. *Let (V, ω) be a $2n$-dimensional symplectic vector space and $V = L_1 \oplus L_2$, where L_1, L_2 are Lagrangian subspaces of V. Then the space $\mathcal{J}(V, \omega, L_1, L_2)$ of complex structures on (V, ω) compatible with ω and $J(L_1) = L_2$ is homeomorphic to the space $\mathcal{Q}$ of $n \times n$ symmetric, positive definite matrices. And thus $\mathcal{J}(V, \omega, L_1, L_2)$ is contractible.*

Proof. By the above Lemma, we may assume that $V = \mathbb{R}^{2n}$,

$$\omega = \omega_0 = dx_1 \wedge dy_1 + dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n,$$

$$L_1 = \{(x_1, x_2, \dots , x_n, 0, 0, \dots , 0) \in \mathbb{R}^{2n} | \, x_i \in \mathbb{R}, i = 1, 2, \dots , n\},$$

and

$$L_2 = \{(0, 0, \dots , 0, y_1, y_2, \dots , y_n) \in \mathbb{R}^{2n} | \, y_i \in \mathbb{R}, i = 1, 2, \dots , n\}.$$

Consider a complex structure J as a $2n \times 2n$ matrix. Then J is compatible with ω_0 and $JL_1 = L_2$ if and only if

(1) $J^2 = -Id_{2n}$, $J^t J_0 J = J_0$, $\langle v, -J_0 J v\rangle > 0$ for all $v \neq 0$, $JL_1 = L_2$,

where $J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and $\langle \cdot , \cdot \rangle$ is the standard metric on $\mathbb{R}^{2n}$.

Let $J = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A, B, C, D \in GL(n, \mathbb{R})$. Then by the above equation (1), we have $A = D = 0, C = -B^{-1}$ and B is symmetric and negative definite. So

$$\mathcal{J}(V, \omega, L_1, L_2)$$
$$= \left\{ \begin{pmatrix} 0 & B \\ -B^{-1} & 0 \end{pmatrix} \in GL(2n, \mathbb{R}) | \, B \text{ is symmetric and negative definite} \right\}.$$

Thus $\mathcal{J}(V, \omega, L_1, L_2)$ is contractible. □

At last we provide:

Proof of Theorem 1. Consider the bundle $\mathcal{J}(E, \omega, E_1, E_2)$ whose fibers consist of $\mathcal{J}(E_x, \omega_x, E_{1x}, E_{2x})$ Since its fibre is contractible, there is a global section. So there is a globally defined almost complex structure J on E. □

Finally we prove Theorem 2:

Proof of Theorem 2. If f is a normal embedding then there is a Lagrangian transversal subbundle $\mathbb{L}$ of $f^*T\mathbb{R}^{2n}$. Thus $A = \mathbb{L}$ and $A^{\perp_\omega} = \mathbb{L}$. Since $\mathbb{L}$ is transversal to TM, so $A^{\perp_\omega} + TM = f^*T\mathbb{R}^{2n}$ holds.

Now we prove the converse. Since A is a subbundle of $\mathbb{L}$, we have that $A^{\perp_\omega}$ contains $\mathbb{L}$. Consider $K = \mathbb{L}^{\perp_g} \cap A^{\perp_\omega}$.

Note that $A \cap K \subset \mathbb{L} \cap K = 0$, where 0 denotes the set of zero tangent vectors.

Claim. $A \oplus K$ is a Lagrangian subbundle of $f^*T\mathbb{R}^{2n}$ and transverse to TM.

Proof of Claim. First we show that $A \oplus K$ is transverse to TM. By assumption, $A^{\perp_\omega} + TM = f^*T\mathbb{R}^{2n}$. Note that by construction we have $A^{\perp_\omega} = \mathbb{L} \oplus K$. Therefore, we have $(\mathbb{L} \oplus K) + TM = f^*T\mathbb{R}^{2n}$. Also by construction we have $\mathbb{L} = (\mathbb{L} \cap TM) \oplus A$ and

$$((\mathbb{L} \cap TM) \oplus A \oplus K) + TM = (A \oplus K) + TM = f^*T\mathbb{R}^{2n}.$$

Now consider the ranks of the bundles. Write k for the rank of $\mathbb{L} \cap TM$. Then the rank of A is $n-k$, that of $A^{\perp_\omega}$ is $n+k$ and that of K is k. Thus the rank of $A \oplus K$ is n. This shows that $A \oplus K$ is a vector bundle of rank n transverse to TM.

Next we observe that ω vanishes on $A \oplus K$: It vanishes on A since $A \subset \mathbb{L}$. It vanishes on K since $K \subset \mathbb{L}^{\perp_g}$ and $\mathbb{L}^{\perp_g} = J\mathbb{L}$ is Lagrangian. There is no nontrivial pairing between A and K since $K \subset A^{\perp_\omega}$. □

So f is a normal embedding. □

REMARKS. 1. A close look at the proof of Theorem 2 reveals that the codomain need not be $\mathbb{R}^{2n}$. In fact it can be any symplectic manifold.

2. In view of Theorem 2, one might ask how common it is that a tangent bundle admit a nontrivial subbundle. Note that a subbundle of a finite dimensional vector bundle necessarily induces a decomposition of the bundle once we choose a Riemannian metric of the ambient bundle. Using this fact it is easy to see that not every tangent bundle has a nontrivial subbundle. For example, the tangent bundle TS^{2n} of even dimensional sphere S^{2n} is not decomposable ([2], p. 100).

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