Bull. Korean Math. Soc. **39** (2002), No. 2, pp. 283–287

RESOLUTION OF THE CONJECTURE ON STRONG PRESERVERS OF MULTIVARIATE MAJORIZATION

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Abstract. In this paper, we will investigate the set of linear operators on real square matrices that strongly preserve multivariate majorization without any additional conditions on the operator. This answers earlier conjecture on nonnegative matrices in [3].

1. Introduction

Majorization is a topic of much interest in various areas of mathematics and statistics. If $\mathbf{x}$ and $\mathbf{y}$ are nonincreasing n-vectors of nonnegative real numbers such that $\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i$ for $k = 1, \ldots, n$ with equality at $k = n$, then we say that $\mathbf{x}$ is (*vector*) *majorized* by $\mathbf{y}$ and write $\mathbf{x} \prec \mathbf{y}$. Our interest is in the subject of majorization for matrices. The matrix version of majorization is that an $m \times n$ real matrix A is majorized by B if there is a matrix X such that $A = BX$. We are interested in a specific type of majorization of matrices. One such possibility is to say that A is (*matrix*) *majorized* by B, written $A \prec^{mul} B$, if there exists a *row stochastic matrix* X such that $A = BX$, which was defined in [8]. Another possibility is to say that A is *multivariate majorized* by B if there exists a *doubly stochastic matrix* D such that $A = BD$. This is motivated by the theorem of Hardy-Littlewood and Polya saying that for row vectors $\mathbf{a}$ and $\mathbf{b}$ in R^n, $\mathbf{a}$ is to be *majorized* by $\mathbf{b}$, written $\mathbf{a} \prec \mathbf{b}$, if there exists an $n \times n$ doubly stochastic matrix D such that $\mathbf{a} = \mathbf{b}D$. References on multivariate majorization are found in [1, 2, 4, 7, 13]. Let $\mathbf{DS}(n)$ be the set of $n \times n$ doubly stochastic matrices, i.e., non-negative matrices with all row sums and column sums equal to one. The set

Received June 23, 2000.

2000 Mathematics Subject Classification: 15A04, 15A21, 15A30.

Key words and phrases: majorization, multivariate majorization, strong preserver, doubly stochastic matrix .

This work was partially supported by KRF 2001-DP0005* and the BK21 program**.

$\mathbf{DS}(n)$ is closed under matrix multiplication, transposition, and convex combinations; that is, if A, B are doubly stochastic, so are AB, A^T, and $\alpha A + (1-\alpha)B$ for all $0 \le \alpha \le 1$. By Birkhoff's Theorem (e.g., see [6]), $\mathbf{DS}(n)$ is the convex hull of the set $\mathbf{P}(n)$ of $n \times n$ permutation matrices. There has been a great deal of interest in studying linear maps preserving convex sets, linear maps preserving relations and linear maps preserving certain groups (see [5, 9, 10, 11, 12, 14]). Let $\mathcal{A}$ be a linear space of matrices, T be a linear operator on $\mathcal{A}$ and $\mathcal{R}$ be a relation on $\mathcal{A}$. We say a linear operator T *strongly preserves* $\mathcal{R}$ if

$$\mathcal{R}\,(X,Y) \quad \text{if and only if} \quad \mathcal{R}(T(X),T(Y)).$$

In this paper, we will study linear operators that strongly preserve multivariate majorization. In [3], the authors characterized the linear operators that strongly preserve multivariate majorization with $T(I) = I$ on nonnegative real matrices. They also conjectured that the main theorem is valid without $T(I) = I$. This means that the operators that strongly preserve multivariate majorization, without $T(I) = I$, are the operators of the form $A \to CAP$ where P is a permutation matrix and C is an invertible matrix. We now show it is in fact true.

2. Main results

The characterization of linear operators that strongly preserve multivariate majorization with $T(I) = I$ on nonnegative real matrices was given in [3]. In this section we characterize the linear operators that strongly preserving multivariate majorization without $T(I) = I$. Throughout the paper, let $M_n(\mathbb{R})$ be the set of $n \times n$ real matrices, and $M_n(\mathbb{R}^+)$ be the set of $n \times n$ nonnegative real matrices.

In [3], the authors proved the following.

THEOREM 2.1. ([3], Theorem 3.1) *If $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ strongly preserves multivariate majorization, $T(I) = I$, and $T : M_n(\mathbb{R}^+) \to M_n(\mathbb{R}^+)$ (i.e., T preserves nonnegative matrices) then there exists a permutation matrix P such that $T(X) = P^tXP$ for every $X \in M_n(\mathbb{R})$.*

LEMMA 2.2. ([3], Lemma 2.3) *If T is a linear operator on $M_n(\mathbb{R})$ that strongly preserves multivariate majorization, then T is nonsingular.*

Now, we will study properties for strong preservers of multivariate majorization.

LEMMA 2.3. *Let $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ strongly preserve multivariate majorization. If the image of a permutation matrix is a permutation matrix, then the image of a doubly stochastic matrix is a doubly stochastic matrix.*

Proof. For any doubly stochastic matrix D, we have $D \prec^{\text{mul}} P$ for every permutation matrix P. Then $T(D) \prec^{\text{mul}} T(P) = Q$ for a permutation matrix Q. Thus there is a $S \in \mathbf{DS}(n)$ such that is $T(D) = QS$. Therefore $T(D) \in \mathbf{DS}(n)$. □

LEMMA 2.4. ([3] Theorem 2.8) *If $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ strongly preserves multivariate majorization with $T(I) = I$, then the image of a permutation matrix under T is a permutation matrix.*

COROLLARY 2.5. *If $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ strongly preserves multivariate majorization with $T(I) = I$, then the image of a doubly stochastic matrix under T is a doubly stochastic matrix.*

Proof. By Lemma 2.3 and Lemma 2.4, the proof is complete. □

THEOREM 2.6. *If $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ strongly preserves multivariate majorization with $T(I) = I$, then the followings are equivalent;*

(1) *$T(P) = Q$, where P and Q are permutation matrices;*
(2) *$T(D) = S$ for $D,\ S \in \mathbf{DS}(n)$.*

THEOREM 2.7. *If $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ strongly preserves multivariate majorization with $T(I) = I$, then the image of all $n \times n$ permutation matrices is the set of all $n \times n$ permutation matrices.*

Proof. Since $\mathbf{P}(n)$ is a finite set, S is one to one on $\mathbf{P}(n)$ because of Lemma 2.2 and Lemma 2.4, and we have $T(\mathbf{P}(n)) = \mathbf{P}(n)$. □

Now we prove the conjecture.

THEOREM 2.8. *Let $L : M_n\ (\mathbb{R}^+) \to M_n(\mathbb{R}^+)$ be a linear operator that strongly preserves multivariate majorization, then*

$$L \text{ is of the form } A \mapsto CAP$$

with an invertible matrix $C \in M_n(\mathbb{R}^+)$ and $P \in \mathbf{P}(n)$.

Proof. Suppose L is a linear operator on $M_n(\mathbb{R}^+)$ that strongly preserves multivariate majorization, then L is nonsingular because of Lemma 2.2 and there is an invertible matrix B such that $L(B)$ is invertible. Now we define $T : M_n(\mathbb{R}^+) \to M_n(\mathbb{R}^+)$ by

$$T(X) = \{L(B)\}^{-1}L(BX).$$

Then $T(I) = I$ and T also strongly preserves multivariate majorization.

Since, by Theorem 2.1, $T(X) = P^tXP$ for some $P \in \mathbf{P}(n)$, we have $L(BX) = L(B)P^tXP$. Substitute $A = BX$, then $X = B^{-1}A$, and

$$L(A) = CAP$$

where $C = \{L(B)\}P^tB^{-1}$ is invertible and $P \in \mathbf{P}(n)$. □

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