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Vibration Analysis of Cantilever Plates Undergoing Translationally Accelerated Motion

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This paper presents a modeling method for the vibration analysis of translationally accelerated cantilever plates. An accurate dynamic modeling method, which was introduced in the previous study, is employed to obtain the equations of motion for the vibration analysis. Dimensionless parameters are identified to generalize the conclusions from numerical results. The effects of the dimensionless parameters on the natural frequencies and mode shapes are investigated. Particularly, the magnitude of critical acceleration which causes the dynamic buckling of the structure is calculated. Incidentally, the natural frequency loci veering phenomena are observed and discussed.

Key Words : Vibration Analysis, Accelerated Cantilever Plate, Dimensionless Parameters Natural Frequencies and Mode Shapes, Dynamic Buckling, Natural Frequency Loci Veering

1. Introduction

Flexible structures which have plate-like shapes are often found in several practical engineering examples such as aircraft wing, spacecraft appendages, and solar panels of satellites. Especially, when an aircraft takes off or a rocket launches, its flexible wing structures are accelerated. This acceleration induces inertia forces to the structures which cause the variations of bending stiffness. This results in the variation of the vibration characteristics of the structures. The vibration characteristics of these structures, which vary as the acceleration increases, should be identified accurately for the reliable designs of the structures.

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Earlier pioneering investigations on the vibration of blade structures have been made since 1920's. Southwell, R. (1921), Theodorsen, T. (1935), Schilhansl, M. (1958), Putter, S. and Manor, H. (1978), Bhat, R. (1986) and Yoo, H. and Shin, S. (1998) show portions of their contributions. In these studies, however, the structures are idealized as beams, and only limited to rotating structures. Since some structures behave like plates rather than beams, the modeling for a plate is necessary. Furthermore, translational motion often causes the variations of vibration characteristics. Therefore, the variations of vibration characteristics due to translational motion need to be investigated. Recently, Yoo, H. and Chung, J. (2001) introduced a dynamic modeling method for plates undergoing overall motion. The integrity of the modeling method was proved by comparing its results to those of (Dokainish and Rawtani, 1971; Ramamurti and Kielb, 1984). However, the modeling method has not been applied to the vibration analysis of a plate undergoing translational motion.

The purpose of the present work is to investigate the vibration characteristics of rectangular plates undergoing translationally accelerated motion. The dynamic modeling method, introduced in Yoo and Chung (2001), was employed to obtain the modal formulation, in which dimensionless parameters are identified. Especially, the effects of the dimensionless parameters on the variations of natural frequencies and their associated mode shapes are investigated. This class of problems has never been solved in previous studies. Thus, it constitutes original contribution of the present work.

This paper consists of four Chapters. Following the introduction, in Chapter 2, the equations of motion (obtained by the modeling method introduced in Yoo and Chung (2001)) are presented and the corresponding dimensionless modal equations are derived. In Chapter 3, numerical results based on the modal equations are obtained and discussions are made. Conclusions are given in the final chapter.

2. Formulation for Vibration Analysis

Figure 1 shows a rectangular plate which is characterized by natural length a, width b, and thickness h. The plate is attached to an accelerated rigid hub. ρ is the mass per unit area of the plate, ν is the Poisson's ratio, and D is the bending rigidity. Equations of motion for the accelerated plate can be obtained as follows (see Yoo and Chung (2001) for the detailed procedure of derivation):

$$\sum_{j=1}^{\mu}\Big[\Big(\int_0^a\int_0^b \rho\phi_i\phi_j dxdy\Big)\ddot{q}_j + \Big(\int_0^a\int_0^b D(\phi_{i,xx}\phi_{j,xx}+\phi_{i,yy}\phi_{j,yy}+\nu\phi_{i,yy}\phi_{j,xx} + \nu\phi_{i,xx}\phi_{j,yy}+2(1-\nu)\phi_{i,xy}\phi_{j,xy})\,dxdy\Big)q_j - a_x\Big(\int_0^b\int_0^a \rho(a-x)\,\phi_{i,x}\phi_{j,x}dxdy\Big)q_j - a_y\Big(\int_0^b\int_0^a \rho(b-y)\,\phi_{i,y}\phi_{j,y}dxdy\Big)q_j\Big]=0 \quad (1)$$

$(i=1, 2, \cdots, \mu)$

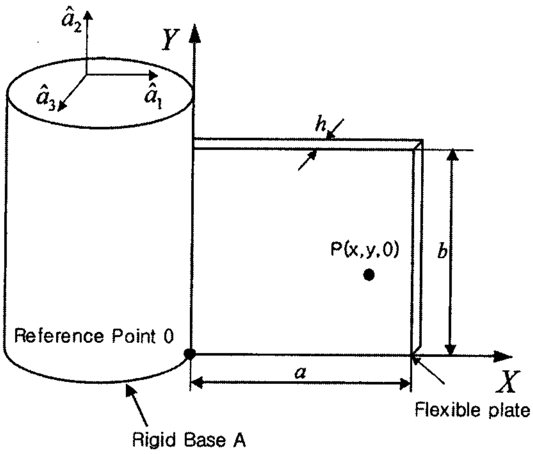


Fig. 1 Configuration of a translationally accelerated rectangular plate

where a_x and a_y denote the X and Y directional accelerations, respectively. In the above and in the following, a subscript following a comma indicates partial differentiation with respect to that subscript. For instance, $\phi_{i,xx}$ denotes the double partial derivative of ϕ_i with respect to x. In the present work, by using the Rayleigh-Ritz method, the lateral displacement w can be approximated to:

$$w(x, y, t)=\sum_{i=1}^{\mu}\phi_i(x, y)\,q_i(t) \quad (2)$$

where μ is the total number of generalized coordinates.

In Eq. (1), two motion-induced stiffness variation terms which are associated with acceleration are shown. These terms play important roles in the modal analysis of the translationally accelerated cantilever plates.

In order to draw general conclusions from numerical results, it is useful to rewrite Eq. (1) in a dimensionless form. For this purpose, the following dimensionless variables are introduced.

$$\tau=\frac{t}{T},\quad \xi=\frac{x}{a},\quad \eta=\frac{y}{b},\quad \vartheta_j=\frac{q_j}{a},\quad \delta=\frac{a}{b},\quad \alpha_x=\frac{a_x}{a_r},\quad \alpha_y=\frac{a_y}{a_r},\quad \varphi_i(\xi, \eta)\equiv\phi_i(x, y) \quad (3)$$

where a_r, a_s, and T are given as

$$T \equiv \sqrt{\frac{\rho a^4}{D}}$$
$$a_r \equiv \frac{a}{T^2} \qquad (4)$$
$$a_s \equiv \frac{b}{T^2}$$

Using these dimensionless variables, a dimensionless form of Eq. (1) is obtained as follows:

$$\sum_{j=1}^{\mu}\Big[\Big(\int_0^1\int_0^1 \varphi_i\varphi_j d\xi d\eta\Big)\ddot{\vartheta}_j$$
$$+\Big(\int_0^1\int_0^1 (\varphi_{i,\xi\xi}\varphi_{j,\xi\xi}+\delta^4\varphi_{i,\eta\eta}\varphi_{j,\eta\eta}+\nu\delta^2\varphi_{i,\eta\eta}\varphi_{j,\xi\xi}$$
$$+\nu\delta^2\varphi_{i,\xi\xi}\varphi_{j,\eta\eta}+2(1-\nu)\,\delta^2\varphi_{i,\xi\eta}\varphi_{j,\xi\eta})\,d\xi d\eta\Big)\,\vartheta_j \qquad (5)$$
$$-\alpha_x\Big(\int_0^1\int_0^1 (1-\xi)\,\varphi_{i,\xi}\varphi_{j,\xi}d\xi d\eta\Big)\vartheta_j$$
$$-\alpha_y\Big(\int_0^1\int_0^1 (1-\eta)\,\varphi_{i,\eta}\varphi_{j,\eta}d\xi d\eta\Big)\vartheta_j\Big]=0$$
$$(i=1,\,2,\,\cdots,\,\mu)$$

From the dimensionless Eq. (5), the eigenvalue problem of a translationally accelerated cantilever plate can be obtained by assuming that the ϑ_j's are harmonic functions of τ (dimensionless time). Defining a column matrix $\underline{\vartheta}$ whose elements are the ϑ_j's, one has

$$\underline{\vartheta}=e^{j\omega\tau}\underline{\Theta} \qquad (6)$$

where j denotes the imaginary number, ω is the dimensionless natural frequency which can be obtained by multiplying the natural frequency of the accelerated plate by T, and $\underline{\Theta}$ is a constant vector characterizing the deflection shape for synchronous motion. Substituting Eq. (6) into Eq. (5) yields the following discrete eigenvalue problem:

$$\omega^2 M\underline{\Theta}=K\underline{\Theta} \qquad (7)$$

where M and K are square matrices of size ($\mu\times\mu$), whose respective elements M_{ij} and K_{ij} are defined as

$$M_{ij}=\int_0^1\int_0^1 \varphi_i\varphi_j d\xi d\eta \qquad (8)$$
$$K_{ij}=\int_0^1\int_0^1 (\varphi_{i,\xi\xi}\varphi_{j,\xi\xi}+\delta^4\varphi_{i,\eta\eta}\varphi_{j,\eta\eta}$$
$$+\nu\delta^2\varphi_{i,\xi\xi}\varphi_{j,\eta\eta}$$
$$+2(1-\nu)\,\delta^2\varphi_{i,\xi\eta}\varphi_{j,\xi\eta})\,d\xi d\eta \qquad (9)$$
$$-\alpha_x\Big(\int_0^1\int_0^1 (1-\xi)\,\varphi_{i,\xi}\varphi_{j,\xi}d\xi d\eta\Big)$$
$$-\alpha_y\Big(\int_0^1\int_0^1 (1-\xi)\,\varphi_{i,\eta}\varphi_{j,\eta}d\xi d\eta\Big)$$

3. Numerical Results

In this section, numerical results are obtained by using the modal equation which is introduced in Chapter 2. Several assumed mode functions which are constructed by using cantilever beam functions and free-free beam functions are employed to obtain the matrices defined in Eqs. (8) and (9). In the present work, 5 cantilever beam functions and 7 free-free beam functions which include 2 rigid body mode functions are employed (see Leissa, A. W. (1969)) to generate 35 plate mode functions. Table 1 shows a typical trend of converging natural frequencies for an accelerated plate in the Y-direction in the case of $\delta=1$, $\alpha_y=4$, and $\alpha_x=0$. The natural frequencies converge rapidly as more modes are added. Therefore, 35 plate mode functions are presumably sufficient to obtain convergence for the lowest 5 modes of the accelerated plate.

Figure 2 shows the variations of the lowest five dimensionless natural frequencies of accelerated plates (with aspect ratio of $\delta=1$) with respect to Y component of dimensionless acceleration. The results show that the natural frequencies decrease as the dimensionless acceleration increases. As expected intuitively, since the accelerated plates move to the direction of buckling of plates, the natural frequencies decrease. An interesting phenomenon observed from Fig. 2 is that the first and the second eigenvalue loci approach each other as the acceleration increases. However, they do not cross but veer away from each other. This

Table 1 Convergence of dimensionless natural frequencies ($\alpha_y=4$, $\alpha_x=0$, $\delta=1$)

No. of X mode	No. of Y mode	1st Freq.	2nd Freq.	3rd Freq.	4th Freq.	5th Freq.
1	3	3.511	8.080	27.032	* * *	* * *
2	3	3.501	7.003	21.276	25.988	31.966
3	3	3.498	6.997	21.249	25.866	31.407
3	4	3.498	6.956	21.241	25.825	30.855
3	5	3.491	6.951	21.177	25.803	30.854
4	6	3.489	6.920	21.168	25.725	30.777
5	7	3.485	6.918	21.131	25.686	30.745
6	7	3.484	6.914	21.127	25.665	30.740

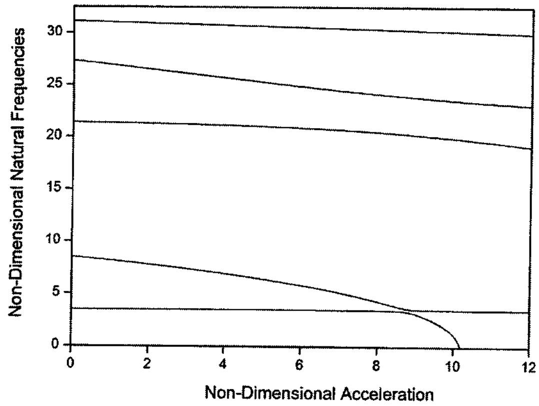
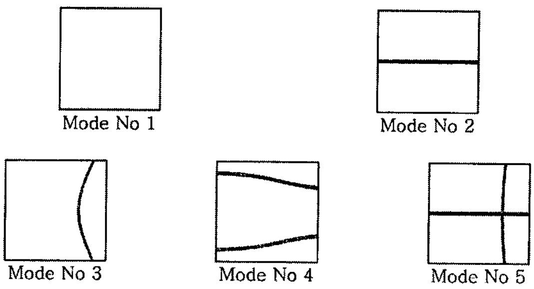
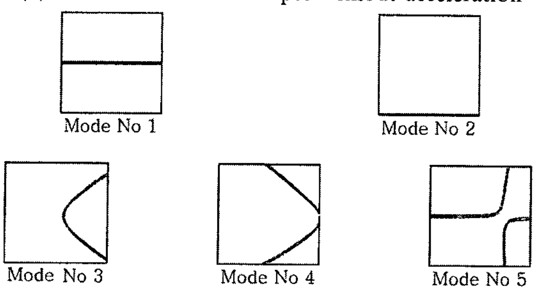


Fig. 2 Variations of non-dimensional natural frequencies vs. Y directional non-dimensional acceleration



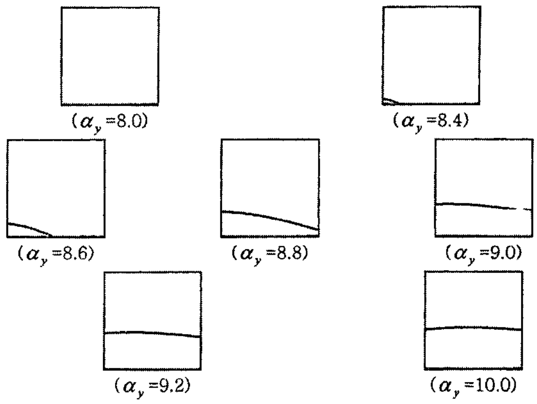
(a) Lowest five mode shapes without acceleration



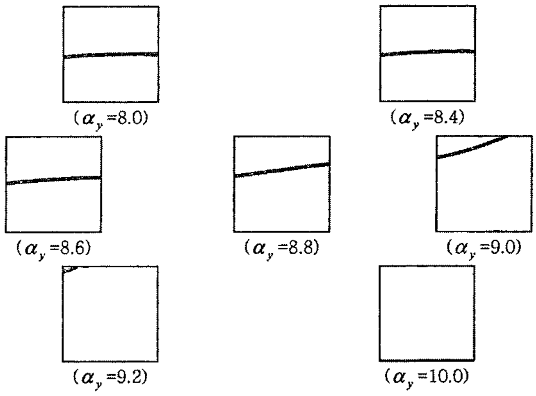
(b) Lowest five mode shapes with acceleration

Fig. 3 Nodal line patterns of lowest five mode shapes with and without acceleration

phenomenon is referred to as eigenvalue loci veering. Each of Figs. 3(a) and 3(b) show the nodal lines of the lowest five mode shapes when the dimensionless acceleration α_y is 0 and 10, respectively. It is shown in Fig. 3(a) that the first mode represents the first bending mode, the second mode represents the first torsion mode, the third mode represents the second bending mode, the fourth mode represents the chordwise bending



(a) Nodal line variations of 1st mode shape



(b) Nodal line variations of 2nd mode shape

Fig. 4 Nodal line variations of 1st and 2nd mode shapes with seven different accelerations

mode, and the fifth mode represents the combination mode of bending and torsion. Comparing these mode shapes of the accelerated plate to those of the non-accelerated plate, the first and second modes seem to switch their shapes. As shown in Fig. 2, the reason for this is that the eigenvalue associated with the second torsion mode decreases faster than that of the first bending mode as the acceleration increases. And, when acceleration exceeds 11, the first torsion natural frequency becomes zero. The accelerated cantilever plate will buckle at the zero natural frequency. Thus, this acceleration will be called the critical acceleration.

As the dimensionless acceleration increases (and the two loci approach each other), the first bending mode becomes more and more like the first torsion mode as shown in Fig. 4(a). On the

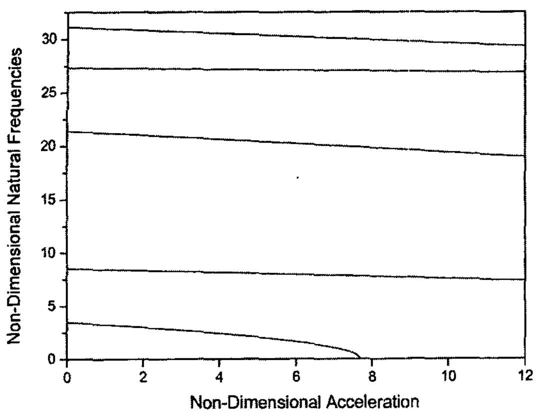
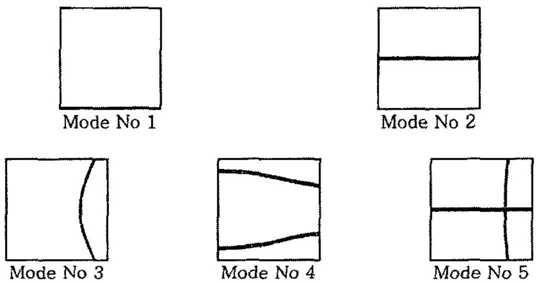
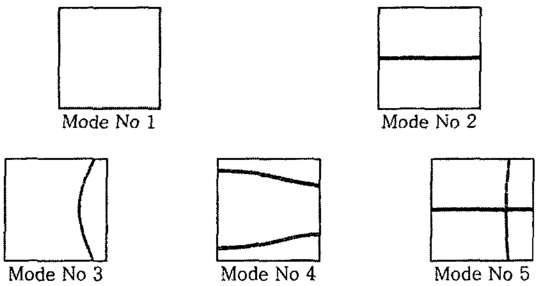


Fig. 5 Variations of non-dimensional natural frequencies vs. X directional non-dimensional acceleration



(a) Lowest five mode shapes without acceleration



(b) Lowest five mode shapes with acceleration

Fig. 6 Nodal line patterns of lowest five mode shapes with and without acceleration

other hand, as shown in Fig. 4(b), the first torsion mode becomes more and more like the first bending mode.

Figure 5 shows the variations of the lowest five dimensionless natural frequencies of accelerating plates (with aspect ratio of $\delta=1$) with respect to X directional dimensionless acceleration. As with Fig. 3, the lowest five natural frequencies decrease as the acceleration increases. However, while

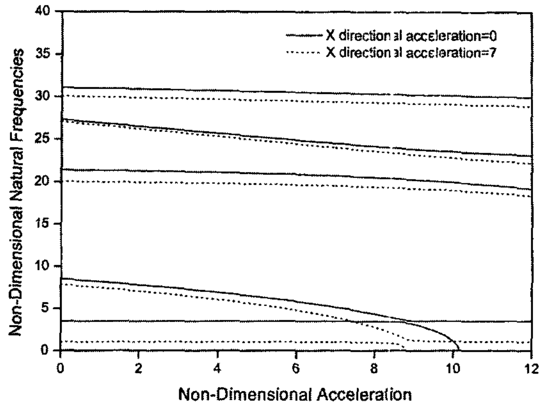


Fig. 7 Variations of non-dimensional natural frequencies vs. Y directional non-dimensional acceleration with or without constant X directional non-dimensional acceleration

dynamic buckling occurs to the first torsion mode in the case of Y directional acceleration, dynamic buckling occurs with the first bending mode in the case of X directional acceleration. Each of Figs. 6(a) and 6(b) shows the nodal lines of the lowest five mode shapes when the dimensionless acceleration is 0 and 7, respectively. Contrary to Fig. 2, the natural frequency veering does not appear in Fig. 5.

Figure 7 shows the variations of the lowest five dimensionless natural frequencies of plates (with aspect ratio of $\delta=1$) with combined acceleration effects. The solid lines represent the results without X directional acceleration and the dotted lines represent those with X directional acceleration. Those two types of lines show similar tendency. The X directional acceleration, however, results in the lower natural frequencies and lower dynamic buckling.

4. Conclusions

In this paper, a modeling method for the modal analysis of translationally accelerated plates is presented and the numerical results are obtained with the modeling method. It is shown that the modal characteristics vary with dimensionless parameters. Natural frequencies and mode shape variations due to acceleration changes are especially investigated and the magnitude of critical acceleration which causes the dynamic buckling is obtained. Incidentally, the natural frequency loci veering and associated mode shape variations are found from the numerical results. The modeling method presented in this paper can be employed for the design of translationally accelerated plate structures.

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References

Bhat, R., 1986, "Transverse Vibrations of a Rotating Uniform Cantilever Beam with Tip Mass as Predicted by Using Beam Characteristic Orthogonal Polynomials in the Rayleigh-Ritz Method," *J. of Sound and Vibration*, Vol. 105, No. 2, pp. 199~210.

Dokainish, M. and Rawtani, S., 1971, "Vibration Analysis of Rotating Cantilever Plates," *Int. J. for Numerical Methods in Engineering*, Vol. 3, pp. 233~248.

Han, B. K., Chung. K. and Han, D. S., 1989, "Vibration Analysis on Plates by Orthogonal Polynominals," *KSME International Journal*, Vol. 3, No. 2, pp. 95~102.

Leissa, A. W., 1969, Vibration of Plates, NASA SP-160.

Putter, S. and Manor, H., 1978, "Natural Frequencies of Radial Rotating Beams," *J. of Sound and Vibration*, Vol. 56, pp. 175~185.

Ramamurti, V. and Kielb, R., 1984, "Natural Frequencies of Twisted Rotating Plates," *J. of Sound and Vibration*, Vol. 97, No. 3, pp. 429~449.

Schilhansl, M., 1958, "Bending Frequency of a Rotating Cantilever Beam," *J. of Applied Mechanics Trans. of ASME*, Vol. 25, pp. 28~30.

Southwell, R. and Gough, F., 1921, "The Free Transverse Vibration of Airscrew Blades," *British A.R.C. Reports and Memoranda*, No. 766.

Theodorsen, T., 1935, "Propeller Vibrations and the Effect of Centrifugal Force," *NASA TN*, No. 516.

Yoo, H. and Chung, J., 2001, "Dynamics of Rectangular Plates Undergoing Prescribed Overall Motion," *J. of Sound and Vibration*, Vol. 239, No. 1, pp. 123~137.

Yoo, H. and Shin, S., 1998, "Vibration Analysis of Rotating Cantilever Beams," *J. of Sound and Vibration*, Vol. 212, No. 4, pp. 807~828.