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# A NOTE ON HOFER'S NORM

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Abstract. We show that when $(M, \omega)$ is a closed, simply connected, symplectic manifold for all $\gamma \in \pi_1(\mathrm{Ham}(M), id)$ the following inequality holds:

$$\|\gamma\| \geq \sup_{\bar{x}} |A(\bar{x})|,$$

where $\|\gamma\|$ is the coarse Hofer's norm, $\bar{x}$ run over all extensions to $D^2$ of an orbit $x(t) = \varphi_t(z)$ of a fixed point $z \in M$, $A(\bar{x})$ the symplectic action of $\bar{x}$, and the Hamiltonian diffeomorphisms $\{\varphi_t\}$ of $M$ represent $\gamma$.

## 1. Introduction

Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold. Then we can associate to a smooth 1-periodic Hamiltonian function $H : S^1 \times M \to \mathbb{R}$ the Hamiltonian vector field $X_H^t : M \to TM$, $t \in S^1$, which is defined by $\omega(X_H^t, \cdot) = -dH_t(\cdot)$ with $H_t(z) = H(t, z)$ for $z \in M$. The Hamiltonian vector field $X_H^t$ generates the Hamiltonian flow $\varphi_t$ via $\frac{d}{dt}\varphi_t = X_H^t \circ \varphi_t$, $\varphi_0 = id$. We denote by $\mathrm{Ham}(M)$ the group of Hamiltonian diffeomorphisms of $M$. Its Lie algebra is the space of Hamiltonian vector fields, which is identified with the space of all smooth Hamiltonian functions on $S^1 \times M$ satisfying the following normalization condition, where in the case of a compact manifold $M$ the function is only unique up to an additive constant. A function $H$ is said to satisfy a normalization condition if $H$ is compactly supported when $M$ is open, for all $t \in S^1$ the mean value of $H(t, \cdot)$ over $M$ vanishes, i.e., $\int_M H(t, z)\omega^n = 0$ when $M$ is closed.

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Now let $(M, \omega)$ be a $2n$-dimensional, closed, simply connected, symplectic manifold. Let $\{\varphi_t\}_{t\in S^1}$, $\varphi_0 = \varphi_1 = id$ be a loop of Hamiltonian diffeomorphisms of $M$, and let $H : S^1 \times M \to \mathbb{R}$ be the associated normalized 1-periodic Hamiltonian function. For a point $z \in M$ consider its orbit $x(t) = \varphi_t(z)$. Since $M$ is simply connected the map $x : S^1 \to M$ is contractible, and hence it can be extended to a map $\bar{x} : D^2 \to M$. Then the symplectic action of $\bar{x}$ is defined by

$$A(\bar{x}) = -\int_{D^2} \bar{x}^*\omega + \int_0^1 H(t, x(t))dt.$$

For an element $\gamma \in \pi_1(\mathrm{Ham}(M), id)$ define its coarse Hofer's norm by

$$\|\gamma\| = \inf_H \max_{t\in S^1,\, z\in M} |H(t,z)|,$$

where $H$ run over all normalized 1-periodic Hamiltonian functions $S^1 \times M \to \mathbb{R}$ generating a loop of Hamiltonian diffeomorphisms which represents $\gamma$. In this notes we induce an inequality between the coarse Hofer's norm for $\gamma \in \pi_1(\mathrm{Ham}(M), id)$ and the absolute value of the symplectic action of $\bar{x}$ defined above.

## 2. Symplectic fibrations

Let $(M, \omega)$ be a $2n$-dimensional, closed, symplectic manifold. Denote by $\mathrm{Symp}(M)$ the group of symplectic diffeomorphisms of $M$. Let $\{\varphi_t\}_{t\in S^1}$, $\varphi_0 = \varphi_1 = id$ be a loop of symplectic diffeomorphisms of $M$ which represents $\gamma \in \pi_1(\mathrm{Symp}(M), id)$. Consider two copies $D_-$ and $D_+$ of the disc $D^2$ bounded by $S^1$, and a map $\Psi : M \times S^1 \to M \times S^1$ given by $(z, t) \mapsto (\varphi_t z, t)$. Choose the orientation of $S^2$ induced from $D_+$. Then we obtain a symplectic fibration $p : P_\gamma = (M \times D_-) \cup_\Psi (M \times D_+) \to S^2$ with fiber $(M, \omega)$. When we consider only loops of Hamiltonian diffeomorphisms of $M$ the symplectic fibration is said to be Hamiltonian symplectic fibration. In [1] there exists a unique class in $H^2(P_\gamma; \mathbb{R})$ such that its restriction to fibers coincides with $[\omega]$ and its top power vanishes. This cohomology class is said to be the coupling class, which can be written as $[\omega'] + c\cdot p^*a$, where $\omega'$ is a 2-form on $P_\gamma$ satisfying $\omega'_x = \omega$ for $x \in S^2$ on the fiber, $a \in H^2(S^2; \mathbb{Z})$ the positive generator, and $c$ a constant.

## 3. Weak coupling

Let $p : P_\gamma \longrightarrow S^2$ be a Hamiltonian symplectic fibration with fiber $(M, \omega)$, which is associated with a loop $\{\varphi_t\}$ of Hamiltonian diffeomorphisms representing an element $\gamma \in \pi_1(\mathrm{Ham}(M), id)$. Let $H : S^1 \times M \to \mathbb{R}$ be the associated normalized 1-periodic Hamiltonian function. Let $c_\gamma \in H^2(P_\gamma; \mathbb{R})$ be the coupling class and $a$ the positive generator of $H^2(S^2; \mathbb{Z})$. The weak coupling construction says that for a sufficiently small $\epsilon > 0$ there exists a smooth family of closed 2-forms $\{\omega_t\}$, $t \in [0, \epsilon)$ on $P_\gamma$ with the following properties:

1. $\omega_0$ is the lift of an area form on $S^2$,
2. $[\omega_t] = tc_\gamma + p^* a$,
3. the restriction of $\omega_t$ to each fiber of $P_\gamma$ coincides with $t\omega$,
4. $\omega_t$ is symplectic for $t > 0$.

See [1] for details.

## 4. Coupling form

Let $p : P_\gamma \longrightarrow S^2$ be a symplectic fibration with a $2n$-dimensional closed symplectic manifold $(M, \omega)$ as its fiber. At each point $(z, x) \in P_\gamma$ denote by $\mathrm{Vert}_{(z,x)} = \ker\, dp(z,x) = T_{(z,x)}(P_\gamma)_x$ the vertical tangent space to the fiber. A connection $\nu$ on $p : P_\gamma \longrightarrow S^2$ is a field of horizontal subspaces $\mathrm{Hor}_{(z,x)} \subset T_{(z,x)}P_\gamma$ such that $TP_\gamma = \mathrm{Vert} \oplus \mathrm{Hor}$. The connection $\nu$ is said to be symplectic if the parallel transport preserves symplectic forms on fibers. The curvature $\rho_\nu$ of $\nu$ is a 2-form on the base which takes values in the Lie algebra of the group of symplectic diffeomorphims of fiber. A symplectic connection is said to be Hamiltonian if its curvature $\rho_\nu$ takes values in the space of Hamiltonian vector fields on the fiber $(M, \omega)$. We call such a connection a Hamiltonian symplectic connection. In view of the identification in Section 1 one can view $\rho_\nu(x)(\xi, \eta)$, $\xi,\ \eta\ \in T_x S^2$, as a Hamiltonian function on the fiber with the normalization condition.

In [1] and [2] the coupling form, say $\delta_\nu$, of a Hamiltonian symplectic connection $\nu$ is the unique closed 2-form on $P_\gamma$ such that the restriction to each fiber coincides with the symplectic form, which gives rise to a connection $\nu$ with horizontal distribution

$$\mathrm{Hor}_{(z,x)} = \mathrm{Vert}_{(z,x)}^{\delta_\nu} = \{\xi \in T_{(z,x)}P_\gamma \mid \delta_\nu(\xi, \eta) = 0\ \forall \eta \in \mathrm{Vert}_{(z,x)}\},$$

and the image of its top power under the fiber integration map

$$\mathrm{FI} \ : \ \Omega^{2(n+1)}(P_\gamma) \to \Omega^2(S^2)$$

vanishes, i.e., $\mathrm{FI}((\delta_\nu)^{n+1})(x)(\xi,\eta) = \int_{(P_\gamma)_x} \iota(\tilde{\xi} \wedge \tilde{\eta})(\delta_\nu)^{n+1} = 0$, where $\xi$, $\eta \in T_xS^2$, and $\tilde{\xi}$, $\tilde{\eta}$ their horizontal lifts to the points of the fiber $(P_\gamma)_x$ over $x$. In order to represent the coupling form $\delta_\nu$ explicitly we refer to [3].

Introduce polar coordinates $(u,t)$ on $D^2$, where $u \in [0,1]$ is the radius, and $t$ is the angle mod 1. Take a monotone cut-off function $\psi$ on $[0,1]$ such that $\psi(u) = 0$ for $u$ near 0, $\psi(u) = 1$ for $u$ near 1, and $|\psi'(u)| \leq 1+\epsilon$ for all $u$ and a small positive number $\epsilon$. Consider a 2-form on $P_\gamma$ which is $\omega$ on $M \times D_-$, and $\omega + \psi'(\cdot)H(t,\varphi_t(\cdot))p^*\tau$ on $M \times D_+$, where $\tau$ is a 2-form on $S^2$ representing the positive generator $a$ of $H^2(S^2;\mathbb{Z})$. It is closed and its restriction to each fiber coincides with the symplectic form $\omega$.

One can check that it gives rise to a connection and that the image of its top power under the fiber integration map vanishes. Moreover, the following calculation using the definition of $\psi$ and the fact that $H$ is normalized shows that the above 2-form is the coupling form, say $\delta_\nu$, which represents the coupling class $c_\gamma \in H^2(P_\gamma;\mathbb{R})$:

$$\begin{aligned}
\int_{P_\gamma} \delta_\nu^{n+1} &= \int_{M\times D_-} \omega^{n+1} + \int_{M\times D_+} [\omega + \psi'(u)H(t,\varphi_t(z))p^*\tau]^{n+1} \\
&= \int_{M\times D_+} \omega^n \wedge \psi'(u)H(t,\varphi_t(z))p^*\tau \\
&= \int_{M\times D_+} d(\omega^n \wedge \psi(u)H(t,\varphi_t(z))\ dt) \\
&= \int_0^1 \int_M H(t,\varphi_t(z))\omega^n dt \\
&= 0.
\end{aligned}$$

It was proved in [1] and [2] that the coupling form can be written as $\delta_\nu = \omega' \oplus -\rho_\nu$, where $TP_\gamma = \mathrm{Vert} \oplus \mathrm{Hor}$ is the splitting associated to the connection $\nu$ on $P_\gamma$, and $\rho_\nu$ the curvature of $\nu$. When the fiber $(M,\omega)$ of the Hamiltonian symplectic fibration $p : P_\gamma \to S^2$ is simply connected we can take an extension $\bar{x} : D^2 \to M$ of a map $x : S^1 \to M$ representing an orbit $x(t) = \varphi_t(z_0)$ of a point $z_0 \in M$. Define a section $s : S^2 \to P_\gamma$ by

$$s(u,t) = \begin{cases} (z_0,u,t) & \text{on} \quad D_+, \\ (\bar{x}(u,t),u,t) & \text{on} \quad D_-. \end{cases}$$

Using $c_\gamma = [\omega'] + c \cdot p^*a$ we find $(s^*c_\gamma, [S^2]) = c$. On the other hand

$$\begin{aligned}(s^*c_\gamma, [S^2]) &= \int_{D_+} s^*\delta_\nu - \int_{D_-} s^*\delta_\nu \\ &= \int_{D_+} \psi'(u)H(t, \varphi_t(z))\tau - \int_{D^2} \bar{x}^*\omega \\ &= [\psi(u)]_0^1 \int_0^1 H(t, \varphi_t(z))dt - \int_{D^2} \bar{x}^*\omega \\ &= A(\bar{x}),\end{aligned}$$

and hence we get the following theorem.

THEOREM 4.1. $c_\gamma = [\omega'] + A(\bar{x}) \cdot p^*a$.

Now consider vector fields $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ on $D_+$ to compute the curvature $\rho_\nu$ of the connection $\nu$. Their horizontal lifts to $P_\gamma$ are $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t} - \psi(u)X_H(t, \varphi_t(z))$. Thus at all $(u,t) \in D^+$ and for all $z \in M$, we finds that

$$\begin{aligned}\rho_\nu(\tfrac{\partial}{\partial u}, \tfrac{\partial}{\partial t})(z) &= [\tfrac{\partial}{\partial u}, \tfrac{\partial}{\partial t} - \psi(u)X_H(t, \varphi_t(z))]^{\mathrm{Vert}} \\ &= -\psi'(u)X_H(t, \varphi_t(z)).\end{aligned}$$

Since $H$ is normalized and $\tau(u,t)(\frac{\partial}{\partial u}, \frac{\partial}{\partial t}) = 1$, $\rho_\nu(z)$ is identified with $-\psi'(\cdot)H(t, \varphi_t(z))\tau$, and hence

$$\begin{aligned}c_\gamma = [\delta_\nu] &= [\omega' \oplus -\rho_\nu] \\ &= [\omega'] + \psi'(\cdot)H(t, \varphi_t(\cdot))p^*a.\end{aligned}$$

From this result and Theorem 4.1 we finds

$$|A(\bar{x})| \le \max_u |\psi'(u)| \cdot \max_{S^1 \times M} |H| \le (1+\epsilon) \max_{S^1 \times M} |H|.$$

Since $\epsilon$ can be taken arbitrarily small, we have $|A(\bar{x})| \le \max_{S^1 \times M} |H|$. Since $H$ and $\bar{x}$ are also arbitrary, we get the following theorem which we want to prove.

THEOREM 4.2. *Let $(M, \omega)$ be a closed, simply connected, symplectic manifold. Then for all $\gamma \in \pi_1(Ham(M), id)$*

$$\|\gamma\| \ge \sup_{\bar{x}} |A(\bar{x})|,$$

*where $\bar{x}$ run over all extensions to $D^2$ of an orbit $x$ in $M$.*

If the manifold $(M, \omega)$ is weakly exact, then the symplectic action $A(\bar{x})$ is independent of the choice of the extensions $\bar{x}$ of the orbit $x$ in $M$. Thus the $A(\bar{x})$ is constant on the extensions $\bar{x}$ of the $x$, and then we denote the action by $A(x)$.

COROLLARY 4.3. *If $(M, \omega)$ is a closed, simply connected, weakly exact, symplectic manifold, then for all $\gamma \in \pi_1(\mathrm{Ham}(M), id)$ we have an inequality:*

$$\|\gamma\| \ge |A(x)|.$$

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