Bull. Korean Math. Soc. **39** (2002), No. 2, pp. 225–236

ANALYTIC SOLUTIONS OF A FUNCTIONAL DIFFERENTIAL EQUATION WITH PROPORTIONAL DELAYS

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ABSTRACT. By means of the method of majorant series, sufficient conditions are obtained for the existence of analytic solutions of a functional differential equation with proportional delays.

1. Introduction

Functional differential equations with proportional delays have been studied to some extent by many authors. In particular, in [3, 6], asymptotic behaviors of solutions of the equation

$$y'(x) = ay(\lambda x) + by(x),$$

are discussed. In [4], the exponential order of the solutions of the equation

$$y'(z) = Ay(z) + By(\lambda z) + Cy'(\eta z)$$

is discussed. In [5], the well-posededness of the equation

$$y'(t) = ay(t) + \sum_{i=1}^{\infty} b_i y(q_i t) + \sum_{i=1}^{\infty} c_i y'(p_i t)$$

is discussed. In [1], Ifantis discusses the existence of analytic solutions of linear homogeneous functional differential equation

$$f^{(K)}(z) + \sum_{i=1}^{K} \phi_i(z) f^{(K-i)}(z) + \sum_{n=1}^{\infty} \alpha_n(z) f(q^n z) = 0.$$

Received March 20, 2001. Revised November 2, 2001.
2000 Mathematics Subject Classification: 34K25.
Key words and phrases: functional differential equation, proportional delay, analytic solution.

In this note, we will obtain sufficient conditions for the existence of analytic solutions of the following equation

$$(1)\quad f^{(K)}(z)+\sum_{i=1}^{K}\phi_i(z)f^{(K-i)}(z)+\sum_{n=1}^{T}\sum_{j=0}^{K}\psi_{nj}(z)f^{(K-j)}(p_{nj}z)=g(z),$$

under the initial conditions

$$(2)\qquad f^{(t)}(0)=\xi_t,\quad t=0,1,2,\dots,K-1.$$

More specifically, we will show that analytic solutions of (1)–(2) exist when ϕ_i, ψ_{nj}, g are analytic in a neighborhood of the origin, or, when $g\equiv 0$ and $z^i\phi_i$, $z^i\psi_{nj}$ are analytic in a neighborhood of the origin.

2. Main results

For the sake of convenience, the generalized factorial function $h^{[n]}(z)$ will be employed: $h^{[0]}(z)=1$ and

$$h^{[n]}(z)=h(z)h(z-1)\cdots h(z-n+1),\quad n=1,2,\dots.$$

Note that when k, n are nonnegative integers such that $k\geq n$,

$$k^{[n]}=k(k-1)\cdots(k-n+1)=\frac{k!}{(k-n)!}.$$

THEOREM 1. *Suppose* $|p_{nj}|\leq\mu<1$ *for* $j=0,\dots,K$ *and* $n=1,2,\dots,T$ *and suppose* $\phi_1,\dots,\phi_k,\psi_{10},\psi_{11},\dots,\psi_{TK}$ as well as g are *analytic functions of the form*

$$\phi_i(z)=\sum_{m=0}^{\infty}\phi_{im}z^m,\quad i=1,2,\dots,K,$$

$$\psi_{nj}(z)=\sum_{m=0}^{\infty}\psi_{njm}z^m,\quad j=0,\dots,K;\ n=1,2,\dots,T,$$

$$g(z)=\sum_{m=0}^{\infty}g_m z^m$$

for $|z|<\gamma$, *and*

$$1+\sum_{n=1}^{T}\psi_{n0}(0)p_{n0}^{m}\neq 0,\quad m=0,1,2,\dots.$$

Then the initial value problem (1)–(2) *has a unique analytic solution in the region defined by* $|z| < \gamma$.

Proof. We will seek an analytic solution of (1) in the form

$$f(z) = \sum_{m=0}^{\infty} f_m z^m, \tag{3}$$

which satisfies the initial conditions (2), that is, $f_m = \xi_m/m!$ for $m = 0, 1, \ldots, K-1$. Substituting (3) into (1), we see that $\{f_m\}_{m=0}^{\infty}$ is formally determined by the difference equation

$$\begin{aligned} g_m &= (K+m)^{[K]}\Big\{1 + \sum_{n=1}^{T} \psi_{n00} p_{n0}^m\Big\} f_{K+m} \\ &\quad + \sum_{i=1}^{K}\sum_{s=0}^{m} (K-i+m-s)^{[K-i]} \phi_{is} f_{K-i+m-s} \\ &\quad + \sum_{n=1}^{T}\sum_{j=1}^{K} (K-j+m)^{[K-j]} \psi_{nj0} p_{nj}^m f_{K-j+m} \\ &\quad + \sum_{n=1}^{T}\sum_{j=0}^{K}\sum_{s=1}^{m} (K-j+m-s)^{[K-j]} \psi_{njs} p_{nj}^{m-s} f_{K-j+m-s} \end{aligned}$$

for $m \geq 0$. In order to show that $f(z)$ is an analytic solution, we pick an arbitrary number τ in $(0, \gamma)$. By Cauchy's inequality, there exists a positive number M such that

$$|\phi_{im}| \leq \frac{M}{\tau^m}, \quad |\psi_{njm}| \leq \frac{M}{\tau^m}, \quad |g_m| \leq \frac{M}{\tau^m}, \quad m = 0, 1, 2, \ldots,$$

for $1 \leq i \leq K$, $0 \leq j \leq K$ and $n \geq 1$. Note that since $|p_{nj}| \leq \mu < 1$ for $j = 0, \ldots, K$ and $n = 1, 2, \ldots, T$, we may pick a positive integer $N \geq K-1$ such that

$$\Gamma_m \equiv 1 - \sum_{n=1}^{T} |\psi_{n00}||p_{n0}|^m > 0, \quad m > N.$$

Define a majorant sequence $\{S_m\}_{m=0}^{\infty}$ of the sequence $\{f_m\}_{m=0}^{\infty}$ as follows: for $0 \leq t \leq N$, let $B_0, \ldots, B_N$ be arbitrary positive numbers which satisfy

$$\begin{aligned}
|f_0| &= \left|\frac{\xi_0}{1!}\right| \le B_0, \dots, \\
|f_{K-1}| &= \left|\frac{\xi_{K-1}}{(K-1)!}\right| \le B_{K-1}, \\
|f_K| &\le B_K, \dots, |f_N| \le B_N,
\end{aligned}$$

and we let $S_t = B_t$ for $0 \le t \le N$; and for $t > N$, we define S_t recursively by

$$\begin{aligned}
\frac{t!\,\Gamma_{t-K}}{(t-K)!} S_t = & \sum_{i=1}^{K} \sum_{s=0}^{t-K} \frac{(t-i-s)!}{(t-K-s)!} \frac{M}{\tau^s} S_{t-i-s} \\
& + \sum_{n=1}^{T} \sum_{j=1}^{K} \frac{(t-j)!}{(t-K)!} M |p_{nj}|^{t-K} S_{t-j} \\
& + \sum_{n=1}^{T} \sum_{j=0}^{K} \sum_{s=1}^{t-K} \frac{(t-j-s)!}{(t-K-s)!} \frac{M}{\tau^s} |p_{nj}|^{t-K-s} S_{t-j-s} + \frac{M}{\tau^{t-K}}.
\end{aligned}$$

It is not difficult to see that the sequence $\{S_m\}$ defined above is positive and majorizes the sequence $\{f_m\}$, that is, $|f_m| \le S_m$ for $m \ge 0$. We assert further that

$$\limsup_{m\to\infty} S_m^{1/m} \le \frac{1}{\tau}.$$

To this end, note that for $m > N$,

$$\begin{aligned}
& \frac{(m+1)!\Gamma_{m+1-K}}{(m+1-K)!} S_{m+1} \\
= & \left[\frac{m!M}{(m+1-K)!} \left(1 + \sum_{n=1}^{\infty} |p_{n1}|^{m+1-K}\right) + \frac{m!}{(m-K)!} \frac{M}{\tau} \sum_{n=1}^{\infty} |p_{n0}|^{m-K} \right] S_m \\
& + M \sum_{i=2}^{K} \frac{(m+1-i)!}{(m+1-K)!} S_{m+1-i} + M \sum_{n=2}^{\infty} \sum_{j=2}^{K} \frac{(m+1-i)!}{(m+1-K)!} S_{m+1-j} \\
& + \frac{1}{\tau} \sum_{i=1}^{K} \sum_{s=1}^{m+1-K} \frac{(m+1-i-s)!}{(m+1-K-s)!} \frac{M}{\tau^{s-1}} S_{m+1-i-s}
\end{aligned}$$

$$+\frac{1}{\tau}\sum_{n=1}^{\infty}\sum_{j=1}^{K}\frac{(m-j)!}{(m-K)!}M|p_{nj}|^{m-K}S_{m-j}$$

$$+\frac{1}{\tau}\sum_{n=1}^{\infty}\sum_{j=0}^{K}\sum_{s=2}^{m+1-K}\frac{(m+1-j-s)!}{(m+1-K-s)!}\frac{M}{\tau^{s-1}}S_{m+1-j-s}+\frac{1}{\tau}\frac{M}{\tau^{m-K}}.$$

Then by straightforward calculations, we obtain

$$\begin{aligned}\frac{S_{m+1}}{S_m} &= \frac{1}{\Gamma_{m+1-K}}\Big[\frac{M}{m+1}\Big(1+\sum_{n=1}^{T}|p_{n1}|^{m+1-K}\Big)+\frac{1}{\tau}\frac{m+1-K}{m+1}\Gamma_{m-K} \\ &\quad +\frac{(m+1-K)M}{(m+1)\tau}\sum_{n=1}^{T}|p_{n0}|^{m-K}+M\sum_{i=2}^{K}\frac{(m+1-i)!}{(m+1)!}\frac{S_{m+1-i}}{S_m} \\ &\quad +M\sum_{n=1}^{T}\sum_{j=2}^{K}\frac{(m+1-j)!}{(m+1)!}|p_{nj}|^{m+1-K}\frac{S_{m+1-j}}{S_m}\Big].\end{aligned}\tag{4}$$

It follows that

$$\frac{S_{m+1}}{S_m}\geq\frac{1}{\Gamma_{m+1-K}}\frac{1}{\tau}\frac{m+1-K}{m+1}\Gamma_{m-K}\geq\frac{1}{\tau}>0$$

for all large m, and that

$$\begin{aligned}\limsup_{m\to\infty}\frac{S_{m+1-t}}{S_m} &= \limsup_{m\to\infty}\Big(\frac{S_{m+1-t}}{S_{m+2-t}}\Big)\Big(\frac{S_{m+2-t}}{S_{m+3-t}}\Big)\cdots\Big(\frac{S_{m-1}}{S_m}\Big) \\ &= \frac{1}{\liminf_{m\to\infty}(S_{m+2-t}/S_{m+1-t})\dots(S_m/S_{m-1})}<\infty,\quad t\geq 2.\end{aligned}$$

By taking superior limits on both sides of (4), we see that

$$\begin{aligned}\limsup_{m\to\infty}\frac{S_{m+1}}{S_m} &\leq \limsup_{m\to\infty}\frac{1}{\Gamma_{m+1-K}} \\ &\quad\times\left[\limsup_{m\to\infty}\frac{M}{m+1}\left(1+\sum_{n=1}^{T}|p_{n1}|^{m+1-K}\right)\right. \\ &\quad+\limsup_{m\to\infty}\frac{1}{\tau}\frac{m+1-K}{m+1}\Gamma_{m-K} \\ &\quad+\limsup_{m\to\infty}\frac{(m+1-K)M}{(m+1)\tau}\sum_{n=1}^{T}|p_{n0}|^{m-K} \\ &\quad+M\sum_{i=2}^{K}\limsup_{m\to\infty}\frac{(m+1-i)!}{(m+1)!}\limsup_{m\to\infty}\frac{S_{m+1-i}}{S_m}\end{aligned}$$

$$+ M \sum_{n=1}^{T} \sum_{j=2}^{K} \limsup_{m\to\infty} \frac{(m+1-j)!}{(m+1)!}$$

$$\times |p_{nj}|^{m+1-K} \limsup_{m\to\infty} \frac{S_{m+1-j}}{S_m} \Bigg]$$

$$= \frac{1}{1}\left[0 + \frac{1}{\tau} + 0 + M\sum_{i=2}^{K} 0 + M\sum_{n=1}^{T}\sum_{j=2}^{K} 0\right] = \frac{1}{\tau}$$

as required.

Since the radius of convergence of the power series $S_0+S_1z+S_2z^2+\ldots$ is τ and since τ is an arbitrary number in $(0,\gamma)$, hence $f(z)$ is analytic for $|z| < \gamma$. The proof is complete. □

We have found analytic solutions for equation (1) when $\phi_1, ..., \phi_k, \psi_{10}, \psi_{11}, ..., \psi_{TK}$ as well as g are analytic functions near the point 0. Next we consider the case when $\phi_1, ..., \phi_k, \psi_{10}, \psi_{11}, ...,$ and ψ_{TK} have weak singularities at 0. More specifically, let us assume that

$$z^i \phi_i(z) = \sum_{m=0}^{\infty} \phi_{im} z^m, \ i = 1, 2, ..., K, \tag{5}$$

and

$$z^i \psi_{nj}(z) = \sum_{m=0}^{\infty} \psi_{njm} z^m \tag{6}$$

for $i = 1, 2, \ldots, K$; $j = 0, 1, \ldots, K$ and $n = 1, 2, \ldots, T$.

THEOREM 2. *Assume that* $g(z) \equiv 0$ *in* (1). *Suppose* $|p_{nj}| \le \mu < 1$ *for* $j = 0, \ldots, K$ *and* $n = 1, 2, \ldots, T$. *Suppose further that* (5) *and* (6) *hold and each power series in the right hand sides of* (5) *and* (6) *converges for* $|z| < \gamma$. *If* d *is a root of the "indicial equation"*

$$I(z) \equiv z^{[K]} + \sum_{i=1}^{K} \phi_{i0} z^{[K-i]} + \sum_{n=1}^{T}\sum_{j=0}^{K} \psi_{nj0} z^{[K-j]} p_{nj}^{z-K+j} = 0$$

which satisfies $I(m+d) \ne 0$ *for all* $m \ge 1$, *then equation* (1) *has a solution of the form*

$$f(z) = z^d \sum_{m=0}^{\infty} f_m z^m, \ f_0 \ne 0, \tag{7}$$

which is analytic for $|z| < \gamma$.

Proof. By substituting (5), (6) and (7) into equation (1), we see that $I(d)f_0 = 0$, and

$$\begin{aligned} I(m+d)f_m &= -\sum_{i=1}^{K}\sum_{s=1}^{m}\phi_{is}(m-s+d)^{[K-i]}f_{m-s} \\ &\quad -\sum_{n=1}^{T}\sum_{j=0}^{K}\sum_{s=1}^{m}\psi_{njs}(m-s+d)^{[K-i]}p_{nj}^{m-s+d-K+j}f_{m-s} \end{aligned}$$

for $m \geq 1$. Pick any number $\tau \in (0,\gamma)$. By Cauchy's inequality, there exists a positive number M such that

$$|\phi_{im}| \leq \frac{M}{\tau^m}, \quad |\psi_{njm}| \leq \frac{M}{\tau^m}, \quad m = 0,1,2,\dots,$$

for $1 \leq i \leq K$, $0 \leq j \leq K$, and $n \geq 1$. Define a majorant sequence $\{S_m\}_{m=0}^{\infty}$ of the sequence $\{f_m\}_{m=0}^{\infty}$ as follows: $S_0 = |f_0|$, and for $m \geq 1$,

$$\begin{aligned} S_m = &\frac{1}{|I(m+d)|} \\ &\times\left\{\sum_{i=1}^{K}\sum_{s=1}^{m}\frac{M}{\tau^s}(m-s+d)^{[K-i]}S_{m-s}\right. \\ &\left.\quad+\sum_{n=1}^{T}\sum_{j=0}^{K}\sum_{s=1}^{m}\frac{M}{\tau^s}(m-s+d)^{[K-j]}p_{nj}^{m-s+d-K+j}S_{m-s}\right\}. \end{aligned}$$

By induction, it is easy to see that $\{S_m\}$ is a positive sequence and $|f_m| \leq S_m$ for $m \geq 0$. Moreover,

$$\begin{aligned} &|I(m+1+d)|S_{m+1} \\ &= \sum_{i=1}^{K}\sum_{s=1}^{m+1}\frac{M}{\tau^s}(m+1-s+d)^{[K-i]}S_{m+1-s} \\ &\quad+\sum_{n=1}^{T}\sum_{j=0}^{K}\sum_{s=1}^{m+1}\frac{M}{\tau^s}(m+1-s+d)^{[K-j]}|p_{nj}|^{m+1-s+d-K+j}S_{m+1-s} \end{aligned}$$

$$= \sum_{i=1}^{K} \frac{M}{\tau}(m+d)^{[K-i]} S_m + \sum_{n=1}^{T}\sum_{j=0}^{K} \frac{M}{\tau}(m+d)^{[K-j]}|p_{nj}|^{m+d-K+j} S_m$$
$$+ \frac{1}{\tau}\sum_{i=1}^{K}\sum_{s=2}^{m+1} \frac{M}{\tau^{s-1}}(m+1-s+d)^{[K-i]} S_{m+1-s}$$
$$+ \frac{1}{\tau}\sum_{n=1}^{T}\sum_{j=0}^{K}\sum_{s=2}^{m+1} \frac{M}{\tau^{s-1}}(m+1-s+d)^{[K-j]}|p_{nj}|^{m+1-s+d-K+j} S_{m+1-s}$$
$$= \frac{M}{\tau}\sum_{i=1}^{K}(m+d)^{[K-i]} S_m + \frac{M}{\tau}\sum_{n=1}^{T}\sum_{j=0}^{K}(m+d)^{[K-j]}|p_{nj}|^{m+d-K+j} S_m$$
$$+ \frac{1}{\tau}|I(m+d)|S_m.$$

As in the proof of Theorem 1, it is then easy to show that

$$\lim_{m\to\infty} \frac{S_{m+1}}{S_m} = \frac{1}{\tau}.$$

Since the radius of convergence of the power series $S_0 + S_1 z + S_2 z^2 + \ldots$ is τ and since τ is an arbitrary number in $(0, \gamma)$, hence $f(z)$ is analytic for $|z| < \gamma$. Hence $f(z)$ is analytic for $|z| < \gamma$. The proof is complete. □

3. Examples

We first consider the initial value problem

(8) $$f^{(K)}(z) = af^{(K-1)}(\lambda z) + bf^{(K-1)}(z) + cf^{(K)}(\beta z) + g(z),$$

(9) $$f^{(m)}(0) = \xi_m,\ m = 0, 1, ..., K-1,$$

where $|\beta| < 1$, $|\lambda| \le 1$ and $1 - c\beta^m \neq 0$ for $m \ge 0$. When $c = 0$, $K = 1$ and $g(z) \equiv 0$, equation (8) reduces to

$$f'(z) = af(\lambda z) + bf(z),$$

which has been considered in [2] and [3]. Here we will assume that

$$g(z) = \sum_{m=0}^{\infty} g_m z^m$$

is analytic for $|z| < \gamma$. By Theorem 1, our problem has a unique solution of the form

$$f(z) = \sum_{m=0}^{\infty} f_m z^m$$

which is analytic for $|z| < \gamma$. To find this solution, we substitute $f(z)$ into (8) and find the difference equation

$$\frac{(m+1)!}{m!}(1-c\beta^m)f_{m+K} = (a\lambda^m + b)\frac{(m+K-1)!}{m!}f_{m+K-1} + g_m, \quad m \geq 0.$$

Since this equation is of the form

$$h_{m+1} = \rho_m h_m + \chi_m, \quad m \geq 0,$$

we easily see that

$$f_{m+K} = \frac{\Pi_{i=0}^m (a\lambda^i + b)}{\Pi_{i=0}^m (i+K)(1-c\beta^i} \frac{\xi_{K-1}}{(K-1)!} + \sum_{n=0}^m \frac{n!}{(m+K)!} \frac{\Pi_{i=n+1}^m (a\lambda^i + b)}{\Pi_{i=K}^m (1-c\beta^i)} g_n$$

for $m \geq 0$.

Next, we consider the equation

$$\text{(10)} \qquad f^{(K)}(x) = \sum_{n=1}^T \psi_n f^{(K)}(p_n x) + g(x)$$

$$\text{(11)} \qquad f^{(m)}(0) = \xi_m, \quad m = 0, 1, \ldots, K-1.$$

Under the conditions that $|p_n| \leq \mu < 1$ for $n \geq 1$, that

$$1 + \sum_{n=1}^\infty \psi_n p_n^m \neq 0, \quad m = 0, 1, 2, \ldots,$$

and that

$$g(x) = \sum_{m=0}^\infty g_m x^m$$

converges in the real interval $(-\gamma, \gamma)$, and $g_m > 0$ for all large m, as well as

$$\lim_{x \to \gamma^-} g(x) = +\infty,$$

we will show that (10)–(11) has a unique solution of the form

$$f(x) = \sum_{m=0}^{K-1} \frac{\xi_m}{m!} x^m + \sum_{m=0}^\infty \frac{m! g_m}{(m+K)!(1-\sum_{n=1}^\infty \psi_n p_n^m)} x^{m+K},$$

which is analytic for $|x| < \gamma$ and $f^{(K)}(x) \sim g(x)$ as $x \to \gamma^-$.

Indeed, Theorem 1 asserts that (10)–(11) has a unique solution of the form

$$f(x) = \sum_{m=0}^{\infty} f_m x^m$$

which is analytic for $|x| < \gamma$. Substituting $f(x)$ into (10), we obtain

$$f_{m+K} = \frac{m! g_m}{(m+K)!(1 - \sum_{n=1}^{T} \psi_n p_n^m)}, \; m \geq 0,$$

as required. Note that

$$f^{(K)}(z) = \sum_{m=0}^{\infty} \frac{g_m}{(1 - \sum_{n=1}^{T} \psi_n p_n^m)} x^m.$$

Since $|p_n| \leq \mu < 1$ for $n \geq 1$, if we let

$$b_m = \frac{g_m}{(1 - \sum_{n=1}^{T} \psi_n p_n^m)},$$

then

$$\frac{g_m}{(1 - \sum_{n=1}^{T} \psi_n p_n^m)} \sim g_m,$$
$$\frac{g_m}{(1 - \sum_{n=1}^{T} \psi_n p_n^m)} - g_m = o(g_m)$$

as $m \to \infty$. Thus for every $\varepsilon > 0$, there exists a positive integer N such that

$$|b_m - g_m| < \varepsilon |g_m| = \varepsilon g_m, \quad m > N,$$

and

$$\Big| \sum_{m=0}^{\infty} (b_m - g_m) x^m \Big| \leq \sum_{m=0}^{N} |b_m - g_m| \gamma^m + \varepsilon \sum_{m=N+1}^{\infty} g_m x^m, \quad 0 \leq x < \gamma.$$

Since $\lim_{s \to \gamma^-} g(x) = +\infty$, we can also find a positive number δ such that

$$\sum_{m=0}^{N} |b_m - g_m| \gamma^m \leq \varepsilon g(x), \quad \gamma - \delta < x < \gamma.$$

Therefore,

$$\Big| \sum_{m=0}^{\infty} (b_m - g_m) x^m \Big| \leq 2\varepsilon g(x), \; \gamma - \delta < x < \gamma,$$

which shows that $f^{(K)}(x) \sim g(x)$ as $x \to \gamma^-$.

As our final example, consider the generalized Bessel equation

(12) $$z^2 \big[f''(z) + \alpha f''(\beta z) \big] + z f'(\beta z) + \big(z^2 - 1/\beta \big) f(\beta z) = 0,$$

where $0 < \beta \leq 1$,

$$\alpha \neq -\frac{\beta^m + m + 1}{\beta^{m-1}(m+1)},\ m \geq 1.$$

Note that $d = 1$ is a solution of the indicial equation

$$z(z-1)(1 + \alpha\beta^{z-2}) + z\beta^{z-1} - \beta^{z-1} = 0.$$

In view of Theorem 2, equation (12) has an analytic solution of the form

$$f(z) = z\sum_{m=0}^{\infty} f_m z^m,\ f_0 \neq 0.$$

By substituting $f(z)$ into (12), we obtain the difference equation

$$f_m = \frac{-\beta^{m-1}}{(m+1)m(1+\alpha\beta^{m-1}) + m\beta^m} f_{m-2},\ m = 2, 3, \dots .$$

A solution of this difference equation is given by $f_{2m-1} = 0$ for $m = 1, 2, \dots$, and

$$f_{2m} = \frac{(-1)^m \beta^{m^2} f_0}{\Pi_{i=1}^m \left[(2i+1)2i(1+\alpha\beta^{2i-1}) + 2i\beta^{2i}\right]} .$$

Hence

$$f(z) = f_0 \sum_{m=0}^{\infty} \frac{(-1)^m \beta^{m^2}}{\Pi_{i=1}^m \left[(2i+1)2i(1+\alpha\beta^{2i-1}) + 2i\beta^{2i}\right]} z^{2m+1}.$$

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