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PROJECTIVE LIMIT OF A SEQUENCE OF BANACH FUNCTION ALGEBRAS AS A FRÉCHET FUNCTION ALGEBRA

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ABSTRACT. Let X be a hemicompact space with (K_n) as an admissible exhaustion, and for each $n \in \mathbb{N}$, A_n a Banach function algebra on K_n with respect to $\|\cdot\|_n$ such that $A_{n+1}|_{K_n} \subset A_n$ and $\|f|_{K_n}\|_n \leq \|f\|_{n+1}$ for all $f \in A_{n+1}$. We consider the subalgebra $A = \{f \in C(X) : f|_{K_n} \in A_n, \forall n \in \mathbb{N}\}$ of $C(X)$ as a Fréchet function algebra and give a result related to its spectrum when each A_n is natural. We also show that if X is moreover noncompact, then any closed subalgebra of A cannot be topologized as a regular Fréchet Q-algebra. As an application, the Lipschitz algebra of infinitely differentiable functions is considered.

1. Introduction

Let X be a compact Hausdorff space. We denote the algebra of all continuous functions on X by $C(X)$ and the uniform norm of $f \in C(X)$ by $\|f\|_X$. Under a norm, a Banach subalgebra of $C(X)$, which contains the constants and separates the points of X, is called a *Banach function algebra* on X. The uniform norm of an element in a Banach function algebra does not exceed from its norm. A Banach function algebra B on X is called *natural* if each complex homomorphism on B is an evaluation homomorphism at some point of X.

By a Fréchet algebra $(A, (p_n))$ we mean a topological algebra A whose topology can be defined by a sequence (p_n) of separating and submultiplicative seminorms, $p_n(fg) \leq p_n(f)p_n(g)$, $f, g \in A$, and which is complete with respect to this topology. Without loss of generality we can assume that $p_n \leq p_{n+1}$ and that $p_n(1) = 1$ if A has unit 1 (see

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[4]). A Fréchet algebra A is called a *Q-algebra* if the set of quasi-regular elements of A is open in A. This is equivalent to say that the set of quasi-regular elements of A has an interior (see [1]).

In this paper, we assume that all algebras are unital.

The spectrum of a commutative Fréchet algebra $(A,(p_n))$, which is denoted by M_A, is the set of all non-zero continuous complex homomorphisms on A, and for each $f \in A$, $\widehat{f} : M_A \to \mathbb{C}$ is the Gelfand transform of f. We always endow M_A with the Gelfand topology. The Fréchet algebra A is called *functionally continuous* if each complex homomorphism on A is continuous. It is unanswered for about 50 years whether or not each Fréchet algebra is functionally continuous (Michael's problem).

DEFINITION 1.1. A Hausdorff space X is called *hemicompact* if there exists a sequence (K_n) of increasing compact subsets of X such that each compact subset of X is contained in some K_n. The sequence (K_n) with this property is called an *admissible exhaustion* of X.

Let $(A,(p_n))$ be a Fréchet algebra. For each n, let A_n be the completion of $A/\ker p_n$ with respect to the norm $p'_n(f+\ker p_n) = p_n(f)$. Then A_n is a Banach algebra, $A = \varprojlim A_n$, projective limit of (A_n), and $M_A = \cup M_{A_n}$ as sets. Moreover, M_A is a hemicompact space with (M_{A_n}) as an admissible exhaustion and $M_{A_n} = \{\phi \in M_A : | \phi(f) | \leq p_n(f), \forall f \in A\}$, $n \in \mathbb{N}$ (see [4]).

DEFINITION 1.2. Let X be a hemicompact space and A a subalgebra of $C(X)$ which contains the constants and separates the points of X. We call A a *Fréchet function algebra* or *Ff-algebra* on X if it is a Fréchet algebra with respect to some topology such that the evaluation homomorphism φ_x at each $x \in X$ is continuous, that is, $\varphi_x \in M_A$.

We can consider each commutative unital semisimple Fréchet algebra as an Ff-algebra on its spectrum. So indeed the class of Ff-algebras and the class of commutative unital semisimple Fréchet algebras are the same.

Now let $(A,(p_n))$ be an Ff-algebra on X. Since $J : X \to M_A$, $x \mapsto \varphi_x$, is a continuous injective map, $\{\varphi_x : x \in K_n\}$ is a compact subset of M_A for each $n \in \mathbb{N}$. So for each n there exists an integer m such that $\{\varphi_x : x \in K_n\} \subset M_{A_m}$. Therefore,

$$(1) \qquad \|f\|_{K_n} = \sup_{x\in K_n} |\varphi_x(f)| \leq \sup_{\varphi\in M_{A_m}} |\varphi(f)| = \|\widehat{f}\|_{M_{A_m}} \leq p_m(f)$$

for all $f \in A$.

For each $n \in \mathbb{N}$, let $i(n) \geq n$ be the smallest integer that $\|f\|_{K_n} \leq p_{i(n)}(f)$ holds for all $f \in A$ and define p''_n on $A|_{K_n}$ by

$$p''_n(f|_{K_n}) = \inf\{p_{i(n)}(g) : g|_{K_n} = f|_{K_n},\ g \in A\}$$

for each $f \in A$. Then p''_n is an algebra norm on $A|_{K_n}$. Let A_{K_n} be the completion of $A|_{K_n}$ with respect to the norm p''_n. Then we have the following result:

THEOREM 1.3 ([6]). *Let $(A, (p_n))$ be an Ff-algebra on X, (K_n) an admissible exhaustion of X and (A_{K_n}) as defined above. Then (A_{K_n}) is a sequence of Banach algebras and A is dense in $\varprojlim A_{K_n}$. Moreover, if* $\ker q_n \subset \ker p_{i(n)}$ *for each positive integer n, then A is algebraically and topologically a projective limit $\varprojlim A_{K_n}$, where q_n is defined by $q_n(f) = \|f\|_{K_n}$.*

THEOREM 1.4 ([6]). *Let $(A, (p_n))$ and $(B, (q_n))$ be Ff-algebras on hemicompact spaces X and Y, respectively, and let $T : (A, (p_n)) \to (B, (q_n))$ be a continuous monomorphism with a dense range. Then the injective adjoint spectral map $T^* : M_B \to M_A$, $\psi \mapsto \psi \circ T$, is surjective and proper, that is, the inverse image of each compact set is compact, if and only if for each $m \in \mathbb{N}$, there exists an integer n such that*

$$\|\widehat{f}\|_{M_{A_m}} \leq q_n(T(f))$$

for all $f \in A$.

2. Main results

Let X be a hemicompact space and (K_n) an admissible exhaustion of X. In this section, we assume that (A_n) is a sequence of Banach function algebras such that for each $n \in \mathbb{N}$, A_n is a Banach function algebra on K_n with respect to $\|\cdot\|_n$, $A_{n+1}|_{K_n} \subseteq A_n$ and $\|f|_{K_n}\|_n \leq \|f\|_{n+1}$ for all $f \in A_{n+1}$. Consider

$$A = \{f \in C(X) : f|_{K_n} \in A_n,\ n \in \mathbb{N}\}.$$

Clearly, A contains the constants and for each $n \in \mathbb{N}$, $p_n(f) = \|f|_{K_n}\|_n$, $f \in A$, defines a submultiplicative seminorm on A. It is easy to check that A is a Fréchet algebra with respect to the topology defined by the sequence (p_n) of seminorms. Moreover, the evaluation map φ_x at each

$x \in X$ is continuous. So if A separates the points of X, then A is an Ff-algebra on X.

Note that if X is compact and if each A_n is inverse closed, that is, $\frac{1}{f} \in A_n$ if $f \in A_n$ and $f(x) \neq 0$ for all $x \in K_n$, then A is a Q-algebra. This is because A is also inverse closed and there is an integer N such that $K_n = X$ for all $n \geq N$. Let $G = \{f \in A : 1 + f \in A^{-1}\}$, where A^{-1} is the set of all invertible elements of A. If $f \in A$ and $p_N(f) < \frac{1}{2}$, then $\|f\|_X \leq \|f|_{K_N}\|_N = p_N(f) < \frac{1}{2}$, since the norm of a Banach function algebra is greater than the uniform norm. Thus $(1+f)(x) \neq 0$ for all $x \in X$. Since A is inverse closed, $1 + f \in A$, that is, $f \in G$. Hence the open neighborhood $V = \{f \in A : p_N(f) < \frac{1}{2}\}$ of the origin is contained in G. So G has an interior point.

THEOREM 2.1. *Let X be a hemicompact space and let $(A_n, \|\cdot\|_n)$ and $(A, (p_n))$ be as defined above. Suppose that A separates the points of X and that for each n, A_n is natural. If $(B, (q_n))$ is an Ff-algebra on X which contains A as a dense subalgebra and the identity map $I : (A, (p_n)) \to (B, (q_n))$ is continuous, then $M_A = M_B$ as sets.*

Proof. Let $i(n)$, p''_n and A_{K_n} be as defined in Theorem 1.3. Here we notice that $i(n) = n$ and if $f, g \in A$ and $f|_{K_n} = g|_{K_n}$, then $\|(f - g)|_{K_n}\|_n = p_n(f-g) = 0$ so that $p_n(f) = p_n(g)$. This shows that for each $f \in A$ $p''_n(f|_{K_n}) = p_n(f) = \|f|_{K_n}\|_n$, and so A_{K_n} is indeed the closure of $A|_{K_n}$ in the Banach function algebra $(A_n, \|\cdot\|_n)$. Therefore, in this case, each A_{K_n} is a Banach function algebra on K_n and $A = \varprojlim A_{K_n}$ by Theorem 1.3.

Since I is a continuous monomorphism with a dense range, $I^* : M_B \to M_A$, defined by $I^*(\varphi) = \varphi|_A$, is an injective continuous map. For each $m \in \mathbb{N}$ and each $f \in A$,

$$\|\widehat{f}\|_{M_{A_{K_m}}} = r_{A_{K_m}}(f|_{K_m}) = r_{A_m}(f|_{K_m}) = \|f\|_{K_m},$$

where $r_{A_m}(f|_{K_m})$ is the spectral radius of $f|_{K_m}$ in A_m and the last equality is a consequence of the naturality of A_m. On the other hand, since $(B, (q_n))$ is an Ff-algebra on X, for each $m \in \mathbb{N}$, there exists an integer $n \in \mathbb{N}$ such that

$$\|f\|_{K_m} \leq \|\widehat{f}\|_{M_{B_n}} \leq q_n(f), \quad f \in B,$$

where B_n is the completion of $B/\ker q_n$ with respect to the norm $q'_n(f + \ker q_n) = q_n(f)$, $f \in B$ (see the inequality (1)). So by Theorem 1.4, I^* is surjective and proper. Thus $M_A = M_B$ as sets. □

REMARK 1.

(a) In Theorem 2.1, if M_A is a k-space, then the restriction of ${I^*}^{-1}$ to each compact subset of M_A is continuous, since I^* is a proper map. So ${I^*}^{-1}$ is continuous on M_A. Hence M_A is homeomorphic to M_B.

(b) The naturality of each A_n cannot be omitted in Theorem 2.1. For example, let $X = [0,1]$, $K_n = X$, $A_n = A(\overline{D})|_{[-1,1]}$, where $\overline{D}$ is the closed unit disk in $\mathbb{C}$ and $A(\overline{D})$ is the uniform Banach algebra of continuous functions on $\overline{D}$ which are analytic on D. For each $f \in A_n$, there is a unique $g \in A(\overline{D})$ such that $g|_{[-1,1]} = f$. Define $\|f\|_n = \|g\|_{\overline{D}}$. Then $A = \{f \in C(X) : f|_{K_n} \in A_n\} = A(\overline{D})|_{[-1,1]}$, $M_A = \overline{D}$, and A is dense in $C([-1,1])$. But $M_{C([-1,1])} = [-1,1]$.

THEOREM 2.2. *Let X be a hemicompact noncompact space with (K_n) as an admissible exhaustion. Let $(A_n, \|\cdot\|_n)$ and $(A,(p_n))$ be as defined in the beginning of this section such that A separates the points of X. Then any closed subalgebra B of the Ff-algebra $(A,(p_n))$ cannot be normable as a regular Banach algebra.*

Proof. Let $\|\cdot\|$ be a norm on B such that $(B, \|\cdot\|)$ is a regular Banach algebra on M_B. Since B is closed in A, $(B,(p_n))$ is a commutative semisimple Fréchet algebra. By the Carpenter's theorem, i.e., each commutative semisimple Fréchet algebra has a unique topology as a Fréchet algebra, the identity map $I : (B, \|\cdot\|) \to (B,(p_n))$ is a homeomorphism. So there exist an $n_0 \in \mathbb{N}$ and an $M > 0$ such that

$$\|f\| \leq M \cdot p_{n_0}(f) \tag{2}$$

holds for all $f \in B$.

Since X is noncompact, one can choose an $x \in X \setminus K_{n_0}$. By the compactness of K_{n_0} in X and hence in M_B and by the regularity of B on M_B, there exists an $f \in B$ with $\widehat{f}(\varphi_x) = 1$ and $\widehat{f}(\varphi_y) = 0$ for all $y \in K_{n_0}$. That is, $f(x) = 1$ and $f|_{K_{n_0}} = 0$. Thus $p_{n_0}(f) = 0$. Now the inequality (2) implies that $\|f\| = 0$ and hence $f = 0$ as an element of B, which is a contradiction. □

REMARK 2. By the same method as the proof of Theorem 2.2, one can show that the closed subalgebra B of A cannot be topologized as a regular Fréchet Q-algebra.

EXAMPLE 2.3. Let (X,d) be a metric space and $0 < \alpha \leq 1$. The collection of all complex bounded Lipschitz functions of order α on X is denoted by $\mathrm{Lip}(X,\alpha)$. It is well-known (see [7]) that $\mathrm{Lip}(X,\alpha)$ with respect to pointwise multiplication is a Banach algebra under the norm $\|\cdot\|_\alpha$, defined by

$$\|f\|_\alpha = \|f\|_X + p_\alpha(f), \quad f \in \mathrm{Lip}(X,\alpha),$$

where $p_\alpha(f) = \sup_{x\neq y} \frac{|f(x)-f(y)|}{d^\alpha(x,y)}$ and $\|f\|_X = \sup_{x\in X} |f(x)|$.

Now let X be a hemicompact metric space, (K_n) an admissible exhaustion of X, and $0 < \alpha \leq 1$. Let $A_n = \mathrm{Lip}(K_n,\alpha)$ and

$$\|f\|_n = \|f\|_{K_n} + \sup_{\substack{x,y\in K_n \\ x\neq y}} \frac{|f(x)-f(y)|}{d^\alpha(x,y)}, \quad f \in A_n.$$

Clearly, $A_{n+1}|_{K_n} \subset A_n$ and $\|f|_{K_n}\|_n \leq \|f\|_{n+1}$, $f \in A_{n+1}$. So by the above argument, $\mathrm{FLip}(X,\alpha) = \{f \in C(X) : f|_{K_n} \in \mathrm{Lip}(K_n,\alpha),\ n \in \mathbb{N}\}$ is an Ff-algebra on X with respect to the topology defined by the sequence (p_n) of seminorms, where $p_n(f) = \|f|_{K_n}\|_n$ for all $f \in \mathrm{FLip}(X,\alpha)$ and all $n \in \mathbb{N}$. Using [7, Proposition 1.4], one can show that $\mathrm{FLip}(X,\alpha)$ is dense in $C(X)$ in the compact-open topology. So by Theorem 2.1, $M_{\mathrm{FLip}(X,\alpha)} = M_{C(X)} = X$. Indeed, one can show that the Gelfand topology on X inherited from $M_{\mathrm{FLip}(X,\alpha)}$ coincides on the metric topology and so $M_{\mathrm{FLip}(X,\alpha)} \cong X$.

EXAMPLE 2.4. Let $0 < \alpha \leq 1$ and X a perfect compact plane set which is a finite union of regular sets. The algebra of all functions f on X which are n-times differentiable and for each k, $0 \leq k \leq n$, $f^{(k)} \in C(X)$ (resp. $f^{(k)} \in \mathrm{Lip}(X,\alpha)$) is denoted by $D^n(X)$ (resp. $\mathrm{Lip}^n(X,\alpha)$) and the algebra of all functions f with derivatives of all orders (resp. $f^{(k)} \in \mathrm{Lip}(X,\alpha)$ $\forall k \in \mathbb{N}$) is denoted by $D^\infty(X)$ (resp. $\mathrm{Lip}^\infty(X,\alpha)$).

It is well-known (see [3, 5]) that for each $n \in \mathbb{N}$, $D^n(X)$ and $\mathrm{Lip}^n(X,\alpha)$ are natural Banach function algebras on X under the norms, defined by

$$\|f\|_n = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_X}{k!}$$

and

$$\|f\|_n = \sum_{k=0}^{n} \frac{\|f^{(k)}\|_X + p_\alpha(f^{(k)})}{k!},$$

respectively.

Now for each $n \in \mathbb{N}$, set $K_n = X$ and $A_n = D^n(X)$ (resp. $\mathrm{Lip}^n(X,\alpha)$). Then $A = \{f \in C(X) : f|_{K_n} \in A_n,\ n \in \mathbb{N}\} = D^\infty(X)$ (resp. $A = \mathrm{Lip}^\infty(X,\alpha) = \cap A_n$) and $(A, (\|\cdot\|_n))$ is an Ff-algebra on X. Moreover, we have the following inclusions:

$$R_0(X) \subseteq \mathrm{Lip}^\infty(X,\alpha) \subseteq \mathrm{Lip}^n(X,\alpha) \subseteq D^n(X) \subseteq D^1(X)$$

and $D^1(X) \subseteq R(X)$, where $R_0(X)$ is the algebra of all rational functions with poles off X and $R(X)$ is the uniform closure of $R_0(X)$ (see [3]). Thus A is dense in $R(X)$, and since each A_n is natural, we have $M_A = M_{R(X)} = X$ by Theorem 2.1. Indeed, by the compactness of X, M_A is homeomorphic to X.

REMARK 3.

(a) Notice that the algebra $\mathrm{FLip}(X,\alpha)$, defined in Example 2.3, is not in general a Banach algebra. Indeed, it is a Banach algebra if and only if X is compact.

(b) In Example 2.4, the algebras $\mathrm{Lip}^\infty(X,\alpha)$ and $D^\infty(X)$ are Q-algebras, since each A_n is inverse closed. Moreover, there is no topology which make these algebras Banach algebras, since $f \mapsto f'$ defines a nontrivial derivation.

Now let (A_n) and $(A,(p_n))$ be as defined before such that A is an Ff-algebra on X. Set $b(A) = \{f \in A : \sup p_n(f) < \infty\}$ and $\|f\|_\infty = \sup p_n(f)$ for each $f \in b(A)$. Then it is not difficult to check that $(b(A), \|\cdot\|_\infty)$ is a Banach algebra. For instance, if $A = \mathrm{FLip}(X,\alpha)$, then $b(A) = \mathrm{Lip}(X,\alpha)$ and $\|f\|_\infty$ is the Banach algebra norm on $\mathrm{Lip}(X,\alpha)$, which was defined earlier, and if $A = \mathrm{Lip}^\infty(X,\alpha)$ then $b(A) = \mathrm{Lip}(X,M,\alpha) = \{f \in \mathrm{Lip}^\infty(X,\alpha) : \sum_{k=0}^\infty \frac{\|f^{(k)}\|_X + p_\alpha(f^{(k)})}{k!} < \infty\}$, and $\|\cdot\|_\infty$ is the summation applied in the definition of $\mathrm{Lip}(X,M,\alpha)$ which makes $\mathrm{Lip}(X,M,\alpha)$ a Banach algebra (see [5]), where $M = (k!)$.

Assume that $b(A) = A$. Since A is semisimple and the identity map $I : (b(A), \|\cdot\|_\infty) \mapsto (A,(p_n))$ is continuous, the identity map I is a homeomorphism. So (A, p_n) is a Banach algebra.

PROPOSITION 2.5. *Let $(A,(p_n))$ be as in Theorem 2.2. If A is regular then it is a Q-algebra if and only if X is compact.*

Proof. Assume that X is compact. Then it is also a compact subset of M_A. If U is an open subset of M_A containing X, then by the regularity of A, there exists an $f \in A$ with $\widehat{f}(\varphi_x) = 0,\ x \in X$, and $\widehat{f}|_{M_A \setminus U} = 1$, which is impossible if $U \neq M_A$. So M_A is the only open subset which contains X. This shows that A is dense in M_A, and so $M_A = X$. In particular, M_A is compact and so A is a Q-algebra (see [1, 6.3-2]).

The converse is a consequence of Remark 2. □

The following theorem is known for a regular Banach function algebra A and a Banach algebra B (see [2]). Applying [1, Proposition 5.6-1], we can obtain the same result when A is a regular Fréchet function algebra and B is a Fréchet algebra.

THEOREM 2.6. *Let $(A, (p_n))$ be a regular Ff-algebra on its spectrum M_A which is locally compact. Let $(B, (q_n))$ be a commutative Fréchet algebra and $\theta : A \to B$ a continuous monomorphism with a dense range. Then $\overline{\theta^*(M_B)} = M_A$.*

Proof. Since $\theta(A)$ is dense in B, $\theta^*(\psi) = \psi \circ \theta \neq 0$ for each $\psi \in M_B$. The continuity of θ shows that $\theta^*(M_B) \subseteq M_A$. Let $S = \overline{\theta^*(M_B)}$ and $\varphi \in M_A \setminus S$. Since A is Gelfand normal (see [1]) and M_A is locally compact, there exists an $f \in A$ with compact support such that $\widehat{f}(\varphi) = 1$ and $\operatorname{supp} \widehat{f} \subset M_A \setminus S$. Let $I = \{f \in A : \widehat{f}|_S = 0\}$, $K = \operatorname{supp} \widehat{f}$, and $J = k(K) = \{f \in A : \varphi(f) = 0,\ \varphi \in K\}$. Then I and J are closed ideals in A with $h(I) \cap h(J) = \phi$, where $h(I)$ is the set of all closed maximal ideals containing I. Hence $I + J = A$ by [1, Proposition 5.6-1], and so there are $h \in J$ and $g \in I$ with $h + g = 1$. Since $h \in J$, $g|_K = 1$. Consequently, $f = fg$. Since $g = 0$ on S, $\theta^*(\psi)(g) = 0$ for each $\psi \in M_B$ so that $\theta(g) \in \operatorname{rad}(B)$. So we show that $\theta(f) = 0$ and hence $f = 0$, which is a contradiction. Suppose that $q_{n_0}(\theta(f)) \neq 0$ for some $n_0 \in \mathbb{N}$. The equality $\theta(f) = \theta(f)\theta(g)$ implies that $\theta(f) = \theta(f)\theta(g^n)$ for each $n \in \mathbb{N}$. So $q_{n_0}(\theta(f)) \leq q_{n_0}(\theta(f)) q_{n_0}(\theta(g^n))$ and hence $q_{n_0}(\theta(g^n)) \geq 1$ for each $n \in \mathbb{N}$. But $\theta(g) \in \operatorname{rad}(B)$ and so $\lim_{n\to\infty} \sqrt[n]{q_{n_0}(\theta(g^n))} = 0$. □

REMARK 4.

(a) In Theorem 2.6, if B is a Q-algebra, then M_B is compact and so $\theta^*(M_B) = M_A$, that is, $M_B = M_A$ as sets.

(b) With the hypotheses of Theorem 2.6, θ^* is not surjective even if A is a regular Banach function algebra. For example, let X be a hemicompact noncompact metric space, $A = \operatorname{Lip}(X, \alpha)$, $B = F \operatorname{Lip}(X, \alpha)$ and θ the canonical inclusion map. Then $\overline{\theta^*(M_B)} = \overline{X} = M_A$.

(c) The regularity of A cannot be omitted in Theorem 2.6. For example, let A be as given in Remark 1, $B = C([-1,1])$, and θ the canonical inclusion map.

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